

NEW EXAMPLES OF COMPACT MANIFOLDS WITH HOLONOMY $\text{Spin}(7)$

ROBERT CLANCY

ABSTRACT. We find new examples of compact $\text{Spin}(7)$ -manifolds using a construction of Joyce [18, 19]. The essential ingredient in Joyce's construction is a Calabi–Yau 4-orbifold with particular singularities admitting an antiholomorphic involution, which fixes the singularities. We search the class of well-formed quasismooth hypersurfaces in weighted projective spaces for suitable Calabi–Yau 4-orbifolds. We find that different hypersurfaces within the same family of Calabi–Yau 4-orbifolds may result in different $\text{Spin}(7)$ -manifolds.

1. INTRODUCTION

The holonomy group of a connected Riemannian manifold is the group of parallel transport maps around piecewise smooth loops based at a point. Berger [4] classified the possible holonomy groups of irreducible, nonsymmetric Riemannian metrics on simply-connected manifolds in 1955.

Theorem 1.1 (Berger). *Suppose M is a simply-connected manifold and g is a Riemannian metric on M , which is irreducible and nonsymmetric. Then one of the following cases holds:*

- (i) $\text{Hol}(g) = \text{SO}(n)$,
- (ii) $n = 2m$ with $m \geq 2$, and $\text{Hol}(g) = \text{U}(m)$ in $\text{SO}(2m)$,
- (iii) $n = 2m$ with $m \geq 2$, and $\text{Hol}(g) = \text{SU}(m)$ in $\text{SO}(2m)$,
- (iv) $n = 4m$ with $m \geq 2$, and $\text{Hol}(g) = \text{Sp}(m)$ in $\text{SO}(4m)$,
- (v) $n = 4m$ with $m \geq 2$, and $\text{Hol}(g) = \text{Sp}(m)\text{Sp}(1)$ in $\text{SO}(4m)$,
- (vi) $n = 7$ and $\text{Hol}(g) = G_2$ in $\text{SO}(7)$, or
- (vii) $n = 8$ and $\text{Hol}(g) = \text{Spin}(7)$ in $\text{SO}(8)$.

The question of whether there existed manifolds with holonomy group G_2 or $\text{Spin}(7)$ would not be resolved for more than 30 years. Bryant [5] in 1987 used the theory of exterior differential systems to show the existence of many metrics with holonomy G_2 and $\text{Spin}(7)$ on small balls in \mathbb{R}^7 and \mathbb{R}^8 , respectively. Then Bryant and Salamon [6] constructed examples of complete metrics with holonomy G_2 and $\text{Spin}(7)$ on non-compact manifolds, which were vector bundles over manifolds of dimensions 3 and 4. In 1994–5 Joyce [16, 17] constructed examples of compact manifolds with holonomy G_2 and $\text{Spin}(7)$ by resolving quotients of tori by finite groups.

Joyce [18] gives a second construction of manifolds with holonomy $\text{Spin}(7)$ whose basic ingredient is a Calabi–Yau 4-orbifold with an antiholomorphic involution. The exact conditions on the Calabi–Yau 4-orbifold are stated in Condition 3.4. In this thesis we will find all examples of suitable Calabi–Yau 4-orbifolds arising as well-formed quasismooth hypersurfaces in weighted

projective spaces. We will then find the Betti numbers of the Spin(7)-manifolds, which result from the construction given in [18].

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2. REVIEW OF Spin(7) GEOMETRY

The material of this section is entirely from [19]. We will recall the basic definitions and properties of Riemannian holonomy groups and then discuss the group Spin(7). In this section M will denote a connected manifold.

Definition 2.1. Let E be a vector bundle over M , and ∇^E a connection on E . Let $p \in M$ be a point. We say γ is a *loop based at p* if $\gamma : [0, 1] \rightarrow M$ is a piecewise-smooth curve with $\gamma(0) = \gamma(1) = p$. If γ is a loop based at p , then the parallel transport map $P_\gamma : E_p \rightarrow E_p$ is an invertible linear map. Define the *holonomy group* $\text{Hol}_p(\nabla^E)$ of ∇^E based at p to be

$$\text{Hol}_p(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } p\} \subset \text{GL}(E_p).$$

Since M is connected, $\text{Hol}_p(\nabla^E)$ and $\text{Hol}_q(\nabla^E)$ are conjugate as subgroups of $\text{GL}(k, \mathbb{R})$, if k is the rank of E and we have chosen identifications $E_p \simeq \mathbb{R}^k \simeq E_q$. We write $\text{Hol}(\nabla^E)$ to mean this conjugacy class of subgroups of $\text{GL}(k, \mathbb{R})$. The following proposition is a very useful property of holonomy groups.

