On the polygonal diameter of the interior, resp. exterior, of a simple closed polygon in the plane

Yaakov S. Kupitz[1] *, *Horst Martini* **, *Micha A. Perles* *

* Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, ISRAEL, kupitz@math.huji.ac.il; perles@math.huji.ac.il

** Faculty of Mathematics, University of Technology, 09107 Chemnitz, GERMANY, martini@mathematik.tu-chemnitz.de

Abstract

We give a tight upper bound on the polygonal diameter of the interior, resp. exterior, of a simple n-gon, $n \geq 3$, in the plane as a function of n, and describe an n-gon ($n \geq 3$) for which both upper bounds (for the interior and the exterior) are attained *simultaneously.*

Keywords: Jordan-Brouwer theorem, Jordan exterior (interior), Jordan's curve theorem, polygonal diameter, raindrop proof, simple closed polygon

MSC(2000): 51M05, 52B70, 57M50, 57N05

1 Introduction

The following is well known

Theorem 1.1. (The Jordan theorem) *Let* $f : [0,1] \to \mathbb{R}^2$ *be a simple closed curve in the plane (*f *is continous,* $f(0) = f(1)$ *and* $f(u) \neq f(v)$ *for* $0 < u < v \leq 1$*). Define* $P =_{\text{def}} \text{image} f = \{f(u) : 0 \leq u \leq 1\}$*, the image of* f*. Then* $\mathbb{R}^2 \setminus P = U_0 \cup U_1$*, where* U_0, U_1 *are connected open, non-empty mutually disjoint sets,* U_0 *is bounded (interior),* U_1 *is unbounded (exterior), and* $P = \text{bd}(U_0) = \text{bd}(U_1)$*.*

[1]Partially supported by the Landau Center at the Mathematics Institute of the Hebrew University of Jerusalem (supported by Minerva Foundation, Germany), and by Deutsche Forschungsgemeinschaft.

The proof of this theorem is not easy; see [3], [8], [11], [9, p. 37 ff.], [1, vol. I, pp. 39-64], [7, pp. 285 ff.], and the survey [5]. When the curve P is polygonal, however, i.e., when f is piecewise affine, the theorem becomes elementary:

Theorem 1.2. (The piecewise affine Jordan theorem) *Let* $p_0, p_1, \ldots, p_{n-1}, p_n = p_0, n \geq 3$*, be (n distinct) points in* $\mathbb{R}^2$*. Assume that the polygon* $P =_{\text{def}} \bigcup_{i=1}^{n} [p_{i-1}, p_i]$ *is simple, i.e., the segments* $[p_{i-1}, p_i]$ *do not intersect except for common endpoints:* $\{p_i\} = [p_{i-1}, p_i] \cap [p_i, p_{i+1}]$ *for* $1 \leq i \leq n-1, \{p_0\} = [p_0, p_1] \cap [p_{n-1}, p_0]$*. Then* $\mathbb{R}^2 \setminus P = U_0 \cup U_1$ *with the same properties of* U_0, U_1 *listed above* (Theorem 1.1).

Definition 1.1. A polygon P satisfying the conditions of Theorem 1.2 is a *simple closed n-gon.* The bounded [resp. unbounded] domain U_0 [resp. U_1] is the *interior* [resp. *exterior*], denoted by int$P$ [resp. ext$P$], of P.

A particularly simple proof of Theorem 1.2 is known as the "raindrop proof", see [4, pp. 267-269], [6, pp. 281-285], [2, pp. 27-29], or [9, pp. 16-18]. We reproduce this proof in a somewhat more complete and formal form than usually given in the literature for later reference to some of its parts.

So we first prove Theorem 1.2 (in Paragraphs 2 and 3 below). Then, squeezing this proof, a *tight* upper bound on the polygonal diameter of int$P$ [resp. ext$P$] (see Definition 3.2 below) is given as a function of n, and an n-gon ($n \geq 3$) for which both upper bounds are attained *simultaneously* is described (see Theorem 4.1 below). The d-dimensional analogue ($d \geq 2$) of this problem was discussed in [10, Theorem 3.2]. There we gave upper bounds on the polygonal diameter of int$\mathcal{C}$, resp. ext$\mathcal{C}$, for a polyhedral $(d-1)$-pseudomanifold $\mathcal{C}$ in $\mathbb{R}^d$ as a function of the number n of its facets and d. The bounds given there are shown to be *almost* tight (see [10, Section 4]), whereas the bounds given here (for $d = 2$) are tight. Another novelty of the present paper is that there is an n-gon P in $\mathbb{R}^2$ for which *both* upper bounds (on the polygonal diameter of int$P$ and ext$P$) are attained (simultanously), as said above, whereas for $d \geq 3$ the examples given in [10, Section 4] (namely one for int$\mathcal{C}$ and another one for ext$\mathcal{C}$) are *different* from each other.

For the sake of the proof of Theorem 1.2, we split it into two statements: Let P be a simple closed polygon in $\mathbb{R}^2$.

(E) (separation): $\mathbb{R}^2 \setminus P$ is the disjoint union of two open sets, int$P$ and ext$P$. The boundary of each one of these sets is P; int$P$ is bounded and ext$P$ is unbounded.

(F) (connectivity): The sets int$P$ and ext$P$ are [polygonally] connected.

We shall prove (E) (Paragraph 2) by constructing a continuous function $f : \mathbb{R}^2 \setminus P \to \{0, 1\}$ which attains both values 0 and 1 in every neighborhood of every point $x \in P$, and defining ext$P = f^{-1}(0)$, int$P = f^{-1}(1)$. Statement (F) (polygonal connectivity of int$P$ and of ext$P$) follows from Theorem 3.1 below.

2 A "raindrop" proof of (E)

The construction of f will be performed in three steps:

Preliminary step: Choosing a "generic" direction.

