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ON NICOLAS CRITERION FOR THE RIEMANN HYPOTHESIS

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Abstract. Nicolas criterion for the Riemann Hypothesis is based on an inequality that Euler totient function must satisfy at primorial numbers. A natural approach to derive this inequality would be to prove that a specific sequence related to that bound is strictly decreasing. We show that, unfortunately, this latter fact would contradict Cramér conjecture on gaps between consecutive primes. An analogous situation holds when replacing Euler totient by Dedekind Ψ function.

1. Introduction

The Riemann Hypothesis (RH), which describes the non trivial zeroes of Riemann ζ function has been qualified of Holy Grail of Mathematics by several authors [1, 8]. There exist many equivalent formulations in the literature [2]. The one of concern here is that of Nicolas [9] that states that the inequality

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \log\log N_k,$$

where

- $\gamma \approx 0.577$ is the Euler Mascheroni constant,
- φ Euler totient function ,
- $N_n = \prod_{k=1}^{n} p_k$ the primorial of order n,

holds for all $k \geq 1$ if RH is true [9, Th. 2 (a)]. Conversely, if RH is false, the inequality holds for infinitely many k, and is violated for infinitely many k [9, Th. 2 (b)]. Thus, it is enough, to confirm RH, to prove this inequality for k large enough. In this note, we show that a natural approach to this goal fails conditionally on a conjecture arguably harder than RH, namely Cramér conjecture [2]

$$p_{n+1} - p_n = O(\log^2 p_n).$$

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Note that under RH, it can only be shown that [3]

$$p_{n+1} - p_n = O(\sqrt{p_n} \log p_n).$$

See [5] for a critical discussion of this conjecture. An important ingredient of our proof is Littlewood oscillation Theorem for Chebyshev θ function [7, Th. 6.3]. An analogous situation holds when replacing Euler totient by Dedekind Ψ function, and replacing Nicolas criterion by [10, Th. 2].

2. An intriguing sequence

General conventions:

(1) We write $\log_2$ for $\log\log$, and $\log_3$ for $\log\log_2$
(2) The formula $f = O(g)$ means that $\exists C > 0$, such that $|f| \leq Cg$.
(3) The formula $a_k \sim b_k$ means that $\forall \epsilon > 0, \exists k_0$, such that $b_k(1-\epsilon) \leq a_k \leq b_k(1+\epsilon)$, if $k > k_0$.

We begin by an easy application of Mertens formula [6, Th. 429]. For convenience define

$$R(n) = \frac{n}{\varphi(n) \log_2 n}.$$

Recall, for future use, $\theta(x)$, Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \leq x} \log p.$$

Proposition 1. *For n going to ∞ we have*

$$\lim R(N_n) = e^{\gamma}.$$

Proof: Put $x = p_n$ into Mertens formula

$$\prod_{p \leq x} (1 - 1/p)^{-1} \sim e^{\gamma} \log(x)$$

to obtain

$$R(N_n) \sim e^{\gamma} \log(p_n),$$

Now the Prime Number Theorem [6, Th. 6, Th. 420] shows that $x \sim \theta(x)$ for x large. This shows that, taking $x = p_n$ we have

$$p_n \sim \theta(p_n) = \log(N_n).$$

The result follows. $\square$

Define the sequence

$$u_n = R(N_n).$$

We have just shown that this sequence converges to e^γ. But Nicolas inequality is equivalent to saying that

$$u_n > e^\gamma.$$

So we observe

Proposition 2. *If u_n is strictly decreasing for n big enough then Nicolas inequality is satisfied for n big enough.*

Proof: Assume $u_n > u_{n+1}$ for $n > n_0$ and that Nicolas inequality is violated for $N > n_0$ that is

$$u_n \leq e^\gamma,$$

then for $n \geq N+1$ we have $u_{n+1} < u_n \leq e^\gamma$. This implies

$$\overline{\lim}\, u_n < e^\gamma,$$

contradicting Proposition 1.

$\square$

We reduce the decreasing character of u_n to a concrete inequality between arithmetic functions.

Proposition 3. *The inequality $u_n > u_{n+1}$ is equivalent to*

$$\log(1 + \frac{\log p_{n+1}}{\theta(p_n)}) > \frac{\log \theta(p_{n+1})}{p_{n+1}}. \tag{1}$$

Proof: The inequality $u_n > u_{n+1}$ can be written as

$$\frac{N_n}{\varphi(N_n)\log_2 N_n} > \frac{N_{n+1}}{\varphi(N_{n+1})\log_2 N_{n+1}}.$$

Note first that

$$\frac{N_{n+1}}{\varphi(N_{n+1})} = \frac{1}{(1-1/p_{n+1})}\frac{N_n}{\varphi(N_n)},$$

so that, after clearing denominators, $u_n > u_{n+1}$ is equivalent to

$$\log_2(N_{n+1})(1-1/p_{n+1}) > \log_2 N_n,$$

or, distributing, to

$$\log_2(N_{n+1}) - \log_2 N_n > \frac{\log_2 N_{n+1}}{p_{n+1}}.$$

Now, to evaluate the LHS we write $N_{n+1} = N_n p_{n+1}$ so that

$$\log_2(N_{n+1}) = \log_2(N_n p_{n+1}) = \log(\log N_n + \log p_{n+1}) = \log_2 N_n + \log(1 + \frac{\log p_{n+1}}{\log N_n}).$$

to obtain

$$\log(1 + \frac{\log p_{n+1}}{\log N_n}) > \frac{\log_2 N_{n+1}}{p_{n+1}}.$$

The result follows then upon letting $\log N_n = \theta(p_n)$. □

In fact, more could be true.