Proposition 2.2. *Let M be a manifold, E a vector bundle over M , and ∇^E a connection on E . Let $p \in M$ be a point. Then the parallel sections of E are in one-to-one correspondence with the fixed points of the action of $\text{Hol}_p(\nabla^E)$ on E_p .*

If (M, g) is a Riemannian manifold we define the holonomy group of g , $\text{Hol}(g)$, to be the holonomy group of the Levi-Civita connection of (M, g) . Note that the holonomy group of a Riemannian manifold comes equipped with a representation on the fibres of the tangent bundle. Therefore when we say that a manifold has holonomy Spin(7) we must also say what representation of Spin(7) we are considering.

Spin(7) can be defined as the simply-connected double cover of $SO(7)$. We however will define it as the stabiliser group of a certain 4-form on \mathbb{R}^8 , which will determine an embedding of Spin(7) in $\text{GL}(8, \mathbb{R})$ and hence the irreducible 8-dimensional representation, to which Berger's theorem refers.

Definition 2.3. Let \mathbb{R}^8 have coordinates (x_1, \dots, x_8) . Let dx_{ijkl} denote the 4-form $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. We define the *Cayley form*, Ω_0 , by

$$\begin{aligned} \Omega_0 = & dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ & + dx_{5678} + dx_{3478} + dx_{3456} + dx_{2468} - dx_{2457} - dx_{2367} - dx_{2358} \end{aligned}$$

Spin(7) is the subgroup of $\text{GL}(8, \mathbb{R})$ preserving Ω_0 .

The 4-form above can be motivated by the structure of the octonions. The relationship between the octonions and the Cayley form can be found in, for example, [14, Section IV.1.C]. It should be noted that the Cayley form given above differs from that in [14] by an orientation-preserving permutation of the coordinates and an overall change in sign.

Since we have defined Spin(7) as the stabilizer group of the Cayley form by Proposition 2.2, if (M, g) is an oriented Riemannian 8-manifold with $\text{Hol}(g) \subseteq \text{Spin}(7)$, then M admits a (not necessarily unique) parallel 4-form Ω such that for any $p \in M$ there exists an oriented isometry $T_p M \rightarrow \mathbb{R}^8$, which takes Ω_p to Ω_0 .

We will define a Spin(7)-manifold to include a choice of Cayley form. This fixes a particular embedding of $\text{Hol}(g) \subseteq \text{Spin}(7)$.

Definition 2.4. A Spin(7)-manifold is a triple (M, Ω, g) where (M, g) is an oriented Riemannian 8-manifold, $\text{Hol}(g) \subseteq \text{Spin}(7)$ and Ω is a parallel 4-form such that for any $p \in M$ there exists an oriented isometry $T_p M \rightarrow \mathbb{R}^8$, which takes Ω_p to Ω_0 .

We can break up the condition of being a Spin(7)-manifold into a topological one, namely the existence of a reduction of the structure group of TM to Spin(7), and an integrability condition on this reduction.

Definition 2.5. Let M be an oriented manifold 8-manifold. A Spin(7)-structure on M is a pair (Ω, g) where g is a Riemannian metric and for any $p \in M$ there exists an oriented isometry $T_p M \rightarrow \mathbb{R}^8$, which takes Ω_p to Ω_0 .

A Spin(7)-structure is equivalent to a reduction of the structure group of TM to Spin(7). The existence of a Spin(7)-structure is a topological property of M as the following result from [21, Th. 10.7] shows.

Proposition 2.6. *Let M be an oriented 8-manifold. M admits a Spin(7)-structure if and only if $w_2(M) = 0$ and*

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0.$$

For M to be a Spin(7)-manifold it must satisfy an extra integrability condition as the following proposition shows from [19, Prop. 10.5.3].

Proposition 2.7. *Let M be an oriented 8-manifold with Spin(7)-structure (Ω, g) . Then $\text{Hol}(g) \subseteq \text{Spin}(7)$ and Ω is the induced 4-form if and only if $d\Omega = 0$. In this case we say the Spin(7)-structure (Ω, g) is torsion-free.*

The construction of Joyce uses a Calabi–Yau 4-orbifold to construct a Spin(7)-manifold. Any Calabi–Yau 4-fold carries an S^1 of torsion-free Spin(7)-structures as we will soon see. We will now define SU(4) as the stabiliser group of a set of tensors and show that SU(4) embeds into Spin(7).

SU(4) can be defined as the stabiliser of a metric, a Kähler form ω_0 and holomorphic volume form θ_0 . If we let (z_1, z_2, z_3, z_4) be coordinates on \mathbb{C}^4 we can write ω_0, θ_0 as

$$\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_4 \wedge d\bar{z}_4) \text{ and } \theta_0 = dz_1 \wedge \cdots \wedge dz_4.$$

We define a Calabi–Yau 4-fold as a quadruple (X, g, ω, θ) consisting of a Kähler manifold (X, g, ω) and a holomorphic $(4, 0)$ -form θ such that $|\theta| \equiv 4$. It can be shown that for any $p \in X$ there exists an isometry $T_p X \rightarrow \mathbb{C}^4$ taking (ω, θ) to (ω_0, θ_0) .