Choose an orthogonal basis (u, v) for $\mathbb{R}^2$ so that no two vertices of P have the same x-coordinate. Intuitively: the polygon P is drawn as a paper; rotate the paper so that no two vertices lie one above the other. Formally: let $L_1, \ldots, L_t$ be all lines spanned by subsets of $\{p_1, \ldots, p_n\}$. For $i = 1, \ldots, t$ let $L_i^0 =_{\text{def}} L_i - L_i$ be the linear (1-dimensional) subspace parallel to L_i. Choose a unit vector $v \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{t} L_i^0$ ("v" for "vertical"). The vector v is our direction "up", and $-v$ is pointing "down". By our choice of v, a line L, spanned by the vertices of P, will meet a line parallel to v in at most one point.

For a point $p \in \mathbb{R}^2 \setminus P$ denote by $R(p)$ the closed vertical "pointing down" half-line $R(p) =_{\text{def}} \{p - \lambda v : 0 \leq \lambda < \infty\}$. $R(p)$ is the path of a "raindrop" emanating from p. We divide $\mathbb{R}^2 \setminus P$ into two disjoint sets

$$\begin{aligned} S_0 &=_{\text{def}} \{p \in \mathbb{R}^2 \setminus P : R(p) \text{ does not meet any vertex of } P\}, \\ S_1 &=_{\text{def}} \{p \in \mathbb{R}^2 \setminus P : R(p) \text{ meets exactly one vertex of } P\}. \end{aligned}$$

(By our choice of v, we have $\mathbb{R}^2 \setminus P = S_0 \cup S_1$.) We shall define f on S_0 (= Step I), then extend it (continuously) to S_1 (= Step II). The following notation will be used: For a set $A \subset \mathbb{R}^2$, $A^+ =_{\text{def}} \{a + \lambda v : a \in A, \lambda \geq 0\}$.

Thus A^+ is the set of points that lie "above" A. If A is closed, then A^+ is closed. Note that (for all $p \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$):

$$R(p) \text{ meets } A \text{ iff } p \in A^+ . \tag{1}$$

Step I: Define f on S_0.

For $p \in S_o$ denote by $r(p)$ the number of edges of P met by $R(p)$, and define $f(p) =_{\text{def}} \text{par}(r(p)) =_{\text{def}} \frac{1}{2}(1-(-1)^{r(p)})$, the parity of $r(p)$ ($f(p) = 0$ if $r(p)$ is even, 1 if $r(p)$ is odd).

Fig. 1: the function $r(p)$ Fig. 2: the parity function $f(p) = \text{par}(r(p))$

Next we show that S_0 is a dense open subset of $\mathbb{R}^2$, and that $f : S_0 \to \{0,1\}$ is a continuous, hence locally constant function. Using vert$P$ for the set of vertices of P, we have in view of (1)

$$S_0 = \mathbb{R}^2 \setminus (P \cup (\text{vert}P)^+) . \tag{2}$$

The set $(\text{vert}P)^+$ is closed, same as P. Thus S_0 is an open subset of $\mathbb{R}^2$. Moreover, the set $P \cup (\text{vert}P)^+$ can be covered by a finite number of lines in $\mathbb{R}^2$. It follows that S_0 is dense in R^2.

Continuity of f: Assume $x \in S_0$. Let ε be the (positive) distance from x to $P \cup (\text{vert}P)^+ (= \mathbb{R}^2 \setminus S_0)$. If $x' \in \mathbb{R}^2$, $\|x - x'\| < \varepsilon$, then the segment $[x, x']$ does not meet $P \cup (\text{vert}P)^+$. Let $e = [p_{i-1}, p_i]$ $(1 \leq i \leq n)$ be any edge of P. The set e^+ is a closed, convex, unbounded and full-dimensional polyhedral subset of $\mathbb{R}^2$, whose boundary consists of the lower edge e and the side edges p_{i-1}^+, p_i^+. Thus $\text{bd}e^+ \subset P \cup (\text{vert}P)^+$, and therefore the segment $[x, x']$ does not meet the boundary of e^+. It follows that $x' \in e^+$ iff $x \in e^+$, i.e., $R(x)$ meets e iff $R(x')$ meets e. This is true for all edges e of P. Therefore $r(x) = r(x')$, hence $f(x) = f(x')$. This shows that the function $f : S_0 \to \{0,1\}$ is locally constant, hence continuous (in S_0).

Step II: Extend f continuously from S_0 to $S_0 \cup S_1 = \mathbb{R}^2 \setminus P$.

Suppose $p \in S_1$. Let p_i be the unique vertex of P that meets $R(p)$, i.e., $p \in p_i^+$. Note that $p \neq p_i$, i.e., $p \in$ relint p_i^+. Let $e_1 = [p_{i-1}, p_i], e_2 = [p_i, p_{i+1}]$ be the two edges of P incident with p_i. Define $L = p + \mathbb{R}v$. L is the vertical line through p. Denote by L^-, L^+ the two closed half-planes of $\mathbb{R}^2$ bounded by L. None of the edges e_1, e_2 is included in L, and they may be either in the same half-plane L^- or L^+, or in different half-planes. Choose the notation so that either (α) $e_1 \subset L^-, e_2 \subset L^+$ (Fig. 3) or (β) $e_1 \cup e_2 \subset L^+$ (Fig. 4).

