Foliations on the moduli space of rank two connections on the projective line minus four points

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Abstract. We look at natural foliations on the Painlevé VI moduli space of regular connections of rank 2 on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$. These foliations are fibrations, and are interpreted in terms of the nonabelian Hodge filtration, giving a proof of the nonabelian Hodge foliation conjecture in this case. Two basic kinds of fibrations arise: from apparent singularities, and from quasiparabolic bundles. We show that these are transverse. Okamoto's additional symmetry, which may be seen as Katz's middle convolution, exchanges the quasiparabolic and apparent-singularity foliations.

1. Introduction

The Painlevé VI equation is the isomonodromic deformation equation for systems of differential equations of rank 2 on $\mathbb{P}^1$ with four logarithmic singularities over $D := \{t_1, t_2, t_3, t_4\}$. Such a system of differential equations is encoded in a vector bundle with logarithmic connection (E, ∇), where E is a vector bundle on $X = \mathbb{P}^1$ and $\nabla : E \to E \otimes \Omega^1_X(\log D)$ is a first order algebraic differential operator satisfying the Leibniz rule of a connection. At a singular point t_i the residue of ∇ is a linear endomorphism of E_{t_i}. The "space of initial conditions for Painlevé VI" is the moduli space of (E, ∇) such that the residues $\mathrm{res}(\nabla, t_i)$ lie in fixed conjugacy classes. The conjugacy class information is denoted $\mathbf{r}$, which for us will just mean fixing two distinct eigenvalues $r_i^{\pm}$ at each point. The isomonodromic evolution equation concerns what happens when the t_i move. However, in this paper we consider only the moduli space so the t_i are fixed.

The associated moduli stack is denoted by $\mathcal{M}^d(\mathbf{r})$. For generic choices of $\mathbf{r}$, all connections are irreducible and the moduli stack is a $\mathbb{G}_m$-gerb over the moduli space $M^d(\mathbf{r})$. Here d denotes the degree of the bundle E, related to $\mathbf{r}$ by the Fuchs relation (2.1). We usually assume that d is odd, essentially equivalent to $d = 1$, because any bundle of degree 1 having an irreducible connection must be of the form $B = \mathcal{O} \oplus \mathcal{O}(1)$. This facilitates the consideration of the parameter space for quasiparabolic structures.

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The object of this paper is to study several natural fibrations on the moduli space. The second author, with Inaba and Iwasaki, have described the structure of $M^d(\mathbf{r})$ as obtained by several blow-ups of a ruled surface over $\mathbb{P}^1$ in [**21, 22**]. The function to $\mathbb{P}^1$ may be viewed as given by the position of an apparent singularity, considered also by Szabo [**47**] and Aidan [**1**]. The first author has considered this fibration too but also looked at the function from $M^d(\mathbf{r})$ to the space of quasiparabolic bundles [**26**], which as it turns out is again $\mathbb{P}^1$ or more precisely a non-separated scheme which had been introduced by Arinkin [**2**]. The third author has defined a decomposition of $M^d(\mathbf{r})$ obtained by looking at the limit of $(E, u\nabla)$ as $u \to 0$ into the moduli space of semistable parabolic Higgs bundles [**46**].

We compare these pictures by examining precisely the condition of stability depending on parabolic weight parameters. A choice of one of the two residues r_i^- is made at each point, and the eigenspace provides a 1-dimensional subspace $P_i \subset E_{t_i}$. The collection $(E, P_\bullet)$ is a quasiparabolic bundle [**42**]. Given that $E \cong B = \mathcal{O} \oplus \mathcal{O}(1)$, we can write down a parameter space for all quasiparabolic structures on B. The moduli stack for such quasiparabolic bundles is the stack quotient by $A = Aut(B)$.

Specifying two parabolic weights $\alpha_i^\pm$ at each point transforms the quasiparabolic structures into parabolic ones for which there is a notion of stability. There is a collection of 8 inequalities concerning the parabolic weights appearing in Proposition 4.2: (a), (b) and 6 of type (c), see also (6.1) (6.2) (6.3). Depending on these inequalities, the underlying parabolic bundle will either be stable, or unstable. The space of parabolic weights is therefore divided up into a stable zone, and 8 unstable zones.

The different unstable zones are permuted by the operation of performing two elementary transformations. Doing two at a time keeps the condition that the underlying bundle has odd degree. Up to these permutations, we can assume that we are in the (a)-unstable zone $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2$ where $\epsilon_i = (\alpha_i^+ - \alpha_i^-)/2$. In this case, the subbundle $\mathcal{O}(1) \subset E = \mathcal{O} \oplus \mathcal{O}(1)$ is destabilizing. It determines an apparent singularity, which is the unique point at which the subbundle osculates to the direction of the connection. The position of this apparent singularity gives the map to $\mathbb{P}^1$. We point out in Theorem 5.6 that, in this unstable zone, this map is the same as the map taking (E, ∇) to the limiting α-stable Higgs bundle. This furnishes the comparison between the Higgs limit decomposition, and the fibration of [**21, 22**].

This comparison allows us to prove the foliation conjecture of [**46**] in this case. The Higgs limit decomposition is, from the definition, just a decomposition of the moduli space into disjoint locally closed subvarieties, which are Lagrangian for the natural symplectic structure. The foliation conjecture posits that this decomposition should be a foliation in the case when the moduli space is smooth. For the unstable zone, the decomposition is just the collection of connected components of the fibers of the smooth morphism of [**21, 22**] to $\mathbb{P}^1$, so it is a foliation.

We next turn our attention to the stable zone. The quasiparabolic bundles which support an irreducible connection with given residues are exactly the simple ones, and the quotient of the set of simple quasiparabolic structures by the automorphism group is the non-separated scheme $\mathcal{P}$ which is like $\mathbb{P}^1$ but has two copies of each t_i. This is the same as the space of leaves in the fibration corresponding to the unstable zone. It has also appeared in Arinkin's work [**2**] on the geometric Langlands program.

In the stable zone, the limit $\lim_{u\to 0}(E, u\nabla, P_\bullet)$ in the moduli space of α-stable parabolic Higgs bundles is just the underlying parabolic bundle $(E, P_\bullet)$, except at one from each pair of points lying over t_i. Thus, Theorem 6.2 says that in the stable zone, the Higgs limit decomposition is just the decomposition into fibers of the projection $M^d(\mathbf{r}) \to \mathcal{P}$ considered in [**26**], sending $(E, \nabla, P_\bullet)$ to $(E, P_\bullet)$. As before, this interpretation allows us to prove the foliation conjecture of [**46**] in this case.

Putting these together, we obtain a proof of the foliation conjecture for the moduli space of parabolic logarithmic connections of rank 2 on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ with any generic residues and any generic parabolic weights. The genericity condition is non-resonance plus a natural condition which has been introduced by Kostov, ruling out the possibility of reducible connections. The combination of these two conditions will be called "nonspeciality".

In Section 7 we point out that this discussion gives the same results for the case of local systems on a root stack. These correspond to parabolic logarithmic connections on $\mathbb{P}^1$ whose residues and weights are the same rational numbers. In the root stack interpretation, the Higgs limit decomposition may be tied back to the same thing on a compact curve, a cyclic covering of $\mathbb{P}^1$ branched over t_1, t_2, t_3, t_4.

In Section 8 we show that the two different kinds of fibrations, obtained from apparent singularities and from the quasiparabolic structure, are strongly transverse: generic fibers intersect once. A similar picture has been described by Arinkin and Lysenko [**4**] when we switch to trace-free connections (and $\deg(E) = 0$).

In Section 9 we recall the additional Okamoto symmetry, and the fact pointed out by the first author in [**26**] that it interchanges the two different types of fibrations considered above. Then in Section 10, we propose a possible explanation by interpreting Okamoto's additional symmetry as Katz middle convolution. This interpretation is now well known, see Arinkin-Lysenko [**4**], and Boalch [**7**] [**8**].

We calculate, concentrating on the case of finite order monodromy, that a middle convolution with suitably chosen rank 1 local system interchanges the stable and unstable zones. Assuming a compatibility of higher direct images which is not yet proven, the middle convolution will preserve the Higgs limit decomposition and this property would imply that it permutes the two different kinds of foliations.

As a part of the numerous ongoing investigations of the rich structure of these moduli spaces, the present discussion points out the role of the different regions in the space of parabolic weights. Nevertheless, a number of further questions remain open in this direction, such as what happens along the hyperplanes of special values of residues and/or parabolic weights. We hope to address these in the future.

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2. Moduli stacks of parabolic logarithmic λ-connections

Let $X := \mathbb{P}^1$, with a divisor consisting of four distinct points $D := \{t_1, t_2, t_3, t_4\}$, and put $U := X - D$. Let $\mathcal{M}^d \to \mathbb{A}^1$ denote the moduli stack [**21, 22**] of logarithmic λ-connections of rank two and degree d with quasiparabolic structure on (X, D). For a scheme S, an object of $\mathcal{M}(S)$ is a quadruple $(\lambda, E, \nabla, P_\bullet)$ where $\lambda : S \to \mathbb{A}^1$, E is a rank 2 vector bundle on $X \times S$ of degree d on the fibers $X \times \{s\}$, $P_\bullet = (P_1, P_2, P_3, P_4)$ is a collection of rank 1 subbundles

$$P_i \subset E|_{\{t_i\} \times S},$$

and

$$\nabla : E \to E \otimes_{\mathcal{O}_{X\times S}} \Omega^1_{X\times S/S}(\log D \times S),$$

is a logarithmic λ-connection on $X \times S/S$ preserving P_i. This means that ∇ is a map of sheaves satisfying $\nabla(ae) = a\nabla(e) + \lambda d(a)e$, inducing a residue endomorphism

$$\mathrm{res}(\nabla, t_i) : E_{\{t_i\}\times S} \to E_{\{t_i\}\times S}$$

which is required to preserve P_i. The groupoid $\mathcal{M}^d(S)$ has these objects, and morphisms are isomorphisms of bundles with λ-connection preserving the quasi-parabolic structure.

For $\lambda \in \mathbb{A}^1$ let $\mathcal{M}^d_\lambda$ denote the fiber of $\mathcal{M}^d \to \mathbb{A}^1$ over λ. For $\lambda = 1$ it is the moduli stack of logarithmic connections, and the fibers are all the same for $\lambda \neq 0$. For $\lambda = 0$ it is the moduli stack of Higgs bundles. In both cases, quasiparabolic structures are included.

The value of λ is determined by ∇, so it may be left out of the notation, writing if necessary $\lambda = \lambda(\nabla)$.

Given a point $(E, \nabla, P_\bullet) \in \mathcal{M}^d(S)$, we get two residue eigenvalues:

- $\mathrm{res}^-_i(E, \nabla)$ is the scalar by which $\mathrm{res}(\nabla, t_i)$ acts on P_i, and
- $\mathrm{res}^+_i(E, \nabla)$ is the scalar by which $\mathrm{res}(\nabla, t_i)$ acts on E_{t_i}/P_i.

These satisfy the Fuchs relation

$$\sum_{i=1}^{4} (\mathrm{res}^+_i(E, \nabla) + \mathrm{res}^-_i(E, \nabla)) + \lambda \deg(E) = 0. \tag{2.1}$$

Let $\mathcal{N}^d \to \mathbb{A}^1$ be the bundle of possible residual data satisfying the Fuchs relation for $\deg(E) = d$, so

$$\mathcal{N}^d = \{(\lambda, r^+_1, r^-_1, \ldots, r^+_4, r^-_4) | r^+_1 + \cdots + r^-_4 + \lambda d = 0\}.$$

The residues give a map

$$\Psi : \mathcal{M}^d \to \mathcal{N}^d$$

relative to $\mathbb{A}^1$. If $\mathbf{r}(\lambda) : \mathbb{A}^1 \to \mathcal{N}^d$ is a section denoted $\lambda \mapsto (\lambda, r^+_1(\lambda), \ldots, r^-_4(\lambda))$, let $\mathcal{M}^d(\mathbf{r}(\lambda))$ be the pullback of this section in $\mathcal{M}^d$. It is the moduli stack of $(E, \nabla, P_\bullet)$ such that the eigenvalue of the residue of ∇ acting on E_{t_i}/P_i (resp. P_i) is $r^+_i(\lambda(\nabla))$ (resp. $r^-_i(\lambda(\nabla))$) for $i = 1, 2, 3, 4$.

Note that in [**21**, §2.2] the notation is slightly different: the parameter we call λ here is replaced by ϕ but which has a somewhat more general meaning, and the residues are denoted there by λ_i which correspond to our r^-_i. In [**21**] it is assumed that $r^-_i + r^+_i \in \mathbb{Z}$ but that normalization doesn't make for any loss of generality.

Suppose $r^+_i \neq r^-_i$ for $i = 1, \ldots, 4$. Then $\mathcal{M}^d_1(\mathbf{r})$ may also be viewed as the moduli stack of logarithmic connections (E, ∇) with $\deg(E) = d$ and such that the eigenvalues of $\mathrm{res}(\nabla, t_i)$ are $r^\pm_i$, but without specifying $P_\bullet$. The eigenvalue condition is a closed condition, just saying that

$$\begin{aligned} \mathrm{Tr}(\mathrm{res}(\nabla, t_i)) &= r^+_i + r^-_i, \\ \det(\mathrm{res}(\nabla, t_i)) &= r^+_i r^-_i. \end{aligned}$$

Because of the hypothesis that the eigenvalues are distinct, the rank one subspace $P_i \subset E_{t_i}$ is uniquely determined as the r_i^--eigenspace of $\mathrm{res}(\nabla, t_i)$.

Let $\mathcal{M}_1^d(\mathbf{r})^{\mathrm{irr}} \subset \mathcal{M}_1^d(\mathbf{r})$ be the open substack parametrizing irreducible connections. It is a $\mathbb{G}_m$-gerb over its coarse moduli space

$$\mathcal{M}_1^d(\mathbf{r})^{\mathrm{irr}} \to M_1^d(\mathbf{r})^{\mathrm{irr}}.$$

The group $\mathbb{G}_m$ acts on $\mathcal{M}^d$ by

$$u : (\lambda, E, \nabla, P_\bullet) \mapsto (u\lambda, E, u\nabla, P_\bullet).$$

It is compatible with the standard action on the λ-line $\mathbb{A}^1$. The action on the residues is

$$\mathrm{res}_i^{\pm}(E, u\nabla) = u\,\mathrm{res}_i^{\pm}(E, \nabla).$$

Therefore if $\mathbf{r}(\lambda) = \lambda\mathbf{r}$ is a section such that $r_i^{\pm}(\lambda) = \lambda r_i^{\pm}$ then the action restricts to an action on $\mathcal{M}^d(\lambda\mathbf{r})$.

Over the open set $\lambda \neq 0$ the Artin stack $\mathcal{M}^d$ is of finite type, but if $\lambda = 0$ is included then it is only locally of finite type, since the collection of Higgs bundles of degree d with no semistability condition is unbounded.

Introducing parabolic weights allows us to consider a semistability condition [**21, 22**], but is also motivated by the growth rates of harmonic metrics [**43**]. A vector of parabolic weights denoted α is a collection of real numbers

$$\alpha = (\alpha_1^-, \alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \alpha_3^+, \alpha_4^-, \alpha_4^+)$$

with

$$\alpha_i^- \le \alpha_i^+ \le \alpha_i^- + 1.$$

Notice that we don't require that these lie in any particular interval, in fact it will be convenient to choose different intervals for different points t_i sometimes.

This phenomenon, which goes back to Manin's comments figuring in [**14**], is related to Mochizuki's notation [**29**] ${}_cE$ for a parabolic structure based at a real number c. A given parabolic sheaf $E_\bullet$ in a neighborhood of t_i according to the definitions of [**43, 28**], will yield a weighted parabolic bundle $({}_{c_i}E, P_i, \alpha_i^{\pm})$ in the present (and original [**42**]) sense, for each choice of $c_i \in \mathbb{R}$. The parabolic weights $\alpha_i^{\pm}$ are the weights of $E_\bullet$ which are contained in the interval $(c-1, c]$. In the other direction, a given $(E, P_i, \alpha_i^{\pm})$ as we are considering here, will come from a unique parabolic sheaf $E_\bullet$ by the construction ${}_{c_i}E$ using any choice of cutoff number c_i between α_i^+ and $\alpha_i^- + 1$. Since the choice of c_i doesn't have any effect for most of our considerations, we leave it out of the notation.

If α is a choice of weights, define

$$\deg^{\mathrm{par}}(E, \nabla, P_\bullet) := \deg(E) - \sum_{i=1}^{4} (\alpha_i^+ + \alpha_i^-).$$

If $F \subset E$ is a rank one subbundle preserved by ∇, let $\sigma(i, F)$ be either $-$, if $F_{t_i} \subset P_i$, or $+$ otherwise. Then put

$$\deg^{\mathrm{par}}(F) := \deg(F) - \sum_{i=1}^{4} \alpha_i^{\sigma(i,F)}.$$

Say that $(E, \nabla, P_\bullet)$ is α-semistable if, for any rank one subbundle preserved by ∇ we have

$$\deg^{\mathrm{par}}(F) \le \frac{\deg^{\mathrm{par}}(E, \nabla, P_\bullet)}{2};$$

say that it is α-stable if the strict inequality $<$ always holds. These stability and semistability conditions are open on $\mathcal{M}$, and let

$$\mathcal{M}^{d,\alpha} \subset \mathcal{M}^d, \quad \mathcal{M}^{d,\alpha}(\mathbf{r}(\lambda)) \subset \mathcal{M}^d(\mathbf{r}(\lambda))$$

be the open substacks of α-semistable points. As usual denote by a subscript the fiber over $\lambda \in \mathbb{A}^1$.

By geometric invariant theory [**21**] there is a universal categorical coarse moduli space

$$\mathcal{M}^{d,\alpha} \to M^{d,\alpha}$$

where $M^{d,\alpha}$ is a quasiprojective variety. This induces on the closed substack a universal categorical quotient

$$\mathcal{M}^{d,\alpha}(\mathbf{r}(\lambda)) \to M^{d,\alpha}(\mathbf{r}(\lambda))$$

where $M^{d,\alpha}(\mathbf{r}(\lambda))$ is also the closed subscheme of $M^{d,\alpha}$ defined as the inverse image of the same section $\mathbf{r}$ under the morphism

$$M^{d,\alpha} \to \mathcal{N}^d$$

which exists by the categorical quotient property.