Proposition 2.8. *Let (X, g, ω, θ) be a Calabi–Yau 4-fold. Define a 4-form by $\Omega = \frac{1}{2}\omega \wedge \omega + \text{Re } \theta$, then (Ω, g) is a torsion-free Spin(7)-structure on X .*

Proof. Let $p \in X$ and identify ω_p and θ_p with the standard forms on \mathbb{C}^4 . Identifying \mathbb{C}^4 with \mathbb{R}^8 via $z_j = x_{2j-1} + ix_{2j}$ and comparing the expressions for Ω_p and Ω_0 we see that (Ω, g) defines a Spin(7)-structure. Since (X, g, ω, θ) is a Calabi-Yau manifold we have $d\omega = d\theta = 0$, which implies $d\Omega = 0$. \square

The proposition above describes a particular embedding of $SU(4) \hookrightarrow \text{Spin}(7)$. Let (X, g, ω, θ) be a Calabi-Yau 4-fold. Then the 4-form $\Omega_\phi = \frac{1}{2}\omega \wedge \omega + \text{Re}(e^{i\phi}\theta)$ for $\phi \in [0, 2\pi)$ also defines a torsion-free Spin(7)-structure on X and a different embedding of $SU(4) \hookrightarrow \text{Spin}(7)$.

3. CONSTRUCTION OF Spin(7)-MANIFOLDS

The essential idea in Joyce's constructions of manifolds with exceptional holonomy is that of resolving the singularities of orbifolds within a particular holonomy group. We will therefore review the definitions of orbifolds and discuss Riemannian metrics and their holonomy groups on orbifolds. We will then give a short overview of the construction of manifolds with holonomy Spin(7) from Calabi-Yau 4-orbifolds. We direct the reader to [18] and [19, Ch. 10] for the details of the construction.

3.1. Orbifolds.

Definition 3.1. An *orbifold* is a singular manifold M of dimension n whose singularities are locally isomorphic to quotient singularities \mathbb{R}^n/G for finite subgroups $G \subset \text{GL}(n, \mathbb{R})$, such that if $1 \neq \gamma \in G$, then the subspace V_γ of \mathbb{R}^n fixed by γ has $\dim V_\gamma \leq n - 2$.

We say a point p in M is an *orbifold point with orbifold group* G if M is locally isomorphic to \mathbb{R}^n/G at p with G non-trivial.

Definition 3.2. A *Riemannian metric* g on an orbifold M is a Riemannian metric in the usual sense on the nonsingular part of M and where M is locally isomorphic to \mathbb{R}^n/G , the metric g can be identified with the quotient of a G -invariant Riemannian metric defined on an open set of 0 in \mathbb{R}^n . We define the *holonomy group* $\text{Hol}(g)$ of g to be the holonomy group of the restriction of g to the nonsingular part of M .

If p is an orbifold point of M with orbifold group G then we have an inclusion of groups

$$G \subseteq \text{Hol}(g).$$

Therefore for an orbifold to have holonomy Spin(7) we must have that each orbifold group G lies in (the conjugacy class of subgroups of) Spin(7).

Many results for manifolds carry over with small modifications to orbifolds. In particular the Calabi conjecture holds for compact Kähler orbifolds. As a consequence we have the following theorem [19, Th. 6.5.6], which we can use to find Calabi-Yau metrics on orbifolds.

Theorem 3.3. *Let X be a compact complex orbifold with $c_1(X) = 0$ admitting Kähler metrics. Then there is a unique Ricci-flat Kähler metric in every Kähler class on X .*

3.2. Resolution of singularities. Given a Riemannian orbifold (M, g) with holonomy $\text{Hol}(g)$ we would like to know whether we can find a resolution (\hat{M}, \hat{g}) of M such that $\text{Hol}(\hat{g}) \subseteq \text{Hol}(g)$.

If $\text{Hol}(g) \subset \text{SU}(n)$ and then we can use complex geometry to determine whether resolutions exist which do not change the holonomy group. Suppose X is a Gorenstein algebraic variety so the canonical sheaf $\mathcal{O}(K_X)$ is invertible. A resolution $\pi : \hat{X} \rightarrow X$ is called *crepant* if $\pi^*(\mathcal{O}(K_X)) = \mathcal{O}(K_{\hat{X}})$. The importance of crepant resolutions is that a crepant resolution of a Calabi–Yau orbifold is a Calabi–Yau manifold. The question of existence and uniqueness of crepant resolutions of quotient singularities \mathbb{C}^n/G for finite groups $G \subset \text{SU}(n)$ is a difficult one, to which we will return later.