Fig. 3: case α

Fig. 4: case β

A glance on Figures 3 and 4 shows that for a point x in the vicinity of p, but not lying on L, the parity of $r(x)$ is the same in either side of L. Hence we can extend the definition of f to p by defining $f(p)$ to be this parity. To make this into a formal argument consider the closed set $\triangle =_{\text{def}} P \cup (\text{vert}P \setminus \{p_i\})^+$. This set includes the boundary of e^+, for every edge e of P, except for e_1^+ and e_2^+. It also includes the boundaries of e_1^+ and e_2^+, except for $p_i^+ \setminus \{p_i\}$, and it does not contain the point p. Put $\varepsilon =_{\text{def}}$ dist$(p, \triangle) > 0$, and define $U =_{\text{def}} \{x \in \mathbb{R}^2 : \|x - p\| < \varepsilon\} = \text{int}B^2(p, \varepsilon)$. Note that if $x \in U$, then the closed interval $[p, x]$ misses $\triangle$. Now make the following observations.

(I) If e is any edge of P, other than e_1 and e_2, then the interval $[p, x]$ does not meet the boundary of e^+, and therefore p and x are either both in e^+, or both not in e^+.

(II) If, say, $e_1 \subset L^-$ and $x \in \text{int}L^-$ then, moving along the interval $[p, x]$ from p to x, we start at a point $p \in p_i^+ \subset \text{bd}e_1^+$, move into int$e_1^+$, and do not hit the boundary of e_1^+ again. Therefore $x \in \text{int}e_1^+$. The same holds with L^- replaced by L^+, and/or e_1 replaced by e_2. It follows that in case (α): if $x \in U \setminus L$, then x belongs to exactly one of the sets e_1^+, e_2^+. And it follows that in case (β): if $x \in U \cap \text{int}L^-$, then x belongs to none of the sets e_1^+, e_2^+; if $x \in U \cap L^+$, then x belongs to both of them.

(III) If $p_j \in \mathrm{vert}P \setminus \{p_i\}$, then $p_j^+ \subset \triangle$, and therefore $x \notin p_j^+$, who-ever $x \in U$.

(IV) If $x \in U \setminus L$, then clearly $x \notin p_i^+$. If $x \in U \cap L$, then the interval $[p, x]$ lies on L, contains a point $p \in p_i^+ \setminus \{p_i\}$ and does not meet p_i; therefore $x \in p_i^+ \setminus \{p_i\}$ $(= \mathrm{relint} p_i^+)$. From these observations we infer:

(A) $U \setminus L \subset S_0$ and f is constant on $U \setminus L$.

(B) $U \cap L \subset S_1$.

Now define $f(p)$ to be the constant value that f takes on $U \setminus L$. Clearly, if we apply the same procedure to any point $p' \in U \cap L$, we will end up with a value $f(p')$ equal to the value $f(p)$ just defined. (Note that any ε'-neighborhood of p' $(\varepsilon' > 0)$ contains points of $U \setminus L$.) Thus we have extended f to a locally constant, hence continuous function $f : \mathbb{R}^2 \setminus P \to \{0, 1\}$.

To complete the proof of statement (E), we define, as indicated after (F) above, the sets $\mathrm{ext}P =_{\mathrm{def}} f^{-1}(0)$ and $\mathrm{int}P =_{\mathrm{def}} f^{-1}(1)$. These are clearly two disjoint open sets in $\mathbb{R}^2$, whose union is $\mathrm{dom}f = \mathbb{R}^2 \setminus P$. Note that $\mathbb{R}^2 \setminus \mathrm{conv}P \subset \mathrm{ext}P$ and, therefore, $\mathrm{int}P \subset \mathrm{conv}P$. Thus $\mathrm{ext}P$ is unbounded and $\mathrm{int}P$ is bounded.

We still have to show that every point of P is a boundary point of both $\mathrm{int}P$ and $\mathrm{ext}P$ (and therefore $\mathrm{int}P \neq \emptyset, \mathrm{ext}P \neq \emptyset$). Since the boundaries of $\mathrm{int}P$ and of $\mathrm{ext}P$ are closed sets, it suffices to show that the common boundary points of $\mathrm{int}P$ and $\mathrm{ext}P$ are dense in P.

For any vertex $p_i\,(1 \leq i \leq n)$ the intersection of the vertical line $p_i + \mathbb{R}v$ with an edge e of P is at most a singleton. Thus $e \setminus \cup\{p_i + \mathbb{R}v : 1 \leq i \leq n\}$ is dense in e, and $P \setminus \cup\{p_i + \mathbb{R}v : 1 \leq i \leq n\}$ is dense in P. If $x \in P \setminus \cup\{p_i + \mathbb{R}v : 1 \leq i \leq n\}$, then x belongs to the relative interior of some edge e of P. If $\varepsilon > 0$ is sufficiently small, then the points $x + \varepsilon v, x - \varepsilon v$ are both in S_0, the half-line $R(x + \varepsilon v)$ meets e, in addition to all edges met by $R(x - \varepsilon v)$. Thus $r(x + \varepsilon v) = 1 + r(x - \varepsilon v)$, and $f(x + \varepsilon v) \neq f(x - \varepsilon v)$, i.e., $\{f(x - \varepsilon v), f(x + \varepsilon v)\} = \{0, 1\}$. Thus x is a common boundary point of $\mathrm{int}P$ and $\mathrm{ext}P$. This finishes the proof of (E).

3 Proof of (F)

Put $I_i =_{\rm def} [p_{i-1}, p_i], 1 \le i \le n$, the edges of P, and for $i = 1, 2, \ldots, n$ let u_i be a unit vector perpendicular to affI_i. Choose the orientation of u_i in such a way that for each point $b \in$ relintI_i and for all sufficiently small positive value of ε, $b + \varepsilon u_i \in$ ext$P$ and $b - \varepsilon u_i \in$ int$P$. Define $u_{i,i+1} =_{\rm def} u_i + u_{i+1}$, $1 \le i \le n$ (the indices are taken modulo n, i.e., $p_n = p_0, u_{n+1} = u_1, u_{n,n+1} = u_{n,1} = u_n + u_1$).