These moduli spaces are constructed by Inaba, Iwasaki and the second author [**21, 22**], see also Nitsure [**32**] for plain logarithmic connections which can be viewed as the case $\alpha_i^+ = \alpha_i^-$, Maruyama-Yokogawa [**28**] for parabolic bundles, Konno [**25**], Boden-Yokogawa [**9**], Nakajima [**31**], Schmitt [**41**] and others for parabolic Higgs bundles, and the papers of Arinkin and Lysenko [**2, 3, 4**] as well as following papers such as Oblezin [**34**], which treat explicitly the rank two case we are considering here.

The space of initial conditions of Painlevé VI was first introduced in [**35**] by blowing up of rational surfaces along accessible singularities of Painlevé VI equations. More geometric or deformation theoretic descriptions of Okamoto spaces of initial conditions are given by Sakai [**40**] and by Saito-Takebe-Terajima [**39**]. We note that one can identify Okamoto spaces of initial conditions or their natural compactifications, Okamoto-Painlevé pairs, in [**39**] with the moduli spaces of α-stable parabolic connections (see [**22**, Theorem 4.1]).

The global family of rank 2 stable parabolic connections over the space of local exponents constructed in [**21**] really depends on the choice of stability condition from the choice of parabolic weights. However if the local exponents are Kostov-generic, all connections are irreducible, so stability does not depends on weights. Even in this case, if the local exponents are resonant, then the fiber of $M^d \to \mathcal{N}^d$ over that point $\mathbf{r}$ is independent of the parabolic weights, but the total family of connections are not biregular isomorphic near the nighborhood of the fiber, rather a flop phenomenon occurs.

The elementary transformation at the point t_i, may be defined as follows, see [**21**, §3]. Set $\widetilde{E} := \ker(E \to E_{t_i}/P_i)$, let $\widetilde{\nabla}$ be the induced λ-connection, and put $\widetilde{P}_j = P_j$ for $j \neq i$ whereas $\widetilde{P}_i := (E_{t_i}/P_i)(-t_i)$ in the exact sequence

$$0 \to \widetilde{P}_i \to \widetilde{E}_{t_i} \to P_i \to 0.$$

Then

$$\varepsilon_i(E, \nabla, P_\bullet) := (\widetilde{E}, \widetilde{\nabla}, \widetilde{P}_\bullet).$$

Note that $\deg(\widetilde{E}) = \deg(E) - 1$ so

$$\varepsilon_i : \mathcal{M}^d \to \mathcal{M}^{d-1}.$$

These transformations are some of the “Bäcklund transformations” in the classical theory of Painlevé VI and Garnier equations [**21**], and for more general systems they are called “Gabber transformations” by Esnault and Viehweg [**17**], see also Machu [**27**].

Suppose $r_i^{\pm}(E)$ are the residues of ∇ at t_i. A section of $\widetilde{E}$ projecting into $\widetilde{P}_i$ is of the form ze for e a section of E projecting to something nonzero modulo P_i, and z a coordinate at t_i. We can assume that $\nabla(e) = r_i^+(E)e \cdot d\log z$, in which case

$$\nabla(ze) = z\nabla(e) + \lambda e \cdot dz = (r_i^+(E) + \lambda)(ze) \cdot d\log z.$$

On the other hand a section projecting to $\widetilde{E}_{t_i}/\widetilde{P}_i$ is just a section of E projecting to P_i. Thus the new residues are

$$r_i^+(\widetilde{E}) = r_i^-(E), \quad r_i^-(\widetilde{E}) = r^+(E) + \lambda.$$

This transformation defines a function $\varepsilon_i : \mathcal{N}^d \to \mathcal{N}^{d-1}$, such that

$$\Psi(\varepsilon_i(E, \nabla, P_\bullet)) = \varepsilon_i \Psi(E, \nabla, P_\bullet).$$

The natural transformations ε_i on $\mathcal{M}^d$ and $\mathcal{N}^d$ are invertible, because there are natural transformations going in the other direction.

The following well-known fact helps by giving a normal form for the bundles.

Lemma 2.1. *Suppose $(E, \nabla, P_\bullet) \in (\mathcal{M}_1^1)^{\mathrm{irr}}$ is an irreducible logarithmic connection on a bundle of degree 1. Then $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof. Recall that $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, so $\Omega^1_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}(2)$ since D has 4 points. The bundle E has degree 1 and rank 2, with a logarithmic connection

$$\nabla : E \to E \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}(\log D) \cong E(2).$$

Suppose $L \subset E$ is a line subbundle of E with $\deg(L) \geq 2$. Then $\deg(E/L) \leq -1$, so the $\mathcal{O}_X$-linear map $L \to (E/L)(2)$ induced by ∇ must be zero. This says that ∇ preserves L, but that contradicts the hypothesis of irreducibility. $\square$

If $\mathbf{r}$ is a generic collection of residues then any element of $\mathcal{M}_1^1(\mathbf{r})$ is irreducible (see Lemma 3.2 below), so the previous lemma then applies everywhere.

Suppose $(E, \nabla, P_\bullet) \in (\mathcal{M}_1^1)^{\mathrm{irr}}$ is an irreducible connection, and a collection of weights α is specified. Then we obtain a parabolic vector bundle $(E, P_\bullet, \alpha)$. The underlying bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$ is fixed, by Lemma 2.1. We would like to know whether the parabolic bundle is semistable, and if not, what is its destabilizing subbundle.

3. Parametrization of parabolic structures

Motivated by the previous lemma, we now investigate the moduli stack of quasiparabolic structures on the bundle $B := \mathcal{O} \oplus \mathcal{O}(1)$. Let x denote the usual coordinate on $X = \mathbb{P}^1$.

Let $\mathcal{Q}$ denote the space of quasiparabolic structures on B over the collection of four points t_1, t_2, t_3, t_4. Assume that $t_i \neq \infty$, let e be the unit section of $\mathcal{O}$, and let $f \in \mathcal{O}(1)$ be the unit section vanishing at ∞. Thus $e(t_i), f(t_i)$ form a basis for B_{t_i}.

With respect to this basis, a parabolic structure at t_i consists of a line $P_i \subset \mathbb{C}^2$, corresponding hence to a point in $\mathbb{P}^1$. Therefore

$$\mathcal{Q} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Use coordinates u_1, u_2, u_3, u_4 which are allowed to take the value ∞. A point (P_1, P_2, P_3, P_4) is given by coordinates (u_1, u_2, u_3, u_4) where

$$P_i = \langle e(t_i) + u_i f(t_i) \rangle,$$

the case $u_i = \infty$ corresponding to $P_i = \langle f(t_i) \rangle$.

Let $A := \mathrm{Aut}(B)$. It acts on $\mathcal{Q}$. A general element of A may be written as a quadruple (a, b, c, s) with $a, s \in \mathbb{C}^*$ and $b, c \in \mathbb{C}$, acting by

$$e \mapsto s(ae + (b + cx)f), \quad f \mapsto sf.$$

The elements $(1, 0, 0, s)$ provide a central $\mathbb{G}_m \hookrightarrow A$ corresponding to scalar multiplication acting trivially on $\mathcal{Q}$. So A acts through the quotient which has parameters (a, b, c). We have

$$(a, b, c)(e(t_i) + u_i f(t_i)) = ae(t_i) + (b + ct_i + u_i) f(t_i)$$

so in terms of the coordinates this says that (a, b, c) acts by

$$(u_1, u_2, u_3, u_4) \mapsto \left(\frac{b + ct_1 + u_1}{a}, \frac{b + ct_2 + u_2}{a}, \frac{b + ct_3 + u_3}{a}, \frac{b + ct_4 + u_4}{a} \right).$$

In other words, $(1, b, c)$ act by translation by $b(1, 1, 1, 1) + c(t_1, t_2, t_3, t_4)$ and $(a, 0, 0)$ acts by scalar multiplication by a^{-1}.

These actions fix any values of the coordinates $u_i = \infty$. This corresponds to the fact that $\mathcal{O}(1)$ is the destabilizing subbundle of B so it is fixed by the automorphism group, and the conditions $u_i = \infty \Leftrightarrow P_i \in \mathcal{O}(1)_{t_i}$ are preserved by the action of A.

The open subset $\mathbb{C}^4 \subset \mathcal{Q}$ corresponding to finite values of u_i is preserved by the action of A. There, the quotient stack has the form

$$\mathbb{C}^2/\mathbb{C}^*,$$

indeed $\mathbb{C}^4$ modulo the translation action of the $(1, b, c)$ is $\mathbb{C}^2$, on which the elements $(a, 0, 0)$ act by scalar multiplication. We can make this more invariant in the following way. The open set $\mathbb{C}^4$ may be written as

$$\mathbb{C}^4 = \bigoplus_{i=1}^{4} \mathcal{O}(1)_{t_i}.$$

Consider the exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(1) \to \bigoplus_{i=1}^{4} \mathcal{O}(1)_{t_i} \to 0,$$

which on the level of cohomology gives

$$0 \to H^0(\mathcal{O}(1)) \to \bigoplus_{i=1}^{4} \mathcal{O}(1)_{t_i} \to H^1(\mathcal{O}(-3)) \to 0.$$

The image of $H^0(\mathcal{O}(1))$ is the $\mathbb{C}^2$ along which the translations $(1, b, c)$ take place. Therefore, the quotient $\mathbb{C}^2$ is naturally identified with $H^1(\mathcal{O}(-3)) \cong H^0(\mathcal{O}(1))^*$ so we can write

$$\mathcal{Q}/A \supset \mathbb{C}^4/A \cong H^0(\mathcal{O}(1))^*/\mathbb{C}^*.$$

Similar considerations hold for the strata such as (u_1, u_2, u_3, ∞) and permutations, $(u_1, u_2, \infty, \infty)$ and permutations, and so on.

The moduli space may be given a finer stratification, according to how subbundles of the form $\mathcal{O} \hookrightarrow B$ and $\mathcal{O}(-1) \hookrightarrow B$ meet the P_i. These conditions come into play for the stability conditions at various values of the weight parameters α.

Quasi-parabolic structures may also be interpreted in terms of projective geometry. Let $\mathbb{P}(B) \to \mathbb{P}^1$ be the $\mathbb{P}^1$-bundle of lines in the fibers of B. Think of the base $\mathbb{P}^1$ as the space of lines $\ell \subset V$ in a 2-dimensional vector space V. The bundle B associates to ℓ the space $B_\ell = \mathbb{C} \oplus (V/\ell)$, and a line $L \subset B_\ell$ is a 2-dimensional subspace $\tilde{L} \subset \mathbb{C} \oplus V$ such that $\ell \subset \tilde{L}$. Hence, $\mathbb{P}(B)$ may be seen as the variety of flags

$$0 \subset \ell \subset \tilde{L} \subset \mathbb{C} \oplus V$$

such that $\ell \subset V$, or equivalently

$$0 \subset \tilde{L}^\perp \subset \ell^\perp \subset \mathbb{C} \oplus V^*$$

such that $\mathbb{C} \subset \ell^\perp$. In this way, $\tilde{L}^\perp$ may be viewed as a point in $\mathbb{P}(\mathbb{C} \oplus V^*) = \mathbb{P}^2$, and $\ell^\perp$ is a line contining $\tilde{L}^\perp$ and the origin. The origin here means $\mathbb{C} \subset \mathbb{C} \oplus V^*$. This describes $\mathbb{P}(B)$ as the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at the origin.

The space of lines through the origin is our original $\mathbb{P}^1$, and the map $\mathbb{P}(B) \to \mathbb{P}^1$ is the projection centered at the origin. If T is a line through the origin corresponding to a point $t \in \mathbb{P}^1$ then the fiber $\mathbb{P}(B)_t$ is just the line T itself.

The four points t_1, t_2, t_3, t_4 correspond to four fixed lines passing through the origin which will be denoted T_1, T_2, T_3, T_4. The above discussion can be summed up as follows.

Lemma 3.1. *A quasiparabolic structure on the bundle B is the specification of a quadruple of points (U_1, U_2, U_3, U_4) in $\mathbb{P}^2$ such that $U_i \in T_i$. Thus a more invariant expression for the parameter space is*

$$\mathcal{Q} = T_1 \times T_2 \times T_3 \times T_4.$$

The coordinates u_i are obtained by trivializations of T_i, with $u_i = \infty$ corresponding to the origin $0 \in T_i$.

The automorphism group of B acts as the subgroup of automorphisms of $\mathbb{C} \oplus V$ which fix the origin in $\mathbb{P}^2$, and which act trivially on the space of lines passing through the origin. It has the matrix representation

$$A = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right\}.$$

Next consider the addition of a logarithmic connection to a parabolic structure parametrized as above. We say that a collection of residules $\mathbf{r} = (r_1^\pm, \ldots, r_4^\pm) \in \mathcal{N}_1^d$ is Kostov-generic if, for any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{+, -\}$

$$r_1^{\sigma_1} + r_2^{\sigma_2} + r_3^{\sigma_3} + r_4^{\sigma_4} \notin \mathbb{Z}. \tag{3.1}$$

Say that $\mathbf{r}$ is non-resonant if

$$r_i^+ - r_i^- \notin \mathbb{Z}. \tag{3.2}$$

Say that $\mathbf{r}$ is nonspecial if it is Kostov-generic and nonresonant, and special otherwise. These conditions are introduced in [**21**, Definition 2.4], with the terminology "generic" meaning nonspecial.

The special $\mathbf{r}$ form a collection of hyperplanes in $\mathcal{N}_1^d$ which are the reflection hyperplanes for the affine D_4 Weyl group, this group of operations acts on the moduli space by the Okamoto symmetries. These include elementary transformations, plus an additional symmetry to be discussed at the end of the paper.

The following property is well-known.

Lemma 3.2. *Suppose* $\mathbf{r} = (r_1^{\pm}, \ldots, r_4^{\pm}) \in \mathcal{N}_1^d$ *is Kostov-generic. Then for any* $(E, \nabla, P_\bullet) \in \mathcal{M}_1^d(\mathbf{r})$*, the bundle with connection* (E, ∇) *is irreducible.*

Proof. See [**21**, Lemma 2.1]. If $F \subset E$ is a subbundle with compatible connection ∇_F then the residues of F are $r_i^{\sigma_i}$ for some $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{+, -\}$. The Fuchs relation for F says

$$r_1^{\sigma_1} + r_2^{\sigma_2} + r_3^{\sigma_3} + r_4^{\sigma_4} = -\deg(F) \in \mathbb{Z},$$

contradicting (3.1). $\square$

A quasi-parabolic bundle is simple if it has no non-scalar endomorphisms preserving the parabolic subspaces P_i. This is an open condition; denote by $\mathcal{Q}^{\mathrm{simple}} \subset \mathcal{Q}$ the subset of simple quasi-parabolic bundles.

The analogue of Weil's criterion in our case is:

Lemma 3.3. *Suppose* $\mathbf{r} \in \mathcal{N}_1^1$ *is a nonspecial collection of residues, and suppose* $(E, P_\bullet)$ *is a quasi-parabolic bundle with* $\deg(E) = 1$*. Then the following conditions are equivalent:*
—there exists a connection ∇ *on* E*, compatible with the* P_i *and inducing the given residues* r_i^- *on* P_i *and* r_i^+ *on* E_{t_i}/P_i*;*
—$(E, P_\bullet)$ *is an indecomposable quasi-parabolic bundle;*
—$(E, P_\bullet)$ *is a simple quasi-parabolic bundle;*
—$E \cong B$*, there is at most one point with* $u_i = \infty$*, and the* P_i *for* $u_i \neq \infty$ *are not all contained in a single* $\mathcal{O} \subset B$*;*
—$E \cong B$ *and among the points in projective space* $U_i \in T_i \subset \mathbb{P}^2$ *corresponding to the quasiparabolic structure, there are three non-colinear points distinct from the origin.*

Proof. By the nonspeciality condition, if $(E, P_\bullet)$ has a connection with given residues $r_i^{\pm}$, then it must be indecomposable as a quasiparabolic bundle. Indeed, if $E = E_1 \oplus E_2$ were a decomposition into line bundles compatible with $P_\bullet$, then writing ∇ as a matrix the diagonal terms would be connections ∇_1, ∇_2 on E_1, E_2. Compatibility with $P_\bullet$ means that the residue would be either upper or lower triangular at each t_i, so the residues of ∇_1, ∇_2 would be taken from among the residues $r_i^{\pm}$ of ∇. This contradicts the Kostov-genericity condition for $\mathbf{r}$.

So in this case, the Weil criterion [**48, 5, 6, 13**] says that a connection exists if and only if $(E, P_\bullet)$ is indecomposable. For convenience here is the argument. Consider the subsheaf $End(E, P_\bullet) \subset End(E)$ of endomorphisms respecting the parabolic structure. At each t_i we have a map to a skyscraper sheaf

$$End(E, P_\bullet) \to \mathbb{C}^2$$

expressing the action of an endomorphism on P_i and E_{t_i}/P_i. Let $End^{\mathrm{st}}(E, P_\bullet)$ be the subsheaf which is the kernel of these maps at each t_i. It is the subsheaf of endomorphisms which map P_i to 0 and E_{t_i} to P_i. The obstruction to the existence of a logarithmic connection having given residues, is $\beta \in H^1(End^{\rm st}(E,P_\bullet)\otimes\Omega^1_X(\log D))$. There is a trace map $End^{\rm st}(E,P_\bullet) \to \mathcal{O}_X(-D)$ hence

$$H^1(End^{\rm st}(E,P_\bullet)\otimes\Omega^1_X(\log D)) \to H^1(X,\Omega^1_X) \cong \mathbb{C}.$$

The trace of the obstruction is zero if the Fuchs relation holds. The Serre dual of $H^1(End^{\rm st}(E,P_\bullet)\otimes\Omega^1_X(\log D))$ is $H^0(End(E,P_\bullet))$ which is the space of endomorphisms of the quasiparabolic structure $(E,P_\bullet)$.