Conjecture 1. *Inequality (1) holds for all* $n \geq 1$.

A heuristic motivation runs as follows

$$\log(1 + \frac{\log p_{n+1}}{\theta(p_n)}) \approx \frac{\log p_{n+1}}{\theta(p_n)} \approx \frac{\log p_{n+1}}{p_n}.$$

Similarly

$$\frac{\log \theta(p_{n+1})}{p_{n+1}} \approx \frac{\log p_{n+1}}{p_{n+1}}.$$

But, trivially

$$\frac{\log p_{n+1}}{p_n} > \frac{\log p_{n+1}}{p_{n+1}}.$$

Numerical computations confirm Conjecture 1 up to $n \leq 10000$. Unfortunately, Proposition 4 provides a conditional disproof of this conjecture.

3. Background material

We need an easy consequence of Littlewood oscillation theorem.

Lemma 1. *There are infinitely many n such that*

$$\theta(p_n) > k_n = p_n + C\sqrt{p_n}\log_3 p_n,$$

for some constant C independent of n.

Proof: By [7, Th. 6.3], we know there are infinitely many values of x such that

$$\theta(x) > x + C\sqrt{x}\log_3 x.$$

Let p_n be the largest prime $\leq x$. Thus

$$\theta(p_n) = \theta(x) > x + C\sqrt{x} > p_n + C\sqrt{p_n}\log_3 p_n.$$

□

4. More on u_n

Unfortunately, the sequence u_n is not decreasing as the next Proposition shows, conditionally on Cramér conjecture.

Proposition 4. *The inequality $u_n > u_{n+1}$ is violated for infinitely many n's.*

Proof: By Lemma 1 there are infinitely many n such that $\theta(p_n) > k_n$. For these n the RHS of (1) is $> \frac{\log k_{n+1}}{p_{n+1}} > \frac{\log k_n}{p_{n+1}}$.

Using the elementary bound $\log(1+u) < u$ for $0 < u < 1$, we see that the LHS of (1) is $< \frac{\log p_{n+1}}{k_n}$. Combining the bounds on the LHS and the RHS we obtain

$$k_n \log k_n < p_{n+1}\log p_{n+1}.$$

Since the function $x \mapsto x\log x$ is non decreasing for $x >> e$ we obtain $k_n < p_{n+1}$, that is

$$p_{n+1} - p_n > C\sqrt{p_n}\log_3 p_n,$$

which contradicts Cramér conjecture [2]

$$p_{n+1} - p_n = O(\log^2 p_n).$$

□

But is also not increasing, as the next Proposition shows unconditionally.

Proposition 5. *The inequality $u_n < u_{n+1}$ is violated for infinitely many n's.*

Proof: Suppose that $u_n < u_{n+1}$ for n big enough. Then for n large enough we have

$$u_n \leq e^{\gamma}.$$

If RH is true that is a contradiction by [9, Th. 2 (a)]. If RH is false that contradicts [9, Th. 2 (b)]. □

Thus u_n is not a monotone sequence for n big enough.

5. Analogous problem for Dedekind Ψ function

Recall that the Dedekind Ψ function is the multiplicative function defined by

$$\Psi(n) = n \prod_{p|n} (1 + \frac{1}{p}).$$

Define the sequence $v_n = \frac{\Psi(N_n)}{N_n \log_2 N_n}$. We proved in [10] the two statements

- $v_n > \frac{e^{\gamma}}{\zeta(2)}$ for all $n \geq 3$ iff RH is true
- $\lim v_n = \frac{e^{\gamma}}{\zeta(2)}$

Thus, like for the sequence u_n it is natural to wonder if v_n is decreasing.

Proposition 6. *The inequality $u_n > u_{n+1}$ is equivalent to*

$$\log(1 + \frac{\log p_{n+1}}{\theta(p_n)}) > \frac{\log \theta(p_n)}{p_{n+1}} \tag{2}$$

Proof: The inequality $v_n > v_{n+1}$ can be written as

$$\frac{\Psi(N_n)}{N_n \log_2 N_n} > \frac{\Psi(N_{n+1})}{N_{n+1} \log_2 N_{n+1}}.$$

Note first that

$$\frac{\Psi(N_{n+1})}{N_{n+1}} = (1 + 1/p_{n+1}) \frac{\Psi(N_n)}{N_n},$$

so that, after clearing denominators, $v_n > v_{n+1}$ is equivalent to

$$\log_2(N_{n+1}) > \log_2 N_n (1 + 1/p_{n+1}),$$

or, distributing, to

$$\log_2(N_{n+1}) - \log_2 N_n > \frac{\log_2 N_n}{p_{n+1}}.$$

Like in the proof of Proposition we have

$$\log_2(N_{n+1}) = \log_2 N_n + \log(1 + \frac{\log p_{n+1}}{\log N_n}).$$

Combining the last two statements we obtain

$$\log(1 + \frac{\log p_{n+1}}{\log N_n}) > \frac{\log_2 N_n}{p_{n+1}}.$$

The result follows then upon letting $\log N_n = \theta(p_n)$. □

Note that inequality 2 is slightly looser than inequality 1. Still, the analogue of Proposition 4 is true:

Proposition 7. *The inequality $v_n > v_{n+1}$ is violated for infinitely many n's.*

Similarly one can prove the analogue of Proposition 5 by using the arguments in the proof of [10, Th. 2].

Proposition 8. *The inequality $v_n < v_{n+1}$ is violated for infinitely many n's.*

The proofs of Propositions 7 and 8 are completely analogous to the case of Euler φ and are omitted.

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