Lemma 3.1. *If ε is a sufficiently small positive number, then $p_i + \varepsilon u_{i,i+1} \in$ ext$P$, and $p_i - \varepsilon u_{i,i+1} \in$ int$P$ for $1 \le i \le n$.*

Proof: The edges I_i, I_{i+1} lie in two rays (half-lines) L_i, L_{i+1} bounded by p_i, say $L_i = p_i + \mathbb{R}^+ v_i, L_{i+1} = p_i + \mathbb{R}^+ v_{i+1}$, where v_i, v_{i+1} are suitable unit vectors orthogonal to u_i, u_{i+1}, respectively.

(a) (b) (c)

Fig. 5

If ε is a sufficiently small positive number ($0 < \varepsilon <$ dist$(p_i, P \setminus ($relint$(I_i \cup I_{i+1}))$, then $B^2(p_i, \varepsilon) \setminus P = B^2(p_i, \varepsilon) \setminus (L_i \cup L_{i+1})$. The union $L_i \cup L_{i+1}$ divides $B^2(p_i, \varepsilon)$ into two open sectors, $B^2(p_i, \varepsilon) \cap$ int$P$ and $B^2(p_i, \varepsilon) \cap$ ext$P$. If L_i, L_{i+1} are collinear ($v_{i+1} = -v_i$), then each one of these two sectors is an open half disc. In this case $u_i = u_{i+1}$ (Fig. 5(a)), $u_{i,i+1} = 2u_i = 2u_{i+1}$, and the lemma holds trivially. If u_i, u_{i+1} are not collinear, then one of the sectors is larger than a half disc, and the other is smaller. In both cases we have

$$\langle u_i, v_{i+1} \rangle = \langle u_{i+1}, v_i \rangle = \sin \alpha \,, \qquad (3)$$

where α is the central angle of the sector $B^2(p_i, \varepsilon) \cap$ext$P$ at p_i ($0 \le \alpha \le 360^o$).

If $\langle u_i, v_{i+1} \rangle < 0$, then $B^2(p_i, \varepsilon) \cap$ ext$P$ is the larger sector (Fig. 5(b)), and if $\langle u_i, v_{i+1} \rangle > 0$, then $B^2(p_i, \varepsilon) \cap$ int$P$ is the larger sector (Fig. 5(c)). Summing up the equalities

$$\begin{aligned} u_i &= \langle u_i, u_{i+1} \rangle u_{i+1} + \langle u_i, v_{i+1} \rangle v_{i+1} \,, \\ u_{i+1} &= \langle u_{i+1}, u_i \rangle u_i + \langle u_{i+1}, v_i \rangle v_i \end{aligned}$$

and using (3), we find $(1-\langle u_i, u_{i+1}\rangle)\,(u_i+u_{i+1}) = \sin\alpha\,(v_i+v_{i+1})$.

If $u_i \neq u_{i+1}$, then $1-\langle u_i, u_{i+1}\rangle > 0$, and

$$u_{i,i+1} = u_i + u_{i+1} = \frac{\sin\alpha}{1-\langle u_i, u_{i+1}\rangle}\cdot (v_i+v_{i+1})\,.$$

Thus $u_{i,i+1}$ is a positive [resp., negative] multiple of v_i+v_{i+1} when $\sin\alpha > 0$ [resp., $\sin\alpha < 0$]. In both cases, $u_{i,i+1}$ points towards ext$P$, and $-u_{i,i+1}$ towards int$P$. ∎

Lemma 3.2. ("Push away from $\boldsymbol{P}$")

(a) Fix i, $1 \le i \le n$, suppose $b \in \mathrm{relint} I_i$ and u is a vector satisfying $\langle u, u_i\rangle > 0$. Define $I^0 =_{\mathrm{def}} [b, p_i]$, $I^\varepsilon =_{\mathrm{def}} [b+\varepsilon u, p_i+\varepsilon u_{i,i+1}]$ (u_i, u_{i+1} and $u_{i,i+1} = u_i+u_{i+1}$ denote the same vectors as in the previous lemma). If ε is a sufficiently small positive number, then $I^\varepsilon \subset \mathrm{ext}P$ and $I^{-\varepsilon} \subset \mathrm{int}P$. (The required smallness of ε may depend on the choice of the point b and of the vector u.)

(b) Fix $i, 1 \le i \le n$, and define $J^0 =_{\mathrm{def}} [p_i, p_{i+1}] = I_{i+1}, J^\varepsilon =_{\mathrm{def}} [p_i + \varepsilon u_{i,i+1}, p_{i+1} + \varepsilon u_{i+1,i+2}]$. If ε is a sufficiently small positive number, then $J^\varepsilon \in \mathrm{ext}P$ and $J^{-\varepsilon} \in \mathrm{int}P$.

Proof:

(a) First note that I^0 does not meet any edge of P except I_i and I_{i+1}. The same holds for I^ε, provided

$$|\varepsilon| < \min\left(\frac{1}{2}, \frac{1}{\|u\|}\right)\cdot \mathrm{dist}\left(I^0, P \setminus (\mathrm{relint}(I_i \cup I_{i+1}))\right)\,.$$

By Lemma 3.1, $p_i + \varepsilon u_{i,i+1} \in \mathrm{ext}P$ and $p_i - \varepsilon u_{i,i+1} \in \mathrm{int}P$, provided ε is positive and sufficiently small. To complete the proof, it suffices to show that $I^\varepsilon \cap I_i = \emptyset$ and $I^\varepsilon \cap I_{i+1} = \emptyset$ (for sufficiently small $|\varepsilon|$, $\varepsilon \neq 0$).

As for I_i : $\langle u_i, u\rangle > 0$ (given) and $\langle u_i, u_{i,i+1}\rangle = 1 + \langle u_i, u_{i+1}\rangle > 0$. Therefore, for any $\varepsilon \neq 0$ both endpoints of I^ε lie (strictly) on the same side of the line affI_i, hence $I_i \cap I^\varepsilon = \emptyset$.