If $(E,P_\bullet)$ is indecomposable, then any endomorphism has the form $c+\varphi$ where $c\in\mathbb{C}$ is a scalar constant and φ is nilpotent. The pairing of c with β is $cTr(\beta)=0$. On the other hand, φ preserves a filtration and acts by 0 on the graded pieces. The initial connections on an open cover, used to define the obstruction, can be chosen compatibly with the filtration, so β comes from a class with coefficients in the endomorphisms respecting the filtration [**5**]. As φ acts trivially on the graded pieces, $Tr(\varphi\beta)=0$. This shows that β paired with any endomorphism is zero, which by Serre duality implies that $\beta=0$. So there exists a connection with the given residues.

If $(E,P_\bullet)$ is simple then it is indecomposable.

In the present case the converse is true too. Suppose $(E,P_\bullet)$ is indecomposable. If $E\cong\mathcal{O}(m)\oplus\mathcal{O}(1-m)$ with $m\geq 2$ then one can choose the copy of $\mathcal{O}(1-m)$ to pass through any P_i not contained in $\mathcal{O}(m)_{t_i}$, which decomposes the parabolic bundle. Thus $E\cong B=\mathcal{O}\oplus\mathcal{O}(1)$. Furthermore, if two or more of the P_i are equal to $\mathcal{O}(1)_{t_i}$ then we can choose the $\mathcal{O}\subset B$ to pass through the ≤ 2 remaining other P_i again giving a decomposition. This shows that there is at most one $u_i=\infty$, and at least three $u_i\in\mathbb{C}$. Similarly if the P_i with $u_i\neq\infty$ are all contained in a $\mathcal{O}\subset B$ then this decomposes the quasiparabolic bundle.

In the projective space interpretation of Lemma 3.1, the quasiparabolic structure on $E=B$ corresponds to four points in $\mathbb{P}^2$, $U_i\in T_i\subset\mathbb{P}^2$. The previous paragraph says that the points U_i are not all colinear, which implies that no two can be at the origin, and if one of them is at the origin then the remaining three are not all colinear. Suppose $a\in A$, viewed as an automorphism of $\mathbb{P}^2$ preserving the origin and the U_i. There exists a subset of three U_i which are distinct from the origin and not colinear, and these together with the origin form a frame for $\mathbb{P}^2$. As a preserves the frame, it acts trivially on $\mathbb{P}^2$ so it is a scalar element of A. This shows that $(E,P_\bullet)$ is simple. This discussion also shows the equivalence with the last two conditions. □

Given a parabolic structure consisting of $U_i\in T_i$, there is a conic C passing through the origin and through the U_1,U_2,U_3,U_4. Assuming indecomposability, the conic is unique. Conversely, given a conic passing through the origin, it cuts each line T_i in another point. So, the open set $\mathcal{Q}^{\rm simple}$ is isomorphic to an appropriate open set of the set of conics passing through the origin.

To the conic C we can associate its tangent line at the origin (note that since the T_i are distinct, C cannot be two lines crossing at the origin). This gives a map from the open subset of simple points, to $\mathbb{P}^1$.

There is a tautological universal parabolic structure $P^{\rm univ}_\bullet$ on the trivial bundle $E^{\rm univ}=\mathrm{pr}_2^*(B)$ over $\mathcal{Q}\times X$. Let $\mathcal{H}\to\mathcal{Q}$ be the parameter variety for logarithmic connections on $(E^{\rm univ},P^{\rm univ})$ relative to $\mathcal{Q}$. Thinking of connections as splittings of a certain exact sequence, one can see that $\mathcal{H}$ is a quasiprojective variety. The group A acts on $\mathcal{H}$ over its action on $\mathcal{Q}$, with the moduli stack as quotient, and map to $\mathcal{Q}/\!/A$:

$$\mathcal{M}_1^1 = \mathcal{H}/\!/A \to \mathcal{Q}/\!/A.$$

As before there is a map $\mathcal{H} \to \mathcal{N}_1^1$ and for a collection of residues $\mathbf{r} = (r_1^{\pm}, \ldots, r_4^{\pm})$, let $\mathcal{H}(\mathbf{r}) \subset \mathcal{H}$ denote the inverse image. Thus

$$\mathcal{M}_1^1(\mathbf{r}) = \mathcal{H}(\mathbf{r})/\!/A.$$

Corollary 3.4. *In the situation of the previous lemma, the space of connections on a given simple quasiparabolic bundle $(E, P_\bullet)$ with the specified nonspecial residues, has dimension 1. In fact $\mathcal{H}(\mathbf{r}) \to \mathcal{Q}$ is a smooth fibration over $\mathcal{Q}^{\mathrm{simple}}$ whose fibers are affine lines $\mathbb{A}^1$.*

Proof. The space of connections is the space of splittings of the appropriate sequence, in particular it is a principal homogeneous space on a vector space. Since $(E, P_\bullet)$ is simple the dimensions of all the groups involved are constant as a function of $(u_1, \ldots, u_4) \in \mathcal{Q}^{\mathrm{simple}}$. Semicontinuity theory implies that $\mathcal{H}(\mathbf{r})$ is a smooth fibration. The fiber dimension is 1 by dimension count, hence the fibers are $\mathbb{A}^1$. $\square$

Proposition 3.5. *Fix a nonspecial collection of residues $\mathbf{r} \in \mathcal{N}_1^1$. Then $\mathcal{H}(\mathbf{r})$ is smooth. The quotient $\mathcal{M}_1^1(\mathbf{r}) = \mathcal{H}(\mathbf{r})/\!/A$ is a $\mathbb{G}_m$-gerb over its coarse moduli space $M_1^1(\mathbf{r})$ which is a smooth separated quasiprojective variety and is in fact a fine moduli space. The inverse image of a point $e \in \mathcal{Q}^{\mathrm{simple}}/A$ under the map*

$$\mathcal{M}_1^1(\mathbf{r}) \to \mathcal{Q}^{\mathrm{simple}}/A$$

is a closed substack, a $\mathbb{G}_m$-gerb over a closed subvariety of $M_1^1(\mathbf{r})$.

Proof. By Corollary 3.4, $\mathcal{H}(\mathbf{r})$ is a fibration over $\mathcal{Q}^{\mathrm{simple}}$, so it is smooth. By Lemma 3.2, any point of $\mathcal{H}(\mathbf{r})$ represents an irreducible connection. It follows that the automorphism group of the connection, which is also the stabilizer in A of the action, is $\mathbb{G}_m$. The coarse moduli space exists, and is a fine moduli space, by GIT because for an appropriate choice of parabolic weights all points are stable. See [**21**] for example. Since the stabilizer group is always $\mathbb{G}_m$, the moduli stack is a $\mathbb{G}_m$-gerb over the fine moduli space. If $e \in \mathcal{Q}^{\mathrm{simple}}/A$, then the A-orbit of e is closed in $\mathcal{Q}^{\mathrm{simple}}$ as may be seen directly. Thus, its inverse image is a closed A-invariant subset of $\mathcal{H}(\mathbf{r})$ so it corresponds to a closed substack, lying over a closed subvariety of the fine moduli space. $\square$

A collection of weights $\alpha = (\alpha_1^{\pm}, \alpha_2^{\pm}, \alpha_3^{\pm}, \alpha_4^{\pm})$ for a parabolic structure on a bundle of degree d is called nonspecial if $\alpha_i^- < \alpha_i^+ < \alpha_i^- + 1$, which is analogous to nonresonance, and if it satisfies the Kostov-genericity condition that for any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{+, -\}$,

$$\sum_{i=1}^{4} \alpha_i^{\sigma_i} + \frac{d - \sum_{i=1}^{4}(\alpha_i^+ + \alpha_i^-)}{2} \notin \mathbb{Z}.$$

Lemma 3.6. *If α is a nonspecial collection of weights for degree d, then any $(E, \nabla, P_\bullet) \in \mathcal{M}_\lambda^d$ which is α-semistable, is in fact α-stable.*

Proof. From the Kostov-genericity condition, there can be no rank 1 subsystem with an exact equality between slopes. $\square$

4. The Higgs limit construction

Choose nonspecial collections of residues $\mathbf{r} \in \mathcal{N}_1^d$ and consider the family of moduli stacks

$$\mathcal{M}^d(\lambda\mathbf{r}) \to \mathbb{A}^1.$$

The group $\mathbb{G}_m$ acts over its standard action on $\mathbb{A}^1$.

Given a point $(E, \nabla, P_\bullet)$ in the fiber over $\lambda = 1$, we would like to take the limit of $(E, u\nabla, P_\bullet)$ as $u \to 0$. The limit will be a vector bundle with 0-connection, which is to say a Higgs bundle, i.e. a point in the moduli stack $\mathcal{M}_0^d$. At $\lambda = 0$ the residues go to 0 since, in order to obtain an action of $\mathbb{G}_m$ we had to take the family of residues $\mathbf{r}(\lambda) = \lambda\mathbf{r}$. Thus, the limit should be a point in $\mathcal{M}_0^d(0)$.

Unfortunately, the moduli stack is highly unseparated over $\lambda = 0$, because the existence of an $\mathcal{O}_X$-linear Higgs field doesn't impose as strong a condition as the existence of a connection.

Therefore, there are many different ways to obtain a limit. It is instructive to consider some of the possibilities. These basically come from considering families of gauge transformations depending on u. The first and easiest way is to take the trivial gauge transformations, which is to say we consider the u-connections $u\nabla$ on the fixed quasiparabolic bundle $(E, P_\bullet)$. As $u \to 0$ these approach the zero Higgs field $\theta = 0$, so in this case the limit is just the quasiparabolic bundle $(E, P_\bullet)$ considered as a quasiparabolic Higgs bundle with $\theta = 0$.

Another way of taking the limit is to rescale with respect to the decomposition $E = \mathcal{O} \oplus \mathcal{O}(1)$. Write the connection as a matrix

$$\nabla = \begin{pmatrix} \nabla_0 & \theta \\ \zeta & \nabla_1 \end{pmatrix}$$

where ∇_0 and ∇_1 are logarithmic connections on $\mathcal{O}$ and $\mathcal{O}(1)$ respectively, and $\theta : \mathcal{O}(1) \to \mathcal{O} \otimes \Omega_X^1(\log D)$ and $\zeta : \mathcal{O} \to \mathcal{O}(1) \otimes \Omega_X^1(\log D)$ are $\mathcal{O}_X$-linear operators. Note however that the residues of ∇ are not compatible with the decomposition. Then we can make a gauge transformation rescaling by u on the first component

$$g_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$u\nabla \sim g_u^{-1} \circ u\nabla \circ g_u = \begin{pmatrix} u\nabla_0 & \theta \\ u^2\zeta & u\nabla_1 \end{pmatrix}.$$

In this case the limiting Higgs bundle is $\mathcal{O} \oplus \mathcal{O}(1)$ with Higgs field

$$\nabla_0 = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}, \quad \theta : \mathcal{O}(1) \to \mathcal{O} \otimes \Omega_X^1(\log D).$$

The quasiparabolic structure projects in the limit to one which is compatible with the decomposition.

Other rescalings are possible corresponding to other meromorphic decompositions of the bundle E. In fact, the limiting process works even when the bundle is only filtered, with the limiting bundle being the associated-graded.

In order to get a unique limit we should look for a separated stack or at least a stack having a separated coarse moduli space, and for that reason impose a semistability condition. Fix a nonspecial collection of parabolic weights $\alpha = (\alpha_1^\pm, \ldots, \alpha_4^\pm)$

and consider the moduli family

$$\mathcal{M}^{d,\alpha}(\lambda\mathbf{r}) \to \mathbb{A}^1$$

of α-semistable parabolic logarithmic λ-connections having the given residues. Note that semistability and stability are equivalent since α is chosen to be Kostov-generic.

Proposition 4.1. *For any $(E, \nabla, P_\bullet) \in \mathcal{M}_1^{d,\alpha}(\mathbf{r})$, there exists a unique limit*

$$(F, \theta, Q_\bullet) = \lim_{u\to 0}(E, u\nabla, P_\bullet)$$

in the moduli stack $\mathcal{M}_0^{d,\alpha}(0)$ of parabolic Higgs bundles with vanishing residues.

Proof. See [**46**]. However, the treatment there concerned mostly the case of compact base curve X. Furthermore, in the present case of rank 2, the general iterative procedure of [**46**] is not necessary. So it is perhaps worthwhile to do the existence proof here.

If $(E, P_\bullet)$ is already α-stable as a parabolic vector bundle, then the limit is just $(F, Q_\bullet) = (E, P_\bullet)$ with $\theta = 0$ as in the first example above.

If $(E, P_\bullet)$ is not α-stable, hence also not α-semistable, there is a quasiparabolic line subbundle $(L, R_\bullet) \subset (E, P_\bullet)$ which is maximally destabilizing. Here R_i is either 0 or L_{t_i}, in the second case $R_i = L_{t_i} = P_i$ is required. The parabolic weights are assigned accordingly: $\alpha_{L,i} = \alpha_i^+$ if $R_i = 0$, $\alpha_{L,i} = \alpha_i^-$ if $R_i = L_{t_i}$. This determines the parabolic degree $\deg^{\mathrm{par}}(L, R_\bullet, \alpha_L)$, and the destabilizing condition says that

$$\deg^{\mathrm{par}}(L, R_\bullet, \alpha_L) > \frac{\deg^{\mathrm{par}}(E, P_\bullet, \alpha)}{2}.$$

The quotient E/L similarly has a parabolic structure $R'_\bullet$ and weights $\alpha_{E/L}$, and

$$\deg^{\mathrm{par}}(E/L, R'_\bullet, \alpha_{E/L}) < \frac{\deg^{\mathrm{par}}(E, P_\bullet, \alpha)}{2}.$$

The connection determines an $\mathcal{O}_X$-linear map

$$\theta : L \to (E/L) \otimes \Omega^1_X(\log D),$$

nonzero because otherwise (E, ∇) would be reducible contradicting Lemma 3.2 in view of the genericity assumption for the residues $\mathbf{r}$.

As in the second example described above, after an appropriate gauge rescaling, the limiting Higgs bundle is

$$(F, Q_\bullet) = (L, R_\bullet) \oplus (E/L, R'_\bullet),$$

with Higgs field θ. As $\theta \neq 0$ the only possible θ-invariant subbundle is $(E/L, R'_\bullet)$, and this has slope strictly smaller than the slope of F. So the parabolic Higgs bundle $(F, \theta, Q_\bullet)$ with weights determined by α_L and $\alpha_{E/L}$ is stable.

This shows existence of a limit. For unicity, proceed as in [**46**]. Given two different limits, they correspond to two different families of u-connections on $X \times \mathbb{A}^1$ relative to $\mathbb{A}^1$, isomorphic outside of $u = 0$. Semicontinuity of the space of morphisms between them says that there is a nonzero morphism between the limits at $u = 0$, but since both are α-stable this must be an isomorphism. Thus the limit is unique. □

The limiting Higgs bundle has to be fixed by the action of $\mathbb{G}_m$ scaling the Higgs field, so it is a Higgs bundle corresponding to a variation of Hodge structure [**44**]. The case $\theta = 0$ corresponds to a unitary representation, whereas $L \oplus (E/L)$ with nonzero Higgs field θ corresponds to a variation of Hodge structure with structure group $U(1,1)$ and period map taking values in the unit disc. We don't use this information any further here, but it is suggestive of some interesting questions on the position of real monodromy representations in the overall picture.

The limit process leads to an equivalence relation: two points of $\mathcal{M}_1^d(\mathbf{r})$ are equivalent if their limits are the same. The moduli space is decomposed into equivalence classes which are locally closed subsets, and the foliation conjecture of [**46**] states that these should be the leaves of a foliation. In the present situation we will be able to prove that they are in fact the fibers of a morphism; which morphism it is will depend on the parabolic weight chamber.

The first step in this direction is to describe the possibilities for the limiting Higgs bundle $(F, \theta, Q_\bullet)$. The two examples of limits discussed above will basically cover all of the possibilities, up to making elementary transformations. The first task is to investigate more closely the α-stability condition.

Let $\mu_i := (\alpha_i^+ + \alpha_i^-)/2$ and $\epsilon_i := (\alpha_i^+ - \alpha_i^-)/2$ so

$$\alpha_i^+ = \mu_i + \epsilon_i, \quad \alpha_i^- = \mu_i - \epsilon_i,$$

with $0 < \epsilon_i < \frac{1}{2}$. The parabolic semistability condition for the parabolic bundle (without connection) (E, P_i) is described as follows. Let $\mu_{\rm tot} := \mu_1 + \mu_2 + \mu_3 + \mu_4$, although in fact the values of μ_i and $\mu_{\rm tot}$ won't turn out to make a difference. Let $\epsilon_{\rm tot} := \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$.

Assume that the points t_i are ordered so that $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \epsilon_4$. The conclusion will need to be extended by allowing permutations at the end.

For any sub-line bundle $L \subset E$, let

$$\Sigma(L) := \{i \mid L_{t_i} = P_i\}.$$

Then

$$\deg^{\rm par}(L) = \deg(L) - \mu_{\rm tot} - \epsilon_{\rm tot} + 2 \sum_{i \in \Sigma(L)} \epsilon_i.$$

On the other hand, the parabolic slope of E is $(d - 2\mu_{\rm tot})2$ with $d = \deg(E)$. Therefore, adding $\mu_{\rm tot}$ to both sides of the equation, L contradicts semistability if and only if

$$\deg(L) - \epsilon_{\rm tot} + 2 \sum_{i \in \Sigma(L)} \epsilon_i > d/2.$$

Respectively, L contradicts stability if $\geq$ holds. The left side may alternatively be written $\deg(L) + \sum_{i \in \Sigma(L)} \epsilon_i - \sum_{i \notin \Sigma(L)} \epsilon_i$. Under the hypothesis that the weights are nonspecial, stability and semistability are equivalent, i.e. equality can never hold.