If we wish to construct manifolds with special holonomy then we will not, in general, be able to use algebraic techniques and instead we must rely on finding singularities of a type, which we know we can resolve within a given holonomy group.

3.3. Review of construction. Now suppose Y is a complex 4-orbifold admitting metrics with holonomy $\text{SU}(4)$. We require Y to have isolated singularities $\{p_1, \dots, p_k\}$ modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator of \mathbb{Z}_4 acts as

$$(1) \quad \alpha : (z_1, z_2, z_3, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4).$$

The group \mathbb{Z}_4 lies in $\text{SU}(4)$, which is consistent with Y having holonomy $\text{SU}(4)$. We also require that Y admits an antiholomorphic isometric involution τ with fixed points the finite set $\{p_1, \dots, p_k\}$.

Since τ is antiholomorphic, the complex structure does not descend to the quotient of Y by τ . However we can form a torsion-free $\text{Spin}(7)$ -structure (Ω, g) given by $\Omega = \frac{1}{2}\omega \wedge \omega + \text{Re } \theta$, which is τ -invariant.

Defining $Z = Y/\langle \tau \rangle$ we have that this τ -invariant torsion-free $\text{Spin}(7)$ -structure on Y descends to Z and the orbifold singularities of Z are modelled on \mathbb{R}^8/G where G is a finite subgroup of $\text{Spin}(7)$. We now wish to resolve these singularities by glueing in ALE $\text{Spin}(7)$ -manifold to construct a $\text{Spin}(7)$ -manifold M .

It is shown in [18, Prop. 5.3] that all the singularities are of the same form. If we define coordinates on \mathbb{R}^8 as (x_1, \dots, x_8) and complex coordinates by $z_i = x_{2i-1} + ix_{2i}$ then the singularities are of the form \mathbb{R}^8/G where $G = \langle \alpha, \beta \rangle$ is a finite non-abelian subgroup of $\text{Spin}(7)$ generated by α , which is described in Eq. (1), and β , whose action on \mathbb{C}^4 is given by

$$\beta : (z_1, z_2, z_3, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3).$$

The singularity \mathbb{R}^8/G can be resolved with holonomy contained in $\text{Spin}(7)$ in two ways. Both resolutions X_1, X_2 have the same holonomy as abstract Lie groups but the embeddings into $\text{GL}(8, \mathbb{R})$ (or $\text{Spin}(7)$) are different. If we choose the resolutions of the singularities of Z wisely we can ensure that the holonomy of the resolution of the orbifold Z has holonomy exactly $\text{Spin}(7)$ and not a proper subgroup of it.

The precise necessary conditions on the complex 4-orbifold, Y , are stated below.

Condition 3.4. Let Y be a compact complex 4-orbifold with $c_1(Y) = 0$, admitting Kähler metrics. Let τ be an antiholomorphic involution on Y . We require that Y have isolated singularities $\{p_1, \dots, p_k\}$, with $k \geq 1$, modelled on $\mathbb{C}^4/\mathbb{Z}_4$ as described above and that the fixed point set of τ is $\{p_1, \dots, p_k\}$. We also require that $Y \setminus \{p_1, \dots, p_k\}$ is simply-connected and $h^{2,0}(Y) = 0$.

We will use the following theorem of Joyce to construct examples of Spin(7)-manifolds from appropriate complex 4-orbifolds, which can be found in [18, Th. 5.14].

Theorem 3.5. *Suppose Y satisfies Condition 3.4. Let M be the resulting compact 8-manifold defined in [18, Def. 5.8]. Then there exist torsion-free Spin(7)-structures (Ω, g) on M . We can choose the resolutions of the singularities so that $\text{Hol}(g) = \text{Spin}(7)$.*

4. WEIGHTED PROJECTIVE SPACES AND HYPERSURFACES

Our goal is now to find complex 4-orbifolds which satisfy Condition 3.4. We will use algebraic geometry to find examples of such orbifolds. Hypersurfaces in weighted projective spaces provide a large source of orbifolds with specified (cyclic quotient) singularities. We will therefore begin by reviewing weighted projective spaces, their singularities and hypersurfaces contained in weighted projective spaces. The majority of this section is from [15].

Definition 4.1. Let a_0, \dots, a_n be positive integers with $\text{gcd}(a_0, \dots, a_n) = 1$. The *weighted projective space* $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ with weights a_0, \dots, a_n is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of \mathbb{C}^* given by

$$\lambda : (z_0, \dots, z_n) \mapsto (\lambda^{a_0} z_0, \dots