As for I_{i+1}: If I_{i+1} and I_i lie on the same line ($u_i = u_{i+1}$), then the previous argument shows that $I_{i+1} \cap I^\varepsilon = \emptyset$ for all $\varepsilon \neq 0$ as well. If $u_i \neq u_{i+1}$, consider first the case $\langle u_i, v_{i+1} \rangle < 0$. (Fig. 5(b)). For $\varepsilon > 0, I^\varepsilon$ lies in the open half-plane $\{x \in \mathbb{R}^2 : \langle u_i, x \rangle > \langle u_i, p_i \rangle\}$, whereas I_{i+1} lies in the closed half-plane $\{x \in \mathbb{R}^2 : \langle u_i, x \rangle \leq \langle u_i, p_i \rangle\}$. Therefore $I^\varepsilon \cap I_{i+1} = \emptyset$. For $\varepsilon < 0$,

$$\langle u_{i+1}, p_i + \varepsilon u_{i,i+1} \rangle = \langle u_{i+1}, p_i \rangle + \varepsilon(1 + \langle u_i, u_{i+1} \rangle) < \langle u_{i+1}, p_i \rangle .$$

On the other hand, $\langle u_{i+1}, b \rangle < \langle u_{i+1}, p_i \rangle$ (for any point $b \in \mathrm{relint} I_i$, since $\langle u_{i+1}, v_i \rangle < 0$), and therefore $\langle u_{i+1}, b + \varepsilon u \rangle < \langle u_{i+1}, p_i \rangle$ for sufficiently small $|\varepsilon|, \varepsilon \neq 0$. Thus both endpoints of I^ε lie on the same open side of the line $\mathrm{aff} I_{i+1}$, hence $I^\varepsilon \cap I_{i+1} = \emptyset$.

In the case $\langle u_i, v_{i+1} \rangle > 0$ (Fig. 5(c) above), just repeat the previous argument with the roles of $\varepsilon > 0$ and $\varepsilon < 0$ interchanged.

(b) The proof is similar to that of (a). First, note that J^0 does not meet any edge of P except I_i, I_{i+1} and I_{i+2}. The same holds for J^ε, provided

$$|\varepsilon| < \min\left(\frac{1}{2}, \frac{1}{\|u\|}\right) \cdot \mathrm{dist}\left(J^0, P \setminus \mathrm{relint}(I_i \cup I_{i+1} \cup I_{i+2})\right) .$$

By Lemma 3.1, $p_i + \varepsilon u_{i,i+1}, p_{i+1} + \varepsilon u_{i+1,i+2} \in \mathrm{ext} P$ and $p_i - \varepsilon u_{i,i+1}, p_{i+1} - \varepsilon u_{i+1,i+2} \in \mathrm{int} P$, provided ε is positive and sufficiently small. To complete the proof, it suffices to show that $J^\varepsilon \cap I_i = \emptyset, J^\varepsilon \cap I_{i+1} = \emptyset$ and $J^\varepsilon \cap I_{i+2} = \emptyset$ (for sufficiently small $|\varepsilon|, \varepsilon \neq 0$).

As for $I_{i+1} : \langle u_{i+1}, u_{i,i+1} \rangle = 1 + \langle u_{i+1}, u_i \rangle > 0$ and $\langle u_{i+1}, u_{i+1,i+2} \rangle = 1 + \langle u_{i+1}, u_{i+2} \rangle > 0$. Therefore, for any $\varepsilon > 0$, both endpoints of J^ε lie on the same open side of the line $\mathrm{aff} I_{i+1}$, hence $I_{i+1} \cap J^\varepsilon = \emptyset$.

As for I_i: If I_{i+1} and I_i lie in the same line ($u_i = u_{i+1}$), then the previous argument shows that $I_i \cap J^\varepsilon = \emptyset$ for all $\varepsilon \neq 0$ as well. If $u_i \neq u_{i+1}$, consider first the case $\langle u_i, v_{i+1} \rangle < 0$ (Fig. 5(b)).

For $\varepsilon > 0, J^\varepsilon$ lies in the open half-plane $\{x \in \mathbb{R}^2 : \langle u_{i+1}, x \rangle > \langle u_{i+1}, p_i \rangle\}$, whereas I_i lies in the closed half-plane $\{x \in \mathbb{R}^2 : \langle u_{i+1}, x \rangle \leq \langle u_{i+1}, p_i \rangle\}$. Therefore, $J^\varepsilon \cap I_i = \emptyset$.

For $\varepsilon < 0$, we have $\langle u_i, p_i + \varepsilon u_{i,i+1} \rangle = \langle u_i, p_i \rangle + \varepsilon(1 + \langle u_i, u_{i+1} \rangle) < \langle u_i, p_i \rangle$.

On the other hand, $\langle u_i, p_{i+1}\rangle < \langle u_i, p_i\rangle$ (since $\langle u_i, v_{i+1}\rangle < 0$), and therefore $\langle u_i, p_{i+1} + \varepsilon u_{i+1,i+2}\rangle < \langle u_i, p_i\rangle$ for sufficiently small $|\varepsilon|$. Thus both endpoints of J^ε lie on the same open side of the line affI_i, hence $J^\varepsilon \cap I_i = \emptyset$.

In the case $\langle u_i, v_{i+1}\rangle > 0$ (Fig. 5(c)), just repeat the previous argument with the roles of $\varepsilon > 0$ and $\varepsilon < 0$ interchanged.