Specialize now to the case $E = B = \mathcal{O} \oplus \mathcal{O}(1)$. The parabolic structure is given by a point $(u_1, \ldots, u_4) \in \mathcal{Q}$ as discussed previously, with $P_i = \langle (1, u_i) \rangle$. The semistability condition says

$$\deg(L) + \sum_{i \in \Sigma(L)} \epsilon_i - \sum_{i \notin \Sigma(L)} \epsilon_i \leq 1/2.$$

If $\deg(L) \leq -2$ then noting that $\epsilon_{\rm tot} < 2$ we always have

$$\deg(L) + \sum_{i \in \Sigma(L)} \epsilon_i - \sum_{i \notin \Sigma(L)} \epsilon_i < 0 < d/2 = 1/2,$$

so a line bundle of degree ≤ -2 never contradicts stability.

Consider $L = \mathcal{O}(-1)$. A map $L \to B$ is given by a pair (v, w) where $v = v_0 + v_1 x$ is a linear function and $w = w_0 + w_1 x + w_2 x^2$ is a quadratic function. Then $i \in \Sigma(L)$ if and only if $(1, u_i)$ is proportional to $(v(t_i), w(t_i))$, in other words if

$$w_0 + w_1 t_i + w_2 t_i^2 = u_i(v_0 + v_1 t_i).$$

When $u_i = \infty$ replace this by $(v_0 + v_1 t_i) = 0$. This system of 4 homogeneous equations in 5 unknowns always has a nonzero solution, so there is always an $\mathcal{O}(-1) = L \hookrightarrow B$ such that $L_{t_i} = P_i$ for all $i = 1, 2, 3, 4$. This contradicts semistability if and only if

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 3/2.$$

If this one doesn't contradict semistability then the other ones, with less contact between L and the P_i, will not either. Hence $(E, P_\bullet)$ satisfies the semistability condition for line bundles of degree -1 if and only if

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 \leq 3/2.$$

Consider the other extreme, $L = \mathcal{O}(1)$. There is a unique morphism $L \to B$, and $\Sigma(L)$ is the set of values of i such that $u_i = \infty$. This line subbundle contradicts semistability if and only if

$$1/2 + \sum_{u_i = \infty} \epsilon_i > \sum_{u_i \neq \infty} \epsilon_i.$$

In particular, if $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2$ then L always contradicts semistability. On the other hand, when

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 \geq 1/2.$$

then there exist parabolic structures which are semistable, including at least all of those in $\mathbb{C}^4 \subset \mathcal{Q}$. Note however that some parabolic structures on the boundary can still be unstable.

Turn now to the subbundles of degree 0, $L \hookrightarrow B$. It may be assumed that L is a saturated subbundle, so the inclusion map doesn't go into $\mathcal{O}(1) \subset B$. In other words, the projection $B \to \mathcal{O}$ induces an isomorphism $L \overset{\cong}{\to} \mathcal{O}$ and we may use this isomorphism to trivialize L. Hence the inclusion is given by $(1, v)$ where $v = v_0 + v_1 x$ is a polynomial of degree 1. For a parabolic structure $P_\bullet$ with coordinates (u_1, u_2, u_3, u_4) the condition $L_{t_i} = P_i$ becomes just $v(t_i) = u_i$, i.e.

$$v_0 + v_1 t_i = u_i.$$

For any two indices $i \neq j \in \{1, 2, 3, 4\}$ such that $u_i \neq \infty$ and $u_j \neq \infty$, there is a unique solution (v_0, v_1) to the pair of equations $v(t_i) = u_i$ and $v(t_j) = u_j$. In other words, for any pair of indices $i \neq j$ we can choose L such that $i, j \in \Sigma(L)$. If the u_i are general then $\Sigma(L) = \{i, j\}$ has two elements. On the other hand, for some special values of $u_\cdot$, the set $\Sigma(L)$ can have three or four elements. We consider those cases later on. In the general case, the biggest degree of a subbundle is obtained by choosing $i, j = 1, 2$ when the points are ordered according to decreasing values of ϵ. So, a general parabolic structure will be semistable if

$$\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 \leq 1/2,$$

and all parabolic structures will be unstable if

$$\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 > 1/2.$$

Notice that to prove this last statement, we also need to treat the cases where some $u_i = \infty$. Suppose $\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 > 1/2$, and consider the worst case which is when $u_2 = \infty$. Then

$$1/2 + \epsilon_2 - \epsilon_1 - \epsilon_3 - \epsilon_4 = (1/2 + \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) - 2\epsilon 1 > \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - 1/2 > 0,$$

so this shows that the $\mathcal{O}(1) \subset B$ contradicts stability by the previous discussion.

Proposition 4.2. *For α a nonspecial assignment of parabolic weights, define $\epsilon_i = (\alpha_i^+ - \alpha_i^-)/2$ as above. Suppose one of the following three conditions holds:*
(a) $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2$;
(b) $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 3/2$; or
(c) there exists a renumbering $\{1, 2, 3, 4\} = \{i, j, k, l\}$ such that

$$\epsilon_i + \epsilon_j - \epsilon_k - \epsilon_l > 1/2.$$

Then every parabolic structure $(B, P_\bullet)$ on the bundle $B = \mathcal{O} \oplus \mathcal{O}(1)$ is unstable. If, on the contrary, none of these conditions hold, then a general parabolic structure is stable; however some special parabolic structures might still be unstable.

Proof. The arguments have been done above. □

If there is a destabilizing subbundle, then it is unique; indeed any other distinct destabilizing subbundle would have nonzero projection to the quotient, but this would be a morphism of parabolic line bundles strictly decreasing the parabolic degree, which is impossible.

5. The unstable zones

An unstable zone is when one of the conditions (a), (b) or (c) holds in the previous proposition. In fact (c) contains 6 distinct conditions so there are really 8 different unstable zones. The conditions are mutually exclusive so the different zones are disjoint.

The discussion will be made easier by the fact that pairs of elementary transformations permute the different zones, allowing us to consider a single condition such as (a). The following lemma explains how the parabolic weights should be changed along an elementary transformation.

Lemma 5.1. *Suppose $(\widetilde{E}, \widetilde{P}_\bullet)$ is a quasiparabolic bundle obtained by a single elementary transformation ε_i of $(E, P_\bullet)$ at the point t_i, see page 6. Define parabolic weights at t_i by*

$$\widetilde{\alpha}_i^+ := \alpha_i^-, \quad \widetilde{\alpha}_i^- := \alpha_i^+ - 1, \quad \textit{hence } \widetilde{\epsilon}_i = 1/2 - \epsilon_i,$$

leaving $\widetilde{\alpha}_j^\pm = \alpha_j^\pm$ for $j \neq i$. Then $\widetilde{\alpha}$ is nonspecial if and only if α is, and $(\widetilde{E}, \widetilde{P}_\bullet)$ is $\widetilde{\alpha}$-stable if and only if $(E, P_\bullet)$ was α-stable.

Proof. Whereas $\deg(\widetilde{E}) = \deg(E) - 1$, the change of weights gives back $\deg^{\mathrm{par}}(\widetilde{E}, \widetilde{P}_\bullet, \widetilde{\alpha}) = \deg^{\mathrm{par}}(E, P_\bullet, \alpha)$. Saturated line subbundles of $\widetilde{E}$ correspond to those of E, and this correspondence also preserves parabolic degree, so the stability conditions are equivalent. □

In order to preserve an odd degree of E, we can do two different elementary transformations at t_i and t_j, which changes ϵ_i to $1/2 - \epsilon_i$ and ϵ_j to $1/2 - \epsilon_j$.

Lemma 5.2. *The set of three conditions ((a) or (b) or (c)) is left invariant under any such pair of elementary transformations, and these operations permute the 8 zones transitively. So, up to such transformations, the unstable zones are essentially equivalent.*

PROOF. Direct calculation. □

Suppose $(E, \nabla, P_\bullet)$ is a parabolic connection with weights α, in one of the unstable zones. Up to doing a pair of elementary transformations, we may assume then that we are in zone (a) where the destabilizing subbundle is $\mathcal{O}(1) \subset B$. The limiting parabolic Higgs bundle is $L \oplus L'$ where L is given parabolic weights α_i^+ at t_i, if $u_i \neq \infty$, or α_i^- at t_i if $u_i = \infty$. The parabolic weights for L' are complementary. The Higgs field $\theta : L \to L' \otimes \Omega^1_X(\log D)$ is the piece coming from the connection operator ∇. Noting that $L \cong \mathcal{O}(1)$, $L' \cong \mathcal{O}$ and $\Omega^1_X(\log D) \cong \mathcal{O}(2)$, we see that θ may be viewed as a section of $\mathcal{O}(1)$ or a linear function. Its zero at a point $z \in X$ is interpreted in [**21**] [**22**] [**47**] [**1**] as an "apparent singularity" of the connection.

Definition 5.3. *Let $\mathcal{P}$ be the non-separated scheme obtained by glueing together two copies of $X = \mathbb{P}^1$ by the identity map over the open subset $U = \mathbb{P}^1 - \{t_1, \ldots, t_4\}$. The copies are labeled $\mathcal{P}^+$ and $\mathcal{P}^-$.*

Interestingly enough, this scheme also plays the same role for the stable zone. It appeared in Arinkin's work on the geometric Langlands correspondence [**2**].

In [**22**], Inaba, Iwasaki and the second author define a morphism

$$\Upsilon : \mathcal{M}^1_1(\mathbf{r}) \to \mathcal{P}$$

as follows. Any $(E, \nabla, P_\bullet)$ in $\mathcal{M}^1_1(\mathbf{r})$ has a unique subbundle $L \subset E$ of degree 1. The quotient E/L has degree 0. The connection induces an operator $\varphi : L \to (E/L) \otimes \Omega^1_X(\log D)$. It is an $\mathcal{O}_X$-linear map of line bundles. Comparing degrees of the source and the target, we see that φ has exactly one zero. The position of the zero defines a point in $\mathbb{P}^1$. If located at one of the singular points t_i then we can further ask whether $L_{t_i} \subset P_i \subset E_{t_i}$, if so then the point goes into $\mathcal{P}^-$, if not it goes into $\mathcal{P}^+$.

If the zero of φ is not located at t_i, then the condition that $\mathrm{res}(\nabla, t_i)$ respect the quasiparabolic P_i implies that $P_i \neq L_{t_i}$.

Proposition 5.4. *This pointwise prescription defines a morphism Υ, all fibers of which are trivial $\mathbb{G}_m$-gerbs over $\mathbb{A}^1$. The structure of the moduli space $M^1_1(\mathbf{r})$ is a ruled surface, blown up at two distinct points on each fiber F_i over $t_i \in \mathbb{P}^1$ of Hirzebruch surface $\Sigma_2 \to \mathbb{P}^1$ with subsequently the strict transform of the section at infinity and the fibers F_i, $1 \leq i \leq 4$ removed. The affine fibers of Υ over points of U are the fibers of the ruled surface, over the doubled-up points they are the two exceptional divisors.*

PROOF. See Theorem 4.1 of [**22**]. This picture will be described in further detail in Section 9. □

In order to relate this map with the limit map, we investigate what stable Higgs bundles look like.

Lemma 5.5. *If (E, θ) is an α-stable Higgs bundle in $\mathcal{M}^1_0(0)$ with $\theta = 0$ then $E \cong B$. Therefore, if α is in the unstable zone then $\theta \neq 0$.*

Proof. If $\theta = 0$ then the stability condition is supposed to hold for any subbundle. If E is not of the form $B = \mathcal{O} \oplus \mathcal{O}(1)$ then E has a subbundle of degree 2. For this subbundle, assuming the worst-case scenario $L_{t_i} \not\subset P_i$ for any i, the stability condition as discussed above becomes

$$2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 < \frac{1}{2}.$$

Suppose this holds. It means that $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 3/2$. However, then there is a subbundle of the form $\mathcal{O}(-1) = L' \subset E$ such that $L'_{t_i} = P_i$ for all i. For this subbundle,

$$-1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 1/2$$

contradicting stability. This contradiction shows that $E \cong B$. Furthermore $(E, P_\bullet)$ is an α-stable parabolic bundle, so the prior discussion shows that α has to be in the stable zone.

By contrapositive, if α is in the unstable zone then no stable Higgs bundle with $\theta = 0$ can exist, showing that $\theta \neq 0$. □

Suppose α is in the unstable zone and $(E, \theta, P_\bullet)$ is an α-stable parabolic Higgs bundle in the fixed point set $\mathcal{M}^1_0(0)^{\mathbb{G}_m}$. By the lemma, $\theta \neq 0$. This means that E must be a nontrivial system of Hodge bundles [**44**], which in the rank two case means it is a direct sum of two line bundles

$$E = E^0 \oplus E^1, \quad \theta : E^0 \to E^1 \otimes \Omega^1_X(\log D)$$

with $\theta \neq 0$. It follows that $\deg(E^0) \leq \deg(E^1) + 2$. The quasiparabolic structure is compatible with the $\mathbb{G}_m$-action, so either $P_i \subset E^0$ or $P_i \subset E^1$. The only subbundle preserved by θ is E^1. Let $\Sigma(E^1)$ denote the set of indices $i \in \{1, 2, 3, 4\}$ such that $P_i = E^1_{t_i}$. Then the α-stability condition says that

$$\deg(E^1) - \sum_{i=1}^{4} \epsilon_i + \sum_{i \in \Sigma(E^1)} 2\epsilon_i < 1/2. \tag{5.1}$$

Theorem 5.6. *Suppose $\mathbf{r} \in \mathcal{N}^1_1$ and α is an assignment of parabolic weights, both nonspecial. Suppose that α is in the (a)-unstable zone, i.e. condition (a) of Proposition 4.2 holds. There is a set-theoretical isomorphism, constructibly algebraic but not a morphism of stacks, from the points of $\mathcal{P}$ to the fixed point set of $\mathbb{G}_m$ acting on the moduli space of α-stable strictly parabolic Higgs bundles*

$$\mathbf{V}_\alpha : \mathcal{P} \stackrel{\cong}{\to} (\mathcal{M}^{1,\alpha}_0(0))^{\mathbb{G}_m},$$

such that for any $(E, \nabla, P_\bullet) \in \mathcal{M}^{d,\alpha}_1(\mathbf{r})$ we have

$$\lim_{u \to 0} (E, u\nabla, P_\bullet) = \mathbf{V}_\alpha(\Upsilon(E, \nabla, P_\bullet)).$$

Here the limit is taken in the α-stable Hodge moduli stack $\mathcal{M}^{d,\alpha}(\lambda\mathbf{r}) \to \mathbb{A}^1$.

Proof. In the (a)-unstable zone, $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2$ implies that

$$\left| - \sum_{i=1}^{4} \epsilon_i + \sum_{i \in \Sigma(E^1)} 2\epsilon_i \right| < 1/2,$$

so if $\deg(E^1) \geq 1$ then the α-stability condition (5.1) never holds, while if $\deg(E^1) \leq 0$ then it always holds. Given that $\deg(E^0) + \deg(E^1) = 1$ and $\deg(E^0) \leq \deg(E^1) +$

2, the only possibility is $\deg(E^0) = 1$ and $\deg(E^1) = 0$. In other words, in this case an α-stable system of Hodge bundles must be of the form

$$\mathcal{O}(1) \xrightarrow{\theta} \mathcal{O} \otimes \Omega^1_X(\log D).$$

Thus θ is a section of a line bundle of degree 1, so it has exactly one zero.

The condition that θ be strictly compatible with the parabolic structure means that if $\theta(t_i) \neq 0$ then $P_i = E^1_{t_i}$. However, if $\theta(t_i) = 0$ then P_i can be either $E^1_{t_i}$ or $E^0_{t_i}$. We see that, set theoretically, the set of possible choices for $(E, \theta, P_\bullet)$ is in bijective correspondence with the points of $\mathcal{P}$. This correspondence is the map $\mathbf{V}_\alpha$.

Given $(E, \nabla, P_\bullet) \in \mathcal{M}^{1,\alpha}_0(\mathbf{r})$, the limit $\lim_{u\to 0}(E, u\nabla, P_\bullet)$ is obtained by taking E^0 to be the α-destabilizing subbundle, $E^1 = E/E^0$ and using the map which was previously denoted φ as the Higgs field [**46**]. In view of the definition of Υ described above Proposition 5.4, this gives exactly the required compatibility. □

Corollary 5.7. *The foliation conjecture of* [**46**] *holds for rank two parabolic connections on* $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ *when the residues and parabolic weights are nonspecial, and the parabolic weights are in one of the unstable zones.*

Proof. By doing elementary transformations we can reduce to supposing that α is in the (a)-unstable zone. The decomposition into subspaces according to the position of $\lim_{u\to 0} u()$ is equal to the decomposition into fibers of the map Υ, by the preceding theorem. By Proposition 5.4 which recopies [**22**, Theorem 4.1], this decomposition is the decomposition into fibers of a smooth morphism, in particular it is a foliation. □

6. The stable zone

The stable zone will mean when none of (a), (b) or (c) hold, which is to say, with the nonspeciality hypothesis in effect, that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 1/2; \tag{6.1}$$

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 3/2; \tag{6.2}$$

and for all renumberings $\{1, 2, 3, 4\} = \{i, j, k, l\}$ we have

$$\epsilon_i + \epsilon_j - \epsilon_k - \epsilon_l < 1/2. \tag{6.3}$$

Again this is invariant under elementary transformations. If α is an assignment of parabolic weights in the stable zone, then a general parabolic structure on B will be stable, however special ones might not be stable.

The open subset $\mathcal{Q}^{\mathrm{simple}} \subset \mathcal{Q}$ of simple quasi-parabolic bundles is preserved by the action of the automorphism group A, and

$$\mathcal{Q}^{\mathrm{simple}} // A$$

is the moduli stack of simple quasi-parabolic bundles. Recall from Lemma 3.3 and Corollary 3.4, the image of $\mathcal{H} \to \mathcal{Q}$ is $\mathcal{Q}^{\mathrm{simple}}$.

Lemma 6.1. *The moduli stack* $\mathcal{Q}^{\mathrm{simple}}//A$ *is a* $\mathbb{G}_m$*-gerb over the non-separated scheme* $\mathcal{P}$ *of Definition 5.3. This gerb, which is in fact trivial, is the same as Arinkin's stack* [**2**].