As for I_{i+2}: Since the roles of I_i and I_{i+2} are interchangeable, the statement proved above for I_i applies to I_{i+2} as well. ■

Definition 3.1. Let p be a point in $\mathbb{R}^2 \setminus P$ ($=$ ext$P \cup$int$P$), and I be an edge of P. We say that p *sees* I if, for some point $a \in$ relint I, $[p, a] \cap P = \{a\}$.

Lemma 3.3. *Assume $p \in \mathbb{R}^2 \setminus P$. Then p sees at least one edge of P.*

Proof: Assume, w.l.o.g., that $p \in$ ext$P$. Let q be a point in int$P$. Let U be a neighborhood of q that lies entirely in int$P$. Choose a point $q' \in U$ such that the line aff(p, q') does not meet any vertex of P. (This condition can be met by avoiding a finite number of lines through p.) Then the line segment $[p, q']$ must meet P. Let a be the first point of P on $[p, q']$ (starting from p). Then a is a relative interior point of some edge I of P_i, and $[p, a] \cap P = \{a\}$. ■

Definition 3.2. (poldiam(•)): For a set $S \subset \mathbb{R}^2$ and points $a, b \in S$, denote by $\pi_S(a, b)$ the smallest number of edges of a polygonal path that connects a to b within S ($\pi_S(a, b) =_{\text{def}} \infty$ if no such polygonal path exists). If S is polygonally connected, then $\pi_S(\cdot, \cdot)$ is an integer valued metric on S. The *polygonal diameter* of S is defined as poldiam$(S) =_{\text{def}} \sup\{\pi_S(a, b) : a, b \in S\}$.

To prove (F) in Section 1 above, it suffices to show that poldiam(int$P$)$< \infty$ and poldiam(ext$P$)$< \infty$. The following theorem does it.

Theorem 3.1. (straightforward upper bound on poldiam(int$P$) and poldiam(ext$P$)) *If P is a simple closed n-gon ($n \geq 3$) in $\mathbb{R}^2$, then we have that* poldiam(int$P$) *and* poldiam(ext$P$) *are both* $\leq \lfloor \frac{n}{2} \rceil + 3$.

Proof: Assume that a, b are two points in the same component (int$P$ or ext$P$) of $\mathbb{R}^2 \setminus P$. By Lemma 3.2, $a\,[b]$ sees at least one edge $I'\,[I'']$ of P via

$\mathbb{R}^2 \setminus P$ (possibly $I' = I''$). The set $P \setminus (\text{relint}(I' \cup I''))$ consists of at most two simple polygonal paths P', P'', the shorter one of which, say P', concatenated by I', I'' in both of its endpoints is of the form $\langle J_0, J_1, \ldots, J_m, J_{m+1} \rangle$, where $m \leq \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$, $J_0, J_1 \ldots, J_{m+1}$ are edges of P ($\{J_0, J_{m+1}\} = \{I', I''\}$), J_{i-1} and J_i share a vertex q_i for $i = 1, 2, \ldots, m+1$, a sees via $\mathbb{R}^2 \setminus P$ a point $a' \in \text{relint} J_0$, and b sees via $\mathbb{R}^2 \setminus P$ a point $b' \in \text{relint} J_{m+1}$.

Thus $\langle a, a', q_1, q_2, \ldots, q_m, q_{m+1}, b', b \rangle$ is a polygonal path of $m + 4 \leq \lfloor \frac{n}{2} \rfloor - 1 + 4 = \lfloor \frac{n}{2} \rfloor + 3$ edges that connects a to b and runs along P except for $[a, a']$ and $[b', b]$. By Lemma 3.2, this path can be pushed away from P into $\mathbb{R}^2 \setminus P$, thus producing a polygonal path of $m + 4 \leq \lfloor \frac{n}{2} \rfloor + 3$ edges that connects a to b via $\mathbb{R}^2 \setminus P$. ■

4 Tight upper bounds on poldiam(int$\boldsymbol{P}$) and on poldiam(ext$\boldsymbol{P}$)

Theorem 3.1 gives a upper bound on poldiam(int$P$) [poldiam(ext$P$)] which is somewhat "naive", but sufficient to prove (F) in Section 1 above. Here we "squeeze" the proof of Theorem 3.1 to obtain a tight result.

Theorem 4.1. (Main Theorem) *Let P be a simple closed n-gon in $\mathbb{R}^2, n \geq 3$. Then*

(a) the polygonal diameter of int$P$ *is* $\leq \lfloor \frac{n}{2} \rfloor$*, and the polygonal diameter of* ext$P$ *is* $\leq \lceil \frac{n}{2} \rceil$*;*

(b) for every $n \geq 3$, there is an n-gon P_n for which both *bounds are attained.*

Proof of Theorem 4.1(a): First note that if P is a convex polygon, then poldiam(int$P$) $= 1 \leq \lfloor \frac{n}{2} \rfloor$, and it can be easily checked that poldiam(ext$P$) $= 2 \leq \lceil \frac{n}{2} \rceil$. (If we consider the closures, however, we find that poldiam(cl int$P$) $= 1$, whereas poldiam(cl ext$P$) $= 3$ if P has parallel edges, and equals 2 otherwise.) This settles the case $n = 3$ (P_3 is just a triangle). If $n = 4$ and P is not convex, then ext$P$ is the union of three convex sets (two open half-planes and a wedge), each two having a point in common, and therefore poldiam (ext$P$) $= 2 = \lceil \frac{n}{2} \rceil$. This settles the case $n = 4$ for ext$P$.

In view of the proof of Theorem 3.1 and the foregoing discussion, we can establish the bounds on poldiam(int$P$) and poldiam(ext$P$) as claimed in Theorem 4.1(a) by showing the following:

Theorem 4.2. *Let P be a closed simple n-gon in $\mathbb{R}^2$.*

(i) If $n \geq 4$ and $a, b \in \mathrm{int}P$, then there are two vertices a', b' of P such that a sees a' via int$P$, b sees b' via $\mathrm{int}P$, and a', b' are at most $\lfloor \frac{n}{2} \rfloor - 2$ edges apart on P. (Recall that "a sees a' via $\mathrm{int}P$" means just: $]a, a'[\subset \mathrm{int}P$.)