PROOF. Consider the open set $\mathcal{Q}^i \subset \mathcal{Q}$ consisting of (u_1, u_2, u_3, u_4) such that $u_j \neq \infty$ for $j \neq i$, and the three corresponding points U_j are not colinear. The four open sets $\mathcal{Q}^i$ cover $\mathcal{Q}^{\text{simple}}$ from the discussion of Lemma 3.3. Fix $U_j^0 \in T_j - 0$ such that no three of them is colinear. Any point of $\mathcal{Q}^i$ can be brought by a unique element of A to a point $(U_1, \ldots, U_4)$ such that $U_j = U_j^0$ for $j \neq i$, then the position of $U_i \in T_i \cong \mathbb{P}^1$ provides a coordinate for the quotient $\mathcal{Q}^i/A$. This gives

$$\mathcal{Q}^i/A \cong \mathbb{P}^1$$

for each i. Consider next the intersection $\mathcal{Q}^{ij} = \mathcal{Q}^i \cap \mathcal{Q}^j$. Let U_k and U_l be the other two points. Up to the action of A, they may be supposed to lie on the framing points U_k^0 and U_l^0. Let H be the line passing through U_k^0 and U_l^0. Then $\mathcal{Q}^{ij}$ consists of the choices of $U_i \in T_i - 0 - H \cap T_i$ and $U_j \in T_j - 0 - H \cap T_j$. The group A acts by scaling both of these. Thus, $\mathcal{Q}^{ij}/A \cong \mathbb{G}_m$. Glueing together the two charts $\mathcal{Q}^i/A$ and $\mathcal{Q}^j/A$ along the intersection $\mathcal{Q}^{ij}/A$ is therefore a doubled projective line

$$(\mathcal{Q}^i \cup \mathcal{Q}^j)/A \cong \mathbb{P}^1 \cup^{\mathbb{G}_m} \mathbb{P}^1.$$

It may also be seen as the quotient

$$\left(\mathbb{P}^1 \times \mathbb{P}^1 - \{(0,0), (0,\infty), (\infty, 0), (\infty,\infty)\}\right)/\mathbb{G}_m.$$

To get a global picture, fix $i = 1$. Now $\mathcal{Q}^1/A = \mathbb{P}^1$, a projective line which is identified with T_1 when the other three points are at U_j^0. When we glue in $\mathcal{Q}^2/A \cong \mathbb{P}^1$ this doubles up the origin $0 \in T_1$ as well as the intersection point I_{34} of the line $\overline{U_3^0 U_4^0}$ with T_1. Similarly when we glue in $\mathcal{Q}^3/A \cong \mathbb{P}^1$ it doubles up the origin (in the same way) and the intersection point I_{24}, and when we glue in $\mathcal{Q}^4/A \cong \mathbb{P}^1$ it doubles up the origin and I_{23}. One can see that the quadruple of points $(0, I_{34}, I_{24}, I_{23})$ is equivalent to the original (t_1, t_2, t_3, t_4). Thus

$$\mathcal{Q}^{\text{simple}}/A \cong \mathcal{P}.$$

The gerb is the same as Arinkin's: he was also looking at the moduli stack of quasiparabolic bundles. These $\mathbb{G}_m$-gerbs are in fact trivial, as may be seen directly over each chart $\mathbb{P}^1$ and on the glueing from the fact that $\mathbb{G}_m$-torsors over $\mathbb{G}_m$ or $\mathbb{A}^1$ are trivial. □

The construction using conics described on page 11 gives a more canonical A-invariant morphism from $\mathcal{Q}^{\text{simple}}$ to $\mathbb{P}^1$.

Recall that $\mathcal{H}(\mathbf{r}) \to \mathcal{Q}$ denotes the moduli space of connections on the quasiparabolic bundles parametrized by $\mathcal{Q}$. Keep the hypothesis that $\mathbf{r} \in \mathcal{N}_1^1$ is nonspecial. From Lemma 3.3 it follows that the map may be written as $\mathcal{H}(\mathbf{r}) \to \mathcal{Q}^{\text{simple}}$ with 1-dimensional fibers. We obtain a map

$$\mathcal{M}_1^1(\mathbf{r}) = \mathcal{H}(\mathbf{r})/\!/A \xrightarrow{\Phi} \mathcal{P}.$$

Our main result identifies this map with the quotient by the relation defined by Higgs limits under the $\mathbb{G}_m$-action.

Theorem 6.2. *Suppose $\mathbf{r} \in \mathcal{N}_1^1$ and α is an assignment of parabolic weights, both nonspecial. Suppose that α is in the stable zone, i.e.* (6.1), (6.2) *and* (6.3) *hold. There is a set-theoretical isomorphism, constructibly algebraic but not a morphism of stacks, from the points of $\mathcal{P}$ to the fixed point set of $\mathbb{G}_m$ acting on the moduli space of α-stable strictly parabolic Higgs bundles*

$$\mathbf{V}_\alpha : \mathcal{P} \xrightarrow{\cong} (\mathcal{M}_0^{1,\alpha})^{\mathbb{G}_m}(0),$$

such that for any $(E,\nabla,P_\bullet)\in\mathcal{M}_1^{d,\alpha}(\mathbf{r})$ *we have*

$$\lim_{u\to 0}(E,u\nabla,P_\bullet)=\mathbf{V}_\alpha(\Phi(E,\nabla,P_\bullet)).$$

Here the limit is taken in the α*-stable Hodge moduli stack* $\mathcal{M}^{d,\alpha}(\lambda\mathbf{r})\to\mathbb{A}^1$.

Proof. Recall that $\mathcal{P}=\mathcal{Q}^{\mathrm{simple}}/A$ is the space of A-orbits in the simple quasiparabolic structures, so a point of $\mathcal{P}$ represents an isomorphism class of simple quasiparabolic bundle $(E,P_\bullet)$ and Φ is just the map of forgetting the connection. If $(E,P_\bullet)$ is α-stable, then take $\theta=0$ as Higgs field and set $\mathbf{V}_\alpha(E,P_\bullet):=(E,0,P_\bullet)$. If ∇ is any connection on $(E,P_\bullet)$ then this gives the limiting α-stable Higgs bundle of Proposition 4.1

$$\lim_{u\to 0}(E,u\nabla,P_\bullet)=(E,0,P_\bullet)=\mathbf{V}_\alpha(E,P_\bullet).$$

It remains to define $\mathbf{V}_\alpha$ on the $(E,P_\bullet)$ which are α-unstable. Suppose $(E,P_\bullet)$ is α-unstable, and let $L\subset E$ be the destabilizing subbundle. Since α is in the stable zone, condition (6.2) says that L is never $\mathcal{O}(-1)$. There are two cases: either $L\cong\mathcal{O}$ and there are three $P_i=L_{t_i}$; or $L=\mathcal{O}(1)$ and there is one $P_i=L_{t_i}$. The first case corresponds to three colinear points U_i, while the second case corresponds to some U_i at the origin.

The residues of the Higgs field are 0, which means that $\mathrm{res}(\theta,t_i):E_{t_i}\to P_i$ and $\mathrm{res}(\theta,t_i):P_i\to 0$. So we have

$$\theta:L\to(E/L)\otimes\Omega^1_X(\log D),$$

which is equal to zero at any point where $P_i=L_{t_i}$. If $L\cong\mathcal{O}$ then θ is a section of a line bundle of degree three with three additional zeros at the three points t_i with U_i colinear; if $L\cong\mathcal{O}(1)$ then θ is a section of a line bundle of degree 1 with a single additional zero at the point t_i where U_i is the origin. In both cases, θ becomes a nonzero section of the trivial bundle, in other words it is determined uniquely up to scalar automorphisms of the two component line bundles. This determines the Higgs bundle

$$\mathbf{V}_\alpha(E,P_\bullet):=(L\oplus(E/L),\theta)$$

which will be the limit $\lim_{u\to 0}(E,u\nabla,P_\bullet)$ by the construction of Proposition 4.1, for any connection ∇ on $(E,P_\bullet)$. □

We can be more precise about the possibilities occuring in the above proof. There are two points of $\mathcal{P}$ over each $t_i\in\mathbb{P}^1$. These are the cases when $U_i=0$, and when the other three points U_j,U_k,U_l are colinear. The quasiparabolic structure with $U_i=0$ is unstable if and only if

$$1+\epsilon_i-\epsilon_j-\epsilon_k-\epsilon_l>1/2,$$

in other words

$$\epsilon_j+\epsilon_k+\epsilon_l-\epsilon_i<1/2.$$

The quasiparabolic structure with U_j,U_k,U_l colinear is unstable if and only if

$$\epsilon_j+\epsilon_k+\epsilon_l-\epsilon_i>1/2.$$

In other words, the point t_i corresponds to the hyperplane $\epsilon_j+\epsilon_k+\epsilon_l-\epsilon_i=1/2$ which divides the stable zone into two regions, and the question of which of the two points lying over t_i is unstable depends on which side of this hyperplane we are on.

The resulting 16 subzones are quite probably related to the subzones which will show up as images by the Okamoto symmetry of the various different unstable zones in the last two sections of the paper.

Corollary 6.3. *The foliation conjecture of* [**46**] *holds for rank two parabolic connections on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ when the residues and parabolic weights are nonspecial, and the parabolic weights are in the stable zone.*

Proof. By Theorem 6.2, the pieces of the decomposition according to the Higgs limit are equal to the fibers of the map $M_1^{d,\alpha}(\mathbf{r}) \to \mathcal{Q}^{\mathrm{simple}}/A = \mathcal{P}$. Since this is a smooth map of schemes, even though the target is non-separated, the collection of fibers forms a foliation. $\square$

7. Local systems on root stacks

Consider local systems with monodromy of finite order around the t_i. Fix $n \in \mathbb{N}$ and let

$$Z := X[\frac{D}{n}] \xrightarrow{p} X$$

be the Cadman-Vistoli root stack, which is the universal Deligne-Mumford stack over which the line bundle $\mathcal{O}(D)$ has an n-th root; a good reference is [**10**]. It corresponds to the orbifold obtained by labeling the points $t_i \in X$ with the integer n. The fundamental group $\pi_1(Z, x)$ is also the orbifold fundamental group of X, equivalently it is $\pi_1(U, x)/\langle \gamma_i^n \rangle$ where γ_i are the loops going around t_i.

In this case the DM-stack Z is a quotient stack. Let C_n be the cyclic group of order n with generator c. Choose a homomorphism $g : \pi_1(U, x) \to C_n$ such that $g(\gamma_i)$ is a generator. This exists, for example we can set $g(\gamma_1) = g(\gamma_2) = c$ and $g(\gamma_3) = g(\gamma_4) = c^{-1}$. Then g induces a Galois covering $Y \xrightarrow{q} X$ with Galois group C_n and full degree n ramification over the t_i, lifting to an etale Galois covering of the stack $\tilde{q} : Y \to Z$. This gives

$$Z = Y // C_n.$$

Let $\tilde{t}_i \in Y$ be the unique point lying over $t_i \in X$.

Proposition 7.1. *With the above notations, the following categories are equivalent:*
—local systems on U with finite monodromy of order dividing n around the t_i;
—local systems on Z;
—C_n-equivariant local systems on Y.

Given a local system L on Z corresponding to L_U on U and to a C_n-equivariant local system L_Y on Y, we can associate its *local monodromy* at t_i. This is an object in the category of vector spaces with automorphisms. In terms of L_U it is just the fiber $L_{U,x}$ at the basepoint, together with action of γ_i.

Corresponding to the point t_i is a map $\mathbf{B}(\mathbb{Z}/n) \to Z$ from the one-point classifying stack of the cyclic group $\mathbb{Z}/n$ into Z, and in terms of L the local monodromy is the same as the restriction $L|_{\mathbf{B}(\mathbb{Z}/n)}$, considering a local system over $\mathbf{B}(\mathbb{Z}/n)$ as being the same as a vector space with an automorphism of order n.

In terms of the C_n-equivariant local system L_Y on Y, the local monodromy is the fiber $L_{Y,\tilde{t}_i}$ together with its action of C_n, but this action is viewed as an automorphism using the generating element $g(\gamma_i) \in C_n$.

Given a local system L on Z, its corresponding sheaf of $\mathcal{O}_Z$-modules is denoted $L \otimes \mathcal{O}_Z$. Then

$$E := p_*(L \otimes \mathcal{O}_Z)$$

is a locally free sheaf on X, whose rank is the same as $\mathrm{rk}(L)$. If L corresponds to the C_n-equivariant local system L_Y on the Galois covering Y, with underlying vector bundle $L_Y \otimes \mathcal{O}_Y$, then the C_n-invariant part of the direct image is

$$E = q_*(L_Y \otimes \mathcal{O}_Y)^{C_n} \subset q_*(L_Y \otimes \mathcal{O}_Y),$$

indeed since Y provides local charts for the stack Z this may be taken as the definition of E.

The following proposition is well-known.

Proposition 7.2. *The naturally defined connection on $V|_U$ extends to a logarithmic connection*

$$\nabla : E \to E \otimes \Omega^1_X(\log D).$$

The residue of ∇ at t_i is semisimple and has eigenvalues in $(-1, 0] \cap \frac{1}{n}\mathbb{Z}$. More precisely, suppose that the local monodromy of L at t_i has eigenvalues $e^{\theta_i^j \sqrt{-1}}$ with $0 \leq \theta_i^j < 2\pi$ counted with multiplicity. Then the residue of ∇ at t_i is semisimple with eigenvalues $r_i^j = -\theta_i^j/2\pi$.

This construction sets up an equivalence of categories between the category of local systems L on Z, and the category of vector bundles with logarithmic connection (E, ∇) whose residues are semisimple with eigenvalues in $(-1, 0] \cap \frac{1}{n}\mathbb{Z}$.

In the situation of the proposition, the bundle E also gets a weighted parabolic structure. It consists of a quasiparabolic structure or filtration $P_i^\bullet$ of E_{t_i}, together with weights $\alpha_i^\bullet \in (-1, 0]$. In fact, the filtration is obtained from the decomposition of E_{t_i} into eigenspaces for $\mathrm{res}(\nabla)$ and the j-th graded piece $Gr^j_{P_i}(E_{t_i})$ is just the r_i^j-eigenspace, weighted by $\alpha_i^j = r_i^j$. The index j corresponds to the place of r_i^j in the ordered interval $(-1, 0]$.

In general, the filtration will not be a full flag. Say that the local monodromy of L is non-resonant if the eigenvalues of the monodromy transformation are distinct with multiplicity 1, corresponding to the same non-resonance condition for the residue of the corresponding logarithmic connection. Notice that non-resonance implies $n \geq \mathrm{rk}(L)$, otherwise the number of possible available eigenvalues would be too small. In the non-resonant case, the parabolic filtration is a full flag.

Say that a collection of local monodromy data at all the t_i is Kostov-generic if there is no way of specifying a subset consisting of the same number of eigenvalues at each point, such that the product over all the points is 1. Say that the collection of local monodromy data is nonspecial if it is nonresonant and Kostov-generic. This corresponds to the same condition for the logarithmic connection and also for the parabolic weights.

There is a different characterization of the parabolic structure, obtained by looking at E as $q_*(L_Y \otimes \mathcal{O}_Y)^{C_n}$. Let y be a local coordinate on Y near $\tilde{t}_i$, then $L_Y \otimes \mathcal{O}_Y$ is filtered by the subsheaves $y^k L_Y \otimes \mathcal{O}_Y$. This gives a filtration of E by subsheaves $q_*(y^k L_Y \otimes \mathcal{O}_Y)^{C_n}$. For $k = n$ the subsheaf is equal to $E(-t_i)$, so for $0 \leq k < n$ this defines a subspace $F_i^k \subset E_{t_i}$. The parabolic subspace P_i^j is defined to be $F_i^{-n\alpha_i^j}$ where the α_i^j are the k/n such that the filtration jumps.

In this point of view, any vector bundle on Z or equivalently C_n-equivariant vector bundle on Y leads to a parabolic bundle on (X, D) with weights $\alpha_i^j \in (-1, 0] \cap \frac{1}{n}\mathbb{Z}$. Apply this to vector bundles with λ-connection.

Proposition 7.3. *The above construction provides an equivalence between the categories of:*
—vector bundles with λ-connections on Z;
—C_n-equivariant vector bundles with λ-connections on Y;
—parabolic bundles $(E, P_\bullet^\bullet, \alpha_\bullet^\bullet)$ on (X, D) with weights $\alpha_i^j \in (-1, 0] \cap \frac{1}{n}\mathbb{Z}$ and logarithmic λ-connection

$$\nabla : E \to E \otimes \Omega_X^1(\log D)$$

such that $\mathrm{res}_{t_i}(\nabla)$ respects the filtration $P_i^\bullet$ of E_{t_i} and acts by the scalar $r_i^j = \lambda\alpha_i^j$ on $Gr_{P_i}^j(E_{t_i})$.

For $\lambda = 1$ this correspondence coincides with the correspondences of Propositions 7.1 and 7.2.

Notice that for $\lambda \neq 0$ the parabolic filtration and weights are determined by ∇. On the other hand, at $\lambda = 0$ the requirement becomes just that ∇ acts by 0 on $Gr_{P_i}^j(E_{t_i})$, in other words it respects strictly the parabolic filtration as in [**18**] for example. So for $\lambda = 0$ the connection doesn't determine the weights.

Lemma 7.4. *The correspondence of Proposition 7.3 is compatible with subobjects and preserves the degree, using the parabolic degree for parabolic logarithmic λ-connections. Hence it preserves stability and semistablity, and induces an isomorphism between moduli stacks.*

Suppose now that (E, ∇) is a logarithmic connection on (X, D) whose residues are non-resonant and have rational eigenvalues. Let n be a common denominator for the eigenvalues. By doing elementary transformations we may assume that the eigenvalues of the residue lie in $(-1, 0] \cap \frac{1}{n}\mathbb{Z}$. From the non-resonance condition, the decomposition of E_{t_i} into eigenspaces of dimension 1 induces a full-flag parabolic structure $P_i^\bullet$ at t_i, and the residues of ∇ determine the weights $\alpha_i^j = r_i^j$. The degree $d = \deg(E)$ is determined by the Fuchs relation. We get a point in $\mathcal{M}_1^d(\mathbf{r})$.