(ii) If $n \geq 5$ and $a, b \in \mathrm{ext}P$, then there are two vertices a', b' of P such that a sees a' via $\mathrm{ext}P$, b sees b' via $\mathrm{ext}P$, and a', b' are at most $\lceil \frac{n}{2} \rceil - 2$ edges apart on P,
or: $\pi_{\mathrm{extP}}(a, b) \leq 3$ $\left(\leq \lceil \frac{n}{2} \rceil \text{ for } n \geq 5\right)$.

Remark 4.1. The condition $n \geq 5$ in the first part of Theorem 4 (ii) cannot be relaxed to $n \geq 4$: Let $P_4 = \langle p_0, p_1, p_2, p_3 \rangle$ be a convex quadrilateral, and let $a, b \in \mathrm{ext}P_4$, a close to $[p_0, p_1]$ and b close to $[p_2, p_3]$. Then a and b do not see a common vertex of P_4.

Lemma 4.1. *Let P be a simple closed polygon in $\mathbb{R}^2$. Let $[b', p]$ be an edge of P, a, b two points such that $a \in \mathbb{R}^2 \setminus P$, $b \in]b', p]$ $(=[b', p] \setminus \{b'\})$ and a sees b (via $\mathbb{R}^2 \setminus P$). Then a sees (via $\mathbb{R}^2 \setminus P$) a vertex of P included in $[a, b', b] \setminus [a, b]$.*

Proof: If a sees b' then we are done. Otherwise the polygon $P \setminus]b', p[$ meets the set $[a, b, b'] \setminus [b', b]$. For $0 \leq \lambda \leq 1$, define $b(\lambda) =_{\mathrm{def}} (1 - \lambda)b + \lambda b'$, and let λ_0 be the smallest value of λ, $0 \leq \lambda \leq 1$, such that $[a, b(\lambda)] \cap (P \setminus]b', p[) \neq \emptyset$ $(0 < \lambda_0 \leq 1; \lambda_0 = 1$ is possible). Let c' be the point of $[a, b(\lambda_0)] \cap P$ nearest to a. Then c' is a vertex of P, $c' \in [a, b, b'] \setminus [a, b]$ and a sees c'. ■

Corollary 4.1. *Let P be a simple closed n-gon, $n \geq 3$, in $\mathbb{R}^2$. Every point $a \in \mathbb{R}^2 \setminus P$ sees via $\mathbb{R}^2 \setminus P$ at least two vertices of P.*

Proof: Let R be a ray emanating from a that meets P. By a slight rotation of R around a we may assume that R does not meet any vertex of P, but still $R \cap P \neq \emptyset$. Let b be the first point of R that belongs to P (starting from

a). By assumption $b \in [b', b''[$ for some edge $[b', b'']$ of P. By Lemma 4.1, a sees via $\mathbb{R}^2 \setminus P$ a vertex c' $[c'']$ of P included in $[a, b, b'] \setminus [a, b]$ [included in $[a, b, b''] \setminus [a, b]$], and clearly $c' \neq c''$. ■

Lemma 4.2. *Let P be a simple closed n-gon, $n \geq 4$, in $\mathbb{R}^2$, and let $a \in \mathbb{R}^2 \setminus P$. If every ray emanating from a meets P, then a sees via $\mathbb{R}^2 \setminus P$ two* non-adjacent *vertices of P.*

Remark 4.2. The condition that every ray emanating from a meets P is met by every point $a \in \text{int}P$.

Proof: By Corollary 4.1, a sees a vertex c of P via $\mathbb{R}^2 \setminus P$. Consider the ray $R =_{\text{def}} \{a + \lambda(a - c) : \lambda \geq 0\}$ that emanates from a in a direction *opposite* to c. By our assumption, R meets P. Let b be the first point of R that belongs to P. If b is a vertex of P, then a sees the two vertices b, c via $\mathbb{R}^2 \setminus P$. These vertices are *not adjacent*, since $[c, b] \cap P = \{c, b\}$. Otherwise, if b is not a vertex of P, then b is a relative interior point of an edge $[b', b'']$ of P ($R \cap]b', b''[= \{b\}$). By Lemma 4.1, a sees via $\mathbb{R}^2 \setminus P$ a vertex c' $[c'']$ of P included in $[a, b, b'] \setminus [a, b]$ [included in $[a, b, b''] \setminus [a, b]$]. Clearly, $c' \neq c''$ and c', c'' are non-adjacent in P unless $c' = b'$ and $c'' = b''$. In this case a sees via $\mathbb{R}^2 \setminus P$ both couples of vertices $\{c, b'\}$ and $\{c, b''\}$. At least one of these couples is *non-adjacent* in P, otherwise P would be a triangle, contrary to the assumption that $n \geq 4$. ■

Proof of Theorem 4.2:

(i) Suppose P is a simple closed n-gon, $n \geq 4$, in $\mathbb{R}^2$. Define $S =_{\text{def}} \text{int}P$, and assume $a, b \in S$. If $n = 4, 5$, then cl$S$ ($=P \cup \text{int}P$) is starshaped with respect to a vertex of P. (If $n = 5$, then S can be triangulated by two interior diagonals with a common vertex.) In this case a and b see via S a common vertex a' of P. Define $b' =_{\text{def}} a'$; we find that a', b' are at zero edges apart on P. But $0 \leq 0 = \lfloor \frac{n}{2} \rfloor - 2$ for $n = 4, 5$.