If we assume that the residues are nonspecial, then the parabolic weights are also nonspecial, and our point is stable. We can take the limiting parabolic Higgs bundle

$$\lim_{u \to 0}(E, u\nabla, P_\bullet^\bullet) \in \mathcal{M}_0^{d,\alpha}(\mathbf{r})^{\mathbb{G}_m}.$$

which will be stable too (hence unique up to translation of the $\mathbb{G}_m$-action, see [**46**]).

On the other hand, (E, ∇) has finite order monodromy so it corresponds to a C_n-equivariant vector bundle with connection (E_Y, ∇_Y) on Y. The limit

$$\lim_{u \to 0}(E_Y, u\nabla_Y)$$

is a C_n-equivariant $\mathbb{G}_m$-fixed Higgs bundle on Y. Similarly these correspond to a vector bundle with connection (E_Z, ∇_Z) on the root stack Z and again the limit $\lim_{u \to 0}(E_Z, u\nabla_Z)$ is a $\mathbb{G}_m$-fixed Higgs bundle on Z.

Lemma 7.5. *These three limits are the same via the correspondence of Proposition 7.3.*

The parabolic weights which should be used in order to maintain the correspondence with bundles on the root stack Z or C_n-equivariant bundles on Y, are given by the residues of the connection. These are also given by the local monodromy operators of the local system.

Going back to the case of local systems of rank 2, the parabolic weights determined by the finite order local monodromy will sometimes be in the unstable zone, and sometimes in the stable zone. This is the motivation for our consideration of both zones in the previous discussion. From Corollaries 5.7 and 6.3 we get the foliation conjecture for most irreducible components of the moduli of rank 2 local systems on Z.

Corollary 7.6. *The foliation conjecture of* [**46**] *holds for the moduli of rank* 2 *connections on the orbifold* Z*, at least in the connected components which correspond to nonspecial local monodromy data.*

8. Transversality of the fibrations

Here, we compute the two fibrations defined in the (a)-unstable zone by the map Υ (see Theorem 5.6) and in the stable zone by the map Φ (see Theorem 6.2). We then prove, for Kostov-generic local exponents $\mathbf{r}$, that the two fibrations are strongly transversal: generic fibers intersect at one point. In the next section, we will see that the two fibrations are permuted by an Okamoto symmetry of the moduli space. A similar description is presented at the end of the paper of Arinkin and Lysenko in [**4**].

Let us first recall the classical construction of canonical coordinates (p, q) on the moduli space $\mathcal{M}_1^1(\mathbf{r})$. After twisting by a convenient logarithmic rank one connection (which has no effect on the construction of the two fibrations), we may assume that the local exponents are :

$$(r_1^+, r_1^-, \ldots, r_4^+, r_4^-) = \left(\frac{\kappa_1}{2}, -\frac{\kappa_1}{2}, \ldots, \frac{\kappa_4}{2}, -\frac{\kappa_4}{2} - 1\right) \tag{8.1}$$

(note that the last exponent is shifted by -1 in order to get a degree 1 bundle). We also fix singular points $(t_1, t_2, t_3, t_4) = (0, 1, t, \infty)$. For convenience, denote by $\mathcal{M}_1^1(\boldsymbol{\kappa})$ the moduli space of such connections where $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_4) \in \mathbb{C}^4$ satisfies Kostov-generic conditions :

- $\kappa_i \notin \mathbb{Z}$ for $i = 1, \ldots, 4$,
- $\pm\kappa_1 + \cdots + \pm\kappa_4 \notin 2\mathbb{Z} + 1$ whatever the signs are.

A connection $(E, \nabla, P_\bullet) \in \mathcal{M}_1^1(\boldsymbol{\kappa})$ is therefore irreducible (see Lemma 3.2) and defined on the bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$. Such a connection may be described in the trivialization (e, f) used in section 3 by

$$\nabla : Y \mapsto dY + \Omega Y$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ represents the section $y_1 e + y_2 f$ and $\Omega = A dx$ is a 2×2-matrix of logarithmic 1-forms. Precisely,

$$A = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x-t} + B$$

with $\mathrm{Tr}(A_i)=0$ and $\det(A_i)=-\frac{\kappa_i}{2}$, $B=\begin{pmatrix}0&0\\b&0\end{pmatrix}$ and moreover, the coefficients of the matrix $x(x-1)(x-t)A$ have degree $\begin{pmatrix}2&1\\3&2\end{pmatrix}$. We omit conditions at $x=\infty$ for the eigenvalues for the moment. The subbundle $\mathcal{O}(1)$ generated by $f=\begin{pmatrix}0\\1\end{pmatrix}$ is not ∇-invariant and the $(1,2)$-coefficient vanishes at a single point $x=q\in\mathbb{P}^1$ (possibly ∞). This is the apparent singular point of the scalar equation with respect to the cyclic vector $\mathcal{O}(1)$: q is the image of the map Υ of Theorem 5.6 and we already get the first fibration. Since the automorphism group $\mathrm{Aut}(\mathcal{O}\oplus\mathcal{O}(1))$ fixes the line bundle $\mathcal{O}(1)$, another invariant, under gauge transformations, is given by the $(2,2)$-coefficient of A, in particular its value at $x=q$ whenever $q\neq 0,1,t,\infty$; we define

$$p:=A(2,2)|_{x=q}+\frac{\kappa_1}{2q}+\frac{\kappa_2}{2(q-1)}+\frac{\kappa_3}{2(q-t)}.$$

More abstractly, at the point q where the subbundle $\mathcal{O}(1)$ osculates to the connection, we can compare the connection with a standard one on $\mathcal{O}(1)$ depending on $\boldsymbol{\kappa}$, and p is the difference.

When $(p,q)\in\mathbb{C}^2$, we get an affine chart of the moduli space $M_1^1(\boldsymbol{\kappa})$. In order to reconstruct the connection from p and q, one can use a gauge transformation to normalize the connection as follows. One may first assume that the $(2,1)$-coefficient of $x(x-1)(x-t)A$ has degree one, i.e. $B=0$ and $A_4:=-A_1-A_2-A_3$ be diagonal, say: $A_4=\begin{pmatrix}\frac{-\kappa_4-1}{2}&0\\0&\frac{\kappa_4+1}{2}\end{pmatrix}$ (to do this, we just assume $\kappa_4\neq 0$ and $q\neq 0$). Note that A now defines a trace-free fuchsian system on the trivial bundle $\mathcal{O}\oplus\mathcal{O}$ and the connection ∇ is derived after applying an elementary transformation in the direction $\begin{pmatrix}0\\1\end{pmatrix}$ at $x=\infty$. There still remain diagonal gauge transformations; they can be used to normalize the $(1,2)$-coefficient of A to $\frac{x-q}{x(x-1)(x-t)}$. Once $(p,q)\in\mathbb{C}^2$ are fixed, one deduces that the normalized system is written

$$A_i=\begin{pmatrix}-z_i-\frac{\kappa_i}{2}&\frac{z_i}{u_i}\\-u_i(z_i+\kappa_i)&z_i+\frac{\kappa_i}{2}\end{pmatrix}$$

where

$$\begin{cases}u_1&=&\frac{tz_1}{q}\\u_2&=&\frac{(t-1)z_2}{q-1}\\u_3&=&\frac{t(t-1)z_3}{q-t}\end{cases}\quad\begin{cases}z_1&=&q\frac{E-(\kappa_0-1)^2(t-1)}{t(\kappa_4+1)}\\z_2&=&-(q-1)\frac{E+(\kappa_4+1)(q-t)\tilde{p}-(\kappa_0-1)^2t-(\kappa_0-1)(\kappa_0+\kappa_4)}{(t-1)(\kappa_4+1)}\\z_3&=&(q-t)\frac{E+(\kappa_4+1)t(q-1)\tilde{p}-(\kappa_0-1)^2-(\kappa_0-1)(\kappa_0+\kappa_4)t}{t(t-1)(\kappa_4+1)}\end{cases}$$

and

$$\tilde{p}=p-\frac{\kappa_1}{q}-\frac{\kappa_2}{q-1}-\frac{\kappa_3}{q-t},\quad 2\kappa_0+\kappa_1+\kappa_2+\kappa_3+\kappa_4=1$$

and

$$\begin{aligned}E\;=\;&q(q-1)(q-t)\tilde{p}^2+(\kappa_3(q-t)+\kappa_2t(q-1)-2(\kappa_0-1)(q-1)(q-t))\,\tilde{p}\\&+(\kappa_0-1)^2q-(\kappa_0-1)(\kappa_3+\kappa_2t).\end{aligned}$$

These formulae come from Jimbo-Miwa ([**24**]), although we made slight changes of variables which fit into our notations. At least, they make sense on the Zariski open subset of the moduli space $M_1^1(\boldsymbol{\kappa})$ defined by $(p,q)\in\mathbb{C}^2$ and $q\neq 0,1,t$. This will be enough to compute and compare the two fibrations. We see that $\begin{pmatrix}1\\u_i\end{pmatrix}$ is an eigenvector for the $-\frac{\kappa_i}{2}$ eigenvalue, $i = 1, 2, 3$: these vectors already give us the parabolic struture of the connection over $x = 0, 1, t$. In order to compute the parabolic structure, we consider the unique subbundle $\varphi : \mathcal{O}(-1) \hookrightarrow \mathcal{O} \oplus \mathcal{O}(1)$ that contains the parabolic directions over all 4 points. This line bundle is the destabilizing bundle for the (c)-zone (see section 4). That line bundle also provides the conic discussed in section 3, and the unique zero of the first component of φ will coincide with the parameter Q of the moduli space of parabolic bundles. We find that the line bundle is generated by the section

$$Y = \begin{pmatrix} ((1-t)u_1 + tu_2 - u_3)x - t(u_2 - u_3) \\ (u_1u_2 + (t-1)u_2u_3 - tu_1u_3)x - tu_1(u_2 - u_3) \end{pmatrix}$$

and we get

$$Q = \frac{t(u_2 - u_3)}{u_1u_2 + (t-1)u_2u_3 - tu_1u_3}.$$

Mind that, from the normalization of the connection, the parabolic structure at $x = \infty$ is necessarily lying on the first factor $\mathcal{O}$ of our decomposition $E = \mathcal{O} \oplus \mathcal{O}(1)$; so the section defined above also contains the parabolic structure P_∞. After substitution, we finally obtain

$$Q = q + \frac{1-\kappa_0}{p - \frac{\kappa_1}{q} - \frac{\kappa_2}{q-1} - \frac{\kappa_3}{q-t}} \tag{8.2}$$

In Section 9, we can see that this transformation (8.2) is expressed by a composite of Okamoto symmetries (9.7), which is the conjugate of the extra Okamoto involution s_0 (9.6) by an easy involution.

Clearly, the q-fibration and Q-fibration are strongly transversal whenever $\kappa_0 \neq 1$, which is implied by Kostov-genericity condition. More precisely, let F and L be a general fiber of q-fibration and Q-fibration respectively. Then we see that the intersection number $F \cdot L$ of F and L is one.

Theorem 8.1. *For any $\boldsymbol{\kappa}$ satisfying the Kostov-genericity condition, the two fibrations defined by Φ and Υ are strongly transversal.*

9. Okamoto symmetries

In the article [**37**], Okamoto constructs a group of birational transformations of the moduli space, generated by elementary transformations, permutation of poles t_i, and a rather mysterious extra involution denoted s_0 in what follows. This group is described in many papers. Here we follow notations of [**21, 22**] but also use the presentation of Noumi-Yamada [**33**] for relators.

In order to describe Okamoto symmetries more geometrically, recall first the geometry of the moduli space $M_1^1(\boldsymbol{\kappa})$ and its natural compactification $\overline{M_1^1(\boldsymbol{\kappa})}$ ([**22**]).

In Theorem 4.1 in [**22**] (which we have already mentioned in Proposition 5.4 above), moduli spaces $\overline{M_1^1(\boldsymbol{\kappa})}$ of $\boldsymbol{\alpha}$-stable parabolic ϕ-connections were constructed as follows. We fix the weight $\boldsymbol{\alpha}$ as in [**22**, Theorem 4.1], but for simplicity we will not specify them for a while. Let us consider the Hirzebruch surface of degree 2 which is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$

$$\pi : \Sigma_2 = \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathbb{P}^1.$$

Let C_0 be the unique section of $\pi : \Sigma_2 \longrightarrow \mathbb{P}^1$ with the self-intersection number $(C_0)^2 = -2$ and F the class of a general fiber of π. Moreover we have another class of a section C_1 of π with the condition $C_1 \cdot C_0 = 0$. We see that $C_1 \sim C_0 + 2F$ where $\sim$ means the linear equivalence of divisors.

We fixed four distinct points t_1, t_2, t_3, t_4 in $\mathbb{P}^1$ and consider the fibers $F_i = \pi^{-1}(t_i)$, $1 \leq i \leq 4$. Since the data $\mathbf{r} = \{r_i^{\pm}\}$ are given by $\boldsymbol{\kappa} = \{\kappa_i\}$ as in (8.1) which are nonspecial, we can define two different points $b_i^{\pm}$ in each fiber F_i as follows.

Let e be the unit section of $\mathcal{O}$, and f be the unit section of $\mathcal{O}(2)$ vanishing twice at ∞. denote by q the projective variable of $\mathbb{P}^1$; a point of Σ_2 over $q \neq \infty$ is given by $e(q) + \tilde{p} f(q)$, thus characterized by $\tilde{p} \in \mathbb{P}^1$. In the affine chart $(q, \tilde{p})$, we set

$$\begin{cases} b_1^- &= (0,0) \\ b_1^+ &= (0, t\kappa_1) \end{cases} \quad \begin{cases} b_2^- &= (1,0) \\ b_2^+ &= (1, (1-t)\kappa_2) \end{cases} \quad \begin{cases} b_3^- &= (t,0) \\ b_3^+ &= (t, t(t-1)\kappa_3) \end{cases}$$

Now, let g be the unit section of $\mathcal{O}(2)$ vanishing twice at 0: $e(q) + \tilde{p} f(q) = e(q) + \tilde{p}_\infty g(q)$ where $\tilde{p}_\infty = \frac{\tilde{p}}{q^2}$ whenever $q \neq 0, \infty$. In coordinates $(q, \tilde{p}_\infty)$, we set

$$\begin{cases} b_4^- &= (\infty, -\kappa_0) \\ b_4^+ &= (\infty, -\kappa_0 - \kappa_4) \end{cases}$$

(see Figure 1).

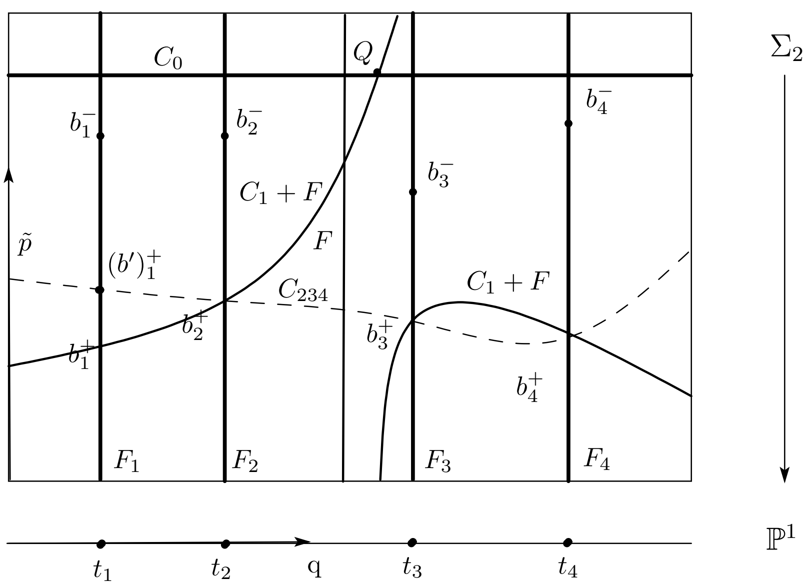


FIGURE 1. Hirzebruch surface Σ_2

Blowing up these 8 points $\{b_i^{\pm}\}_{1 \leq i \leq 4}$ of Σ_2, we obtain a morphism

$$f_{\boldsymbol{\kappa}} : S_{\boldsymbol{\kappa}} = \tilde{\Sigma}_{2,\boldsymbol{\kappa}} \longrightarrow \Sigma_2$$

where $S_{\boldsymbol{\kappa}}$ is a smooth rational surface. We set $E_i^{\pm} = f_{\boldsymbol{\kappa}}^{-1}(b_i^{\pm})$ the exceptional curves of f, and we denote by F_i' the proper transform of F_i. Then one can see that the

Picard group of $S_{\boldsymbol{\kappa}}$ is generated by the classes of $C_0, F, E_1^{\pm}, \cdots, E_4^{\pm}$, and moreover the anti-canonical class $-K_{S_{\boldsymbol{\kappa}}}$ has a unique effective member

$$Y = 2C_0 + F_1' + F_2' + F_3' + F_4'. \tag{9.1}$$

The pair $(S_{\boldsymbol{\kappa}}, Y)$ is an Okamoto-Painlevé pair in the sense of [**39**] (see also [**40**]), which means that the rational surface $S_{\boldsymbol{\kappa}}$ has a rational two form ω (unique up to non-zero constants) whose pole divisor is given by Y with the conditions $Y \cdot C_0 = Y \cdot F_i' = 0$ for $1 \leq i \leq 4$. Precisely, we have

$$\omega = dp \wedge dq \quad \text{with } \tilde{p} = q(q-1)(q-t)p.$$

Note that the complement $S_{\boldsymbol{\kappa}} \setminus Y$ has a holomorphic symplectic structure induced by ω. Then in [**22**], we have the following isomorphisms

$$\begin{array}{ccc} \overline{M_1^1(\boldsymbol{\kappa})} & \simeq & S_{\boldsymbol{\kappa}} \\ \cup & & \cup \\ M_1^1(\boldsymbol{\kappa}) & \simeq & S_{\boldsymbol{\kappa}} \setminus Y. \end{array}$$

The apparent singularity map $\Upsilon : \mathcal{M}_1^1(\boldsymbol{\kappa}) \to \mathcal{P}$ in Section 5 induces a morphism

$$M_1^1(\boldsymbol{\kappa}) \to \mathcal{P} \to \mathbb{P}^1$$

which can be identified with the natural map $\pi_{1,\boldsymbol{\kappa}} = \pi \circ f_{\boldsymbol{\kappa}} : M_1^1(\boldsymbol{\kappa}) \simeq S_{\boldsymbol{\kappa}} \setminus Y \to \Sigma_2 \to \mathbb{P}^1$. One can easily see that $\pi_{1,\boldsymbol{\kappa}}$ can be extended to the natural morphism

$$\pi_{1,\boldsymbol{\kappa}} : \overline{M_1^1(\boldsymbol{\kappa})} \simeq S_{\boldsymbol{\kappa}} \to \mathbb{P}^1.$$

Note that this morphism $\pi_{1,\boldsymbol{\kappa}}$ is the morphism induced by the linear system $|F| \simeq \mathbb{P}^1$. In the Picard group of $\overline{M_1^1(\boldsymbol{\kappa})} \simeq S_{\boldsymbol{\kappa}}$, we have the relations

$$F \sim F_i' + E_i^+ + E_i^- \quad \text{for each } i,\ 1 \leq i \leq 4 \tag{9.2}$$

which correspond to four singular fibers $F_i' + E_i^+ + E_i^-$ of the morphism $\pi_{1,\boldsymbol{\kappa}}$ (see Figure 2). Moreover on a certain Zariski open set of $M_1^1(\boldsymbol{\kappa})$, it coincides with the natural projection $(p, q) \mapsto q$ as in Section 8.