Assume, therefore, that $n \geq 6$, and that a and b do not see a common vertex of P via S. By Lemma 4.2, a sees via S two non-adjacent vertices a', a'' of P. These vertices divide P into two paths P_1, P_2, each having $\leq n - 2$ edges. Applying Lemma 4.2 again, we find that b sees via S two non-adjacent vertices b', b'' of P and $\{a', a''\} \cap \{b', b''\} = \emptyset$.

If both b' and b'' are interior vertices of the same path, say P_1, then they divide P_1 into three parts. The middle part has at least two edges, and the two extreme parts together have at most $n-4$ edges. The shorter extreme part, with endpoints (say) a', b', has at most $\lfloor \frac{n-4}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 2$ edges.

If, however, b' is an interior vertex of P_1 and b'' is an interior vertex of P_2, then they divide P_1 and P_2 into four polygonal paths, each one of which having one endpoint b' or b''. The shortest of these paths has at most $\lfloor \frac{n}{4} \rfloor$ edges. But $\lfloor \frac{n}{4} \rfloor \le \lfloor \frac{n}{2} \rfloor - 2$ for $n \ge 6$.

(ii) Assume $n \ge 5$, define $T = \text{ext}P$, and let $a, b \in T$. Then either

(A1) every ray emanating from a meets P, *or*

(A2) some ray emanating from a misses P.

Similarly, either

(B1) every ray emanating from b meets P, *or*

(B2) some ray emanating from b misses P.

If (A1) and (B1) hold, then both a and b see via T two non-adjacent vertices of P (Lemma 4.2). If $n \ge 6$, this implies that $a[b]$ sees a vertex a' $[b']$ of P such that a', b' are at most $\lfloor \frac{n-4}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 2 \le \lceil \frac{n}{2} \rceil - 2$ or $\lfloor \frac{n}{4} \rfloor \le \lfloor \frac{n}{2} \rfloor - 2 \le \lceil \frac{n}{2} \rceil - 2$ edges apart on P, as in the proof of part (i) above. If $n = 5$, then a sees via T a vertex a' of P, and b sees via T a vertex b' of P, where a' and b' are either equal or adjacent, i.e., a', b' are at most one edge apart on P. But for $n = 5$ one has $1 \le \lceil \frac{n}{2} \rceil - 2$.

If (A2) and (B2) hold, then, due to the compactness of P, we can find rays $R_a = \{a + \lambda u : \lambda \ge 0\}$ and $R_b = \{b + \lambda v : \lambda \ge 0\}$ that miss P, where the direction vectors u and v are *linearly independent*. When λ is sufficiently large, the segment $[a + \lambda u, b + \lambda u]$ misses P. Therefore $\pi_T(a, b) \le 3 \left(\le \lceil \frac{n}{2} \rceil \text{ for } n \ge 5\right)$ if $R_a \cap R_b = \emptyset$, and $\pi_T(a, b) = 2 < 3 \left(\le \lceil \frac{n}{2} \rceil \text{ for } n \ge 5\right)$ if $R_a \cap R_b \ne \emptyset$.

If (A1) and (B2) hold, then a sees via T two non-adjacent vertices a', a'' of P, which divide P into two paths P_1, P_2 (with disjoint relative interiors) each one of which having $\le n-2$ edges. The point b, however, sees two distinct vertices b', b'' of P, which may be adjacent (Corollary 4.1). If $\{a', a''\} \cap \{b', b''\} \ne \emptyset$, then again $\pi_T(a, b) \le 2 < 3 \left(\le \lceil \frac{n}{2} \rceil\right.$ for

$n \geq 5$). If $\{a', a''\} \cap \{b', b''\} = \emptyset$, then b' and b'' are interior vertices of P_1 or P_2, or both. If b' and b'' belong to different paths, then (as in the proof of part (i) above) they divide P_1 and P_2 into four polygonal paths, each having one endpoint b' or b''. The shortest one of these paths has at most $\lfloor \frac{n}{4} \rfloor$ edges. But $\lfloor \frac{n}{4} \rfloor \leq \lceil \frac{n}{2} \rceil - 2$ for $n \geq 5$. If both b' and b'' are interior vertices of the same path, say P_1, then (as in the proof of part (i) above) they divide P_1 into three parts. The two extreme parts together have at most $n - 2 - 1 = n - 3$ edges. The shortest extreme part with endpoints (say) a', b' has at most $\lfloor \frac{n-3}{2} \rfloor$ edges. But $\lfloor \frac{n-3}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor - 1 = \lceil \frac{n}{2} \rceil - 2$ for all $n \in \mathbb{N}$.

The same applies when (A2) and (B1) hold. This finishes the proof of Theorem 4.2. ■

By this also the proof of Theorem 4.1(a) is finished. ■

Proof of Theorem 4.1(b):

We split our examples into two cases, namely even n and odd n, $n \geq 3$.

Example 4.1. $n = 2m$ (even), $m \geq 2$. Figure 6 shows the example for the case $m = 3\,(n = 6)$.

Fig. 6: $m = 3\,(n = 6)$

Here we have $\pi_{\mathrm{int}P}(a, b) = m\,(= 3) = \lfloor \frac{n}{2} \rfloor$ and $\pi_{\mathrm{ext}P}(c, d) = m\,(= 3) = \lceil \frac{n}{2} \rceil$. One can extend the figure inward beyond vertex #4.

Example 4.2. $n = 2m + 1$ (odd), $m \geq 1$. Figure 7 shows the example for the case $m = 3$ ($n = 7$)

Fig. 7: $m = 3\,(n = 7)$

We have $\pi_{\mathrm{int}P}(a, b) = m\,(= 3) = \lfloor \frac{n}{2} \rfloor$ and $\pi_{\mathrm{ext}P}(c, d) = m + 1\,(= 4) = \lceil \frac{n}{2} \rceil$. Again, one can extend the figure inward beyond vertex #4.

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