On the other hand, we also have the natural morphism $\Phi : \mathcal{M}_1^1(\boldsymbol{\kappa}) \to \mathcal{P}$ in Section 6, where in this case $\mathcal{P}$ can be identified with the moduli space Q^{simple}/A of simple quasiparabolic bundles (cf. Lemma 6.1). This induces another natural morphism $\pi_{2,\boldsymbol{\kappa}} : M_1^1(\boldsymbol{\kappa}) \to \mathcal{P} \to \mathbb{P}^1$ which also can be extended to $\pi_{2,\boldsymbol{\kappa}} : \overline{M_1^1(\boldsymbol{\kappa})} \to \mathbb{P}^1$. Let us denote by L the class of general fiber of $\pi_{2,\boldsymbol{\kappa}} : \overline{M_1^1(\boldsymbol{\kappa})} \to \mathbb{P}^1$. Then from the calculation in Section 8, we see that

$$L \sim C_1 + F - E_1^+ - E_2^+ - E_3^+ - E_4^+. \tag{9.3}$$

A general member L of the linear system $|L|$ is isomorphic to $\mathbb{P}^1$ and interesection numbers of related divisors are given as follows:

$$L \cdot C_0 = 1, \quad L \cdot E_i^+ = 1, \quad L \cdot F = 1.$$

For each i, $1 \leq i \leq 4$, let $\{j, k, l\}$ be the complement of i in $\{1, 2, 3, 4\}$. Then there exists an irreducible curve C_{jkl} in Σ_2 such that $C_{jkl} \sim C_1$ and C_{jkl} is passing through 3 points b_j^+, b_k^+, b_l^+. Let $(E')_i^+$ be the proper transform of $C_{jkl} \in \overline{M_1^1(\boldsymbol{\kappa})}$ by the blowing up $\overline{M_1^1(\boldsymbol{\kappa})} \to \Sigma_2$. Then we see that

$$(E')_i^+ \sim C_1 - E_j^+ - E_k^+ - E_l^+,$$

and $((E_i')^+)^2 = -1$, that is, $(E_i')^+$ is a (-1)-exceptional curve. The intersection point of C_{jkl} with F_i is denote by $(b')_i^+$. Note that the morphism $\pi_{2,\boldsymbol{\kappa}}$ has also exactly 4 singular fibers, which correponds to the following linear equivalences of divisors

$$L \sim F_1' + (E')_i^+ + E_i^- \quad \text{for each } i \quad 1 \leq i \leq 4. \tag{9.4}$$

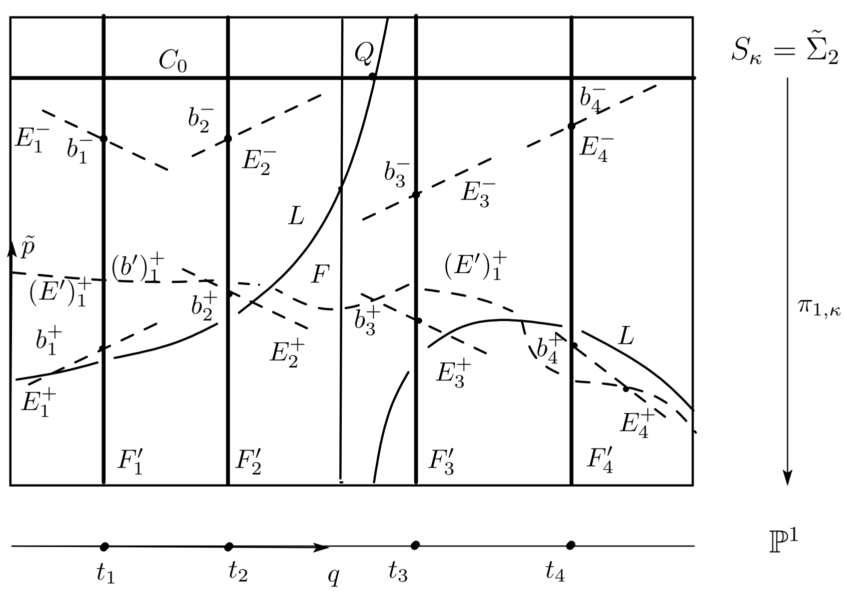
We have the following two fibrations

$$\begin{array}{ccc} \overline{M_1^1(\boldsymbol{\kappa})} & \stackrel{\pi_{2,\boldsymbol{\kappa}}}{\longrightarrow} & \mathbb{P}^1 \\ {\scriptstyle \pi_{1,\boldsymbol{\kappa}}} \big\downarrow & & \\ \mathbb{P}^1 & & \end{array} \tag{9.5}$$

where $\pi_{1,\boldsymbol{\kappa}}$, $\pi_{2,\boldsymbol{\kappa}}$ are corresponding to the linear systems $|F|$ and $|L| = |C_1 + F - E_1^+ - E_2^+ - E_3^+ - E_4^+|$ respectively. These give two different Lagrangian fibrations on the moduli space $M_1^1(\boldsymbol{\kappa})$.

It is interesting to remark that the morphism $\pi_{2,\boldsymbol{\kappa}}$ can be identified with the apparent singularity map $\pi_{1,\boldsymbol{\kappa}'}$ for different data $\boldsymbol{\kappa}'$. In fact, contracting the 8 exceptional curves $(E_i')^+, E_i^-$, we obain the morphism $\overline{M_1^1(\boldsymbol{\kappa})} \to \Sigma_2$, and then the points of blowing ups are corresponding to $(b_i')^+, b_i^-$ on the fiber F_i of the natural fibration of $\Sigma_2 \to \mathbb{P}^1$.

We summarize the results.



$$-K_S \sim Y = 2C_0 + F_1' + F_2' + F_3' + F_4'$$

$$S_{\boldsymbol{\kappa}} \simeq \overline{M_1^1(\boldsymbol{\kappa})}, \quad S_{\boldsymbol{\kappa}} \setminus Y \simeq M_1^1(\boldsymbol{\kappa})$$

$$L \sim C_1 + F - E_1^+ - E_2^+ - E_3^+ - E_4^+$$

$$F \cdot L = 1$$

FIGURE 2. 8 points blowing ups of Hirzebruch surface Σ_2

Proposition 9.1. *The q-fibration and Q-fibration in Section 8 can be identified with the maps $\pi_{1,\boldsymbol{\kappa}}$, $\pi_{2,\boldsymbol{\kappa}}$ in (9.5) respectively. The general fibers of $\pi_{1,\boldsymbol{\kappa}}$ and $\pi_{2,\boldsymbol{\kappa}}$ are given by F and L respectively and they are strongly transversal, that is, $F \cdot L = 1$.*

Next we vary the parameter $\boldsymbol{\kappa}$ and consider the Bäcklund transformations acting on the family of the moduli spaces. From [**21, 22**], after fixing weights $\boldsymbol{\alpha}$, we get a smooth fibration $\boldsymbol{\kappa} : M_1^{1,\boldsymbol{\alpha}} \to \mathbb{C}^4$ with fiber $M_1^{1,\boldsymbol{\alpha}}(\boldsymbol{\kappa})$. The classical group of Bäcklund transformations is an equivariant (with respect to $\boldsymbol{\kappa}$-projection) group of birational transformations (that preserves the isomonodromy flow when we consider t as a variable). In restriction to fibers $M_1^{1,\boldsymbol{\alpha}}(\boldsymbol{\kappa})$ with Kostov-generic $\boldsymbol{\kappa}$, Bäcklund transformations are biregular. The restriction of the Bäcklund transformations group to the action on the parameter space $\boldsymbol{\kappa}$ is faithfull and its image is an affine reflection group, an affine Weyl group of type D_4. Let us describe the generators.

Firstly, one can switch the parabolic structure over t_i to the eigendirection of the other eigenvalue $\frac{\kappa_i}{2}$. By using coordinates (q, p) for a suitable Zariski open set of the moduli spaces, we can describe 4 generators as follows:

$$\begin{cases} s_1 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (-\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p - \frac{\kappa_0}{q}) \\ s_2 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_1, -\kappa_2, \kappa_3, \kappa_4, q, p - \frac{\kappa_0}{q-1}) \\ s_3 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_1, \kappa_2, -\kappa_3, \kappa_4, q, p - \frac{\kappa_0}{q-t}) \\ s_4 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_1, \kappa_2, \kappa_3, -\kappa_4, q, p) \end{cases}$$

One can next permute the poles of the connection by a linear x-transformation in such a way that the cross-ratio t is preserved (we skip here the permutations that do not preserve t parameter).

$$\begin{cases} r_{(12)(34)} : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_2, \kappa_1, \kappa_4, \kappa_3, t\frac{q-1}{q-t}, -(q-t)\frac{(q-t)p+\kappa_0}{t(t-1)}) \\ r_{(13)(24)} : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_3, \kappa_4, \kappa_1, \kappa_2, \frac{q-t}{q-1}, (q-1)\frac{(q-1)p+\kappa_0}{t-1}) \\ r_{(14)(23)} : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_4, \kappa_3, \kappa_2, \kappa_1, \frac{t}{q}, -q\frac{qp+\kappa_0}{t}) \end{cases}$$

One can also apply an even number of elementary transformations centered at parabolics. This has the effect to shift $\boldsymbol{\kappa}$ parameters by integers. We skip the formula of generators which is much too huge; we will describe them in another way just below. By the way, we obtain the group of Schlesinger transformations. So far, the transformations come from geometric transformations on parabolic connections.

Finally, the larger Bäcklund transformation group is generated by the transformations $r_{(ij)(kl)}$ and s_i above and the extra Okamoto involution:

$$s_0 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (\kappa_1 + \kappa_0, \kappa_2 + \kappa_0, \kappa_3 + \kappa_0, \kappa_4 + \kappa_0, q + \frac{\kappa_0}{p}, p) \tag{9.6}$$

The geometric nature (even the Galois group) of the connection is not preserved. This involution exchanges finite and infinite monodromy, reducible and irreducible monodromy, and real monodromy groups $SL(2, \mathbb{R})$ and $SU(2)$. The first author and S. Cantat have described the action of these on the Betti moduli spaces in Appendix B of [**11**].

We have relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ for } i, j \neq 0 \quad \text{and} \quad s_0 s_i s_0 = s_i s_0 s_i$$
$$r_\sigma^2 = 1 \quad \text{and} \quad r_\sigma s_i = s_j r_\sigma \text{ for } \sigma = (ij)(kl)$$

The elementary transformations can be derived by combinations like:

$$r_{(12)(34)} s_3 s_4 s_0 s_1 s_2 s_0 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1 + 1, \kappa_2 + 1, \kappa_3, \kappa_4)$$

(we omit the huge formula in p and q).

Our main remark of the section is that a conjugate of the Okamoto transformation s_0 exchanges the two fibrations. Precisely, recall that the targets of the two maps Υ and Φ are canonically identified as $\mathcal{P}$ (see Sections 5 and 6). After projection $\mathcal{P} \to \mathbb{P}^1$ (identifying pair-wise the non separated points) we respectively get the two maps $\pi_{1,\boldsymbol{\kappa}}, \pi_{2,\boldsymbol{\kappa}}$ or $q, Q : M_1^{1,\boldsymbol{\alpha}} \to \mathbb{P}^1$ computed above (here we consider $\boldsymbol{\kappa}$ as variables). Comparing (8.2) with (9.6), one can then check that the Q-map factors as

$$Q = q(s_1 s_2 s_3 s_4) s_0 (s_1 s_2 s_3 s_4). \tag{9.7}$$

Mind that $s_1 s_2 s_3 s_4$ is an involution so that we also get

$$q = Q(s_1 s_2 s_3 s_4) s_0 (s_1 s_2 s_3 s_4)$$

A similar fact was already observed for $\mathrm{SL}(2,\mathbb{C})$-connections on the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ by Arinkin-Lysenko in [**4**] section 8 and in [**26**]. More precisely, following [**4**], the two maps $\pi_{i,\boldsymbol{\kappa}}$ above, $i = 1, 2$, glue together to define a proper morphism

$$\pi_{1,\boldsymbol{\kappa}} \times \pi_{2,\boldsymbol{\kappa}} \ : \ \overline{M_1^{1,\boldsymbol{\alpha}}} \to \mathbb{P}^1 \times \mathbb{P}^1$$

which is just the blow-up of 8 points along the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$: one first blow-up the 4 points (t_i, t_i), $i = 1, \ldots, 4$, and next blow-up each of the 4 exceptional divisors F_i at one point, the position of which depends on $\boldsymbol{\kappa}$ (see Remark 9.5 below). The anti-canonical divisor $Y = 2C_0 + F_1' + F_2' + F_3' + F_4'$ is therefore defined by the strict transform C_0 of the diagonal and the strict transforms F_i' of the F_i's. This provides an alternate description of our moduli space.

As noticed in Section 5, there are 8 unstable zone for the weights $\boldsymbol{\alpha}$, one of which giving the q-fibration. The other ones give other cyclic vectors and thus other fibrations. They can be deduce from q after applying an even number of elementary transformations at the parabolics P_i. This is also given by Bäcklund transformations. For instance

$$r_{(12)(34)} s_0 s_1 s_2 s_0 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4, q, p) \mapsto (1-\kappa_1, 1-\kappa_2, \kappa_3, \kappa_4, \tilde{q}, \tilde{p}).$$

where

$$\tilde{q} = t(q-1)(q-t) \frac{p^2 + \left(\frac{1-\kappa_1-\kappa_2}{q-1} - \frac{\kappa_3}{q-t}\right) p + \frac{\kappa_0(\kappa_0+\kappa_4)}{(q-1)(q-t)}}{((q-t)p + \kappa_0 + \kappa_4)((q-t)p + \kappa_0)}$$

and

$$\tilde{p} = -\frac{((q-t)p + \kappa_0 + \kappa_4)((q-t)p + \kappa_0)}{t(t-1)p}.$$

There are 16 natural choices for the parabolic structure, corresponding to a choice of one of the two eigenvalues at each point. But switching the parabolic structure over t_i is given by the action of the symmetry s_i, $i = 1, 2, 3, 4$. So the 16 parabolic fibrations are all obtained from the q-fibration after applying conjugates of s_0 by the s_i, $i = 1, 2, 3, 4$. For instance, switching for the other parabolic structure P_i' defined by the r_i^+ eigenspace, we simply get the fibration defined by $q(s_0)$.

Among all affine $\mathbb{A}^1$-fibrations over $\mathcal{P}$ that can be deduced on our moduli space by applying Bäcklund transformations (there are infinitely many) on the q-fibration, the 16 ones above play a special role:

Proposition 9.2. *The* 16 *parabolic fibrations above are the unique affine* $\mathbb{A}^1$-*fibrations on* $M_1^1(\boldsymbol{\kappa})$ *that are strongly transversal to the* q-*fibration and that compactify as* $\mathbb{P}^1$-*fibrations in the natural compactification* $\overline{M_1^1(\boldsymbol{\kappa})}$.

Proof. Recall that the moduli space $M_1^1(\boldsymbol{\kappa})$ can be obtained by removing $Y_{red} = C_0 + F_1' + F_2' + F_3' + F_4'$ from the compactification $\overline{M_1^1(\boldsymbol{\kappa})}$ (see Figure 2). Moreover $\overline{M_1^1(\boldsymbol{\kappa})}$ is obtained by 8 blowing ups at $\{b_i^{\pm}\}_{1 \leq i \leq 4}$ of $\Sigma_2 \to \mathbb{P}^1$ and the q-fibration $\pi_{1,\boldsymbol{\kappa}} : \overline{M_1^1(\boldsymbol{\kappa})} \to \mathbb{P}^1$ is obtained by the linear system $|F|$. Let L' be the divisor class of a general fiber of a fibration strongly transversal to the q-fibration. Since $L' \cdot F = 1$ and the linear system $|L'|$ is base point free by the assumption, one can see that L' can be written as

$$L' = C_1 + nF - \sum_{i=1}^{4} a_i E_i^+ - \sum_{i=1}^{4} b_i E_i^-, \quad n \geq 0.$$

Note that $F \sim F_i' + E_i^+ + E_i^-$ and $F_i' \cdot E_i^{\pm} = 1$. Since L' is numerically effective and $F \cdot F_i' = F \cdot E_i^{\pm} = C_1 \cdot E_i^{\pm} = 0$, $C_1 \cdot F = C_1 \cdot F_i' = 1$, $(E_i^{\pm})^2 = -1$, we see that

$$L' \cdot F_i' = 1 - (a_i + b_i) \geq 0, \quad L' \cdot E_i^+ = a_i \geq 0, \quad L' \cdot E_i^- = b_i \geq 0.$$

Hence $0 \leq a_i, b_i \leq 1, 0 \leq a_i + b_i \leq 1$, or $(a_i, b_i) = (1,0), (0,1), (0,0)$.

On the other hand, since the general fiber of such a fibration $\overline{M_1^1(\boldsymbol{\kappa})} \to \mathcal{P}$ is $\mathbb{P}^1$, if we require the restriction of this fibration to $M_1^1(\boldsymbol{\kappa})$ to be an affine $\mathbb{A}^1$-fibration, we see that $L' \cdot Y_{red} = 1$. Since $C_1 \cdot C_0 = 0$, $C_0 \cdot E_i^{\pm} = 0$, we see that $L' \cdot C_0 = nF \cdot C_0 = n$. Then again by $L' \cdot F_i' = 1 - (a_i + b_i) \geq 0$,

(9.8)

$$1 = L' \cdot Y_{red} = L' \cdot (C_0 + F_1' + F_2' + F_3' + F_4') = n + \sum_{i=1}^{4} (1 - (a_i + b_i)) = n + 4 - \sum_{i=1}^{4} (a_i + b_i)$$

Hence $\sum_{i=1}^4 (a_i + b_i) = n + 3$. Note that $\sum_{i=1}^4 (a_i^2 + b_i^2) = \sum_{i=1}^4 (a_i + b_i) = n + 3$. Moreover $L'^2 = 0$ implies that

$$0 = C_1^2 + 2nC_1 \cdot F - \sum_{i=1}^{4} (a_i^2 + b_i^2) = 2 + 2n - \sum_{i=1}^{4} (a_i^2 + b_i^2) = 2n + 2 - n - 3 = n - 1.$$

Hence we have $n = 1$ and $(a_i, b_i) = (1,0)$ or $(0,1)$ for all i, $1 \leq i \leq 4$. For each choice of $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \{+,-\}^4$, we can consider the divisor class

$$L^{\boldsymbol{\sigma}} = C_1 + F - E_1^{\sigma_1} - E_2^{\sigma_2} - E_3^{\sigma_3} - E_4^{\sigma_4}.$$

Then as in (9.3), the linear system $|L^{\boldsymbol{\sigma}}|$ defines a morphism $\overline{M_1^1(\boldsymbol{\kappa})} \to \mathbb{P}^1$ which gives an affine $\mathbb{A}^1$-fibrations on $M_1^1(\boldsymbol{\kappa}) = \overline{M_1^1(\boldsymbol{\kappa})} \setminus Y_{red} \to \mathbb{P}^1$ which is strongly transversal to the q-fibration. We obtain 16 different fibrations associated to the linear systems $|L^{\boldsymbol{\sigma}}|$ and the above consideration shows that the linear systems $|L^{\boldsymbol{\sigma}}|$ are the only possible strongly transversal fibrations which give affine $\mathbb{A}^1$-fibrations on $M_1^1(\boldsymbol{\kappa})$. $\square$

Now, for any Bäcklund transformation s, one can consider the fibration defined by $q \circ s$. Since s is biregular in restriction to $M_1^1(\boldsymbol{\kappa})$, the resulting $(q \circ s)$-fibration is again an $\mathbb{A}^1$-fibration over $\mathcal{P}$. We can now prove

Corollary 9.3. *Among all* $\mathbb{A}^1$-*fibration of the form* $q \circ s$, *only the* 16 *ones above are transversal to* q.

PROOF. One easily check that the generators s_i and $r_{(ij)(kl)}$ for the Bäcklund transformation group restrict as a biregular transformation of C_0 (mind that they are only birational on $\overline{M_1^1(\boldsymbol{\kappa})}$): the identity for s_i and a Moebius permutation for $r_{(ij)(kl)}$. As a consequence, $q \circ s : C_0 \to \mathbb{P}^1$ is $1:1$ and the linear system defined by the fibers of $q \circ s$ is base point free, even at infinity. We can apply the proposition above to conclude. □

Remark 9.4. *There are many $\mathbb{A}^1$-fibrations on $M_1^1(\boldsymbol{\kappa})$ that are transversal to the q-fibration. For instance, the p-fibration is like this, but its compactification is not base point free: the general fiber intersects Y_{red} exactly at $C_0 \cap F_4'$. The previous statement shows in particular that $p \neq q \circ s$ for any Bäcklund transformation s. One can also find examples of $\mathbb{A}^1$-fibrations transversal to q having arbitrary high intersection number at the base point at infinity. However, all examples which are not of the form $q \circ s$ seem to have only 2 special fibers, not 4 as happens with the 16 ones of the statement.*

Remark 9.5. *In our construction, we have choosen from the beginning the parabolic structure defined by the b_i^-. If we switch to the parabolic structure defined by b_i^+ and copy the Arinkin-Lysenko picture with this new parabolic fibration, then we get the following nice presentation of our moduli space:*

$$\overline{M_1^{1,\boldsymbol{\alpha}}} \to \mathbb{P}^1 \times \mathbb{P}^1$$

is the blow-up of 8 points along the diagonal. The coordinates (x,y) on $\mathbb{P}^1 \times \mathbb{P}^1$ are given by the two fibrations

$$x = q \quad \text{and} \quad y = q + \frac{\kappa_0}{p}.$$

The usual symplectic form is given by

$$dp \wedge dq = \kappa_0 \frac{dx \wedge dy}{(x-y)^2}.$$

The 8 points to blow-up are the 4 ordinary points

$$(x,y) = (0,0), \quad (1,1), \quad (t,t) \quad \text{and} \quad (\infty,\infty)$$

along the diagonal and next the 4 infinitesimal points over, given by the respective slopes

$$\frac{dy}{dx} = 1 + \frac{\kappa_0}{\kappa_1}, \quad 1 + \frac{\kappa_0}{\kappa_2}, \quad 1 + \frac{\kappa_0}{\kappa_3} \quad \text{and} \quad \frac{\kappa_4}{\kappa_0 + \kappa_4}.$$

Indeed, the blow-ups are exactly those ones needed to desingularize our initial parabolic fibration

$$Q = q + \frac{1-\kappa_0}{p - \frac{\kappa_1}{q} - \frac{\kappa_2}{q-1} - \frac{\kappa_3}{q-t}} = x + \frac{1-\kappa_0}{\frac{\kappa_0}{y-x} - \frac{\kappa_1}{x} - \frac{\kappa_2}{x-1} - \frac{\kappa_3}{x-t}}.$$

We note that the fibration given by the dual coordinate $p = \frac{\kappa_0}{y-x}$ (common for both x and y fibrations) is simply given in this picture by the fibration

$$dp = 0 \quad \Leftrightarrow \quad dx = dy.$$

Like in Arinkin-Lysenko's picture, the divisor at infinity Y_{red} is the union of strict transforms of the diagonal and the 4 exceptional divisors produced by the first *blowing-ups. Last, but not least, the Okamoto symmetry is given in this picture by*

$$s_0: \begin{cases} \kappa_i & \mapsto & \kappa_i+\kappa_0 \text{ for } i=1,2,3,4, \\ \kappa_0 & \mapsto & -\kappa_0 \\ (x,y) & \mapsto & (y,x) \end{cases}$$

10. Middle convolution interpretation

We point out here a possible explanation for the existence of a symmetry interchanging the two fibrations, in terms of Katz's middle convolution operation. When applied to local systems of rank 2 on $\mathbb{P}^1$ with 4 singular points having nonresonant local monodromy, the middle convolution operator gives back a new rank 2 system with the same 4 singularities. However, the monodromy transformations are changed.

Let $f_i^{\pm}$ denote the monodromy eigenvalues at t_i. Notice that because the elementary transformations interchange f_i^+ and f_i^- this choice isn't a big constraint.

The middle convolution operation depends on a choice of local system β over $\mathbb{P}^1\times\mathbb{P}^1$ with singularities on four horizontal lines, four vertical lines, and the diagonal. It is given by the 8 monodromy transformations $(a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4)$ with a_i corresponding to the horizontal lines and b_i to the vertical lines. These are subject to the relation

$$a_1a_2a_3a_4 = b_1b_2b_3b_4$$

both products being the inverse of the monodromy on the diagonal.

Now given a rank 2 local system L whose local monodromy transformations have eigenvalues f_i and f_i', assume they are nonspecial. In order to have a convolution with rank as small as possible, a_i should be the inverse of one of the eigenvalues; assume that it is

$$a_i = (f_i)^{-1}.$$

Lemma 10.1. *With the above notations, the middle convolution of L with β is a local system $\mathbf{mc}_\beta(L)$ of rank 2 with local monodromy eigenvalues*

$$b_i \quad \textit{and} \quad b_if_i'(f_1f_2f_3f_4)/f_i.$$

Proof. There are by now a large number of possible references for the middle convolution operation. For the authors' convenience we follow the notations of [**45**]. The local monodromy transformations fit into the vector denoted $\overrightarrow{g}$ with components

$$\overrightarrow{g}_i = [f_i^+]+[f_i^-].$$

The multiplicities are $m_i(f_i) = m_i(f_i') = 1$. In the notations of [**45**] we have $a_i = \beta^{H_i} = f_i^{-1}$, $b_i = \beta^{V_i}$, and

$$\beta^T = (a_1a_2a_3a_4)^{-1} = (b_1b_2b_3b_4)^{-1} = f_1f_2f_3f_4.$$

Then $m_i(f_i) = m_i((\beta^{H_i})^{-1}) = 1$. The components corresponding to exceptional curves on a blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ are

$$\beta^{U_i} := \beta^{H_i}\beta^{V_i}\beta^T = b_i(f_1f_2f_3f_4)/f_i.$$

The number of points is $n=4$ and the initial rank is $r=2$ so the defect is

$$\delta(\beta,\overrightarrow{g}) = (n-2)r - \sum_{i=1}^4 m_i((\beta^{H_i})^{-1}) = 0.$$

This simplifies the formula for the local Katz transformation

$$\kappa_i(\beta, \overrightarrow{g}) = [\beta^{V_i}] + [f_i' \beta^{U_i}].$$

In other words, the monodromy eigenvalues are b_i and with

$$\beta^{U_i} f_i' = b_i f_i' (f_1 f_2 f_3 f_4)/f_i.$$

□

We would like to investigate what this does to the stable and unstable zones, in the case of finite-order local monodromy where the parabolic weights are the same as the rational residues of the connection, which are in turn the angular arguments of the monodromy eigenvalues.

Proposition 10.2. *Fix finite order local monodromy transformations corresponding to residues* $\mathbf{r}$ *and parabolic weights* α*. Assume that they are nonspecial. Let* $\mathbf{r}'$ *and* α' *denote the corresponding values after middle convolution discussed in the previous lemma. The middle convolution operation extends to an operation on the full Hodge moduli stack of* α*-semistable* λ*-connections,*

$$\mathbf{mc}_\beta : \mathcal{M}^{d,\alpha}(\mathbf{r}) \to \mathcal{M}^{d,\alpha'}(\mathbf{r}').$$

It preserves the action of $\mathbb{G}_m$*, hence preserves the operation* $\lim_{u\to 0} u()$*.*

We don't do the proof here. One should be able to show that stability is preserved by saying that direct image preserves harmonic bundles. This is the subject of work in progress by the third author with R. Donagi and T. Pantev; however it should also be a consequence of Sabbah's theory of twistor $\mathcal{D}$-modules [**38**]. Nonetheless this proposition will be used in the following discussion, meaning that the remainder of the paper is for the moment heuristic.

The local monodromy eigenvalues can be written

$$f_i^+ = e^{\sqrt{-1}\theta_i^+}, \quad f_i^- = e^{\sqrt{-1}\theta_i^-},$$

with

$$\theta_i^\pm = 2\pi(\mu_i \pm \epsilon_i),$$

and $0 < \epsilon_i < 1/2$. This corresponds to a logarithmic connection whose residues are $\mu_i \pm \epsilon_i$.

In our main discussion of foliations on the moduli space, we have used the normalization $\deg(E) = 1$. By the Fuchs relation this means

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = -\frac{1}{2}. \tag{10.1}$$

To correspond to a bundle of odd degree, this relation should hold modulo $\mathbb{Z}$.

Now make a choice of which eigenvalue will be used for the middle convolution at each point i.e. along each horizontal line of $\mathbb{P}^1 \times \mathbb{P}^1$, by choosing the identification between $\{f_i, f_i'\}$ and $\{f_i^+, f_i^-\}$. Choose $f_i := f_i^+$ for all i. It should be stressed that the result will depend on this choice, see Remark 10.5 below.

Note that

$$f_1 f_2 f_3 f_4 = -e^{2\pi\sqrt{-1}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}$$

because of the relation (10.1) for the μ_i.

By Lemma 10.1 and the relation (10.1), the eigenvalues of the local monodromy transformation of the middle convolution $\mathbf{mc}_\beta(L)$ at t_i are

$$b_i \text{ and } b_i e^{2\pi\sqrt{-1}y_i}$$

where
$$y_i = -\frac{1}{2} + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - 2\epsilon_i.$$
We should choose a labeling of the eigenvalues as $c_i^{\pm}$, in such a way as to correspond to a logarithmic connection of odd degree.

Let us start off with a local system in one of the unstable zones, for example
$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2. \tag{10.2}$$
In this case we have
$$-1 < y_i < 0.$$
For this case, set
$$c_i^+ := b_i \text{ and } c_i^- := b_i e^{2\pi\sqrt{-1}y_i}.$$
Write $b_i = e^{2\pi\sqrt{-1}z_i}$ and put $\mu_i' := z_i + y_i/2$ and $\epsilon_i' = -y_i/2$. Now
$$c_i^+ = e^{2\pi\sqrt{-1}(\mu_i'+\epsilon_i')},$$
$$c_i^- = e^{2\pi\sqrt{-1}(\mu_i'-\epsilon_i')}.$$
We have
$$\frac{y_1+y_2+y_3+y_4}{2} = -1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4.$$
On the other hand, the equation $b_1b_2b_3b_4 = (f_1f_2f_3f_4)^{-1}$ yields
$$z_1 + z_2 + z_3 + z_4 = \frac{1}{2} - (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) \bmod \mathbb{Z}.$$
Putting these together we get
$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = z_1 + z_2 + z_3 + z_4 + \frac{y_1+y_2+y_3+y_4}{2}$$
$$= \left[\frac{1}{2} - (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)\right] + [-1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4] = -\frac{1}{2}$$
modulo $\mathbb{Z}$. Therefore the given choice corresponds to a logarithmic connection on a bundle of odd degree. We have
$$\begin{aligned}
\epsilon_1' &= \frac{1}{4} + \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2} - \frac{\epsilon_3}{2} - \frac{\epsilon_4}{2} \\
\epsilon_2' &= \frac{1}{4} - \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2} - \frac{\epsilon_3}{2} - \frac{\epsilon_4}{2} \\
\epsilon_3' &= \frac{1}{4} - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2} + \frac{\epsilon_3}{2} - \frac{\epsilon_4}{2} \\
\epsilon_4' &= \frac{1}{4} - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2} - \frac{\epsilon_3}{2} + \frac{\epsilon_4}{2}.
\end{aligned}$$
Notice that these are all in the interval $0 < \epsilon_i' < 1/2$, in view of (10.2).

We can now calculate
$$\epsilon_1' + \epsilon_2' + \epsilon_3' + \epsilon_4' = 1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4$$
so
$$1/2 < \epsilon_1' + \epsilon_2' + \epsilon_3' + \epsilon_4' < 1 < \frac{3}{2}.$$
Also for example
$$\epsilon_1' + \epsilon_2' - \epsilon_3' - \epsilon_4' = \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4$$
which, in view of the assumption (10.2), gives
$$-1/2 < \epsilon_1' + \epsilon_2' - \epsilon_3' - \epsilon_4' < 1/2.$$

Similarly for the other conditions of type (6.3).

Lemma 10.3. *Suppose L is a local system with finite order monodromy, corresponding to an odd degree logarithmic connection whose residues and parabolic weights are in the unstable zone (a) i.e. they satisfy* (10.2)*. Then choosing a rank one local system β with $a_i = f_i^+$, the middle convolution $\mathbf{mc}_\beta(L)$ has local monodromy transformations, again of finite order, lying in the stable zone.*

PROOF. The above calculations give (6.1), (6.2), and (6.3). □

Proposition 10.4. *The same holds for the other unstable zones: the middle convolution with a suitably chosen β goes into the stable zone. In the other direction, if L starts off in the stable zone then for a suitably chosen β the middle convolution will lie in the unstable zone.*

There are 8 unstable zones in all, types (a), (b) and 6 zones of type (c). The calculations are similar to the case (a) treated above. The images by **mc** divide the stable zone up into 8 sub-regions, which presents a computational difficulty for going back in the other direction. However, the fact that **mc** is involutive up to operations of elementary transformations and tensoring with rank 1 systems (which leave stable the distinction between stable and unstable zones), so the fact that the unstable zones go to the stable zone implies that the stable zone goes to the unstable zones.

Remark 10.5. *If, in the example above, we had chosen $f_i = f_i^+$ for $i = 1, 2, 3$ but $f_4 = f_4^-$, then the corresponding choice of β would have left $\mathbf{mc}_\beta(L)$ remaining inside the unstable zone. In general up to the operations of doing pairs of elementary transformations, there are two distinct choices for β and one of them will interchange the two zones.*

The middle convolution operation is one of the additional symmetries considered by Okamoto, although its normalization depends on the choice of a_i and b_i.

The middle convolution is obtained by pullback and higher direct image. These operations preserve the Hodge filtration moduli spaces $\mathcal{M}_{\mathrm{Hod}}$ when well-defined (for example if we assume Kostov genericity). They are compatible with the action of $\mathbb{G}_m$, so they preserve the limiting operation and hence the foliation by subspaces defined by looking at what the limit is. Hence, the middle convolution operation preserves the Higgs limit foliation, and since it exchanges stable and unstable zones, it takes the apparent singularity foliation to the parabolic structure foliation.

This gives a partial explanation of the interchanging phenomenon observed by Arinkin-Lysenko [**4**] and the first author in [**26**] and described in the previous section, although it leaves open the question of why it acts trivially on the quotient space $\mathcal{P}$ by the foliation. It might be possible to answer that by looking more carefully at the direct image operation in the middle convolution, as applied to parabolic Higgs bundles.

Using the middle convolution one can reduce the proof of the foliation conjecture for the stable zone, to the case of the unstable zone which was already known by [**21**]. This gives an alternate method to prove Corollary 6.3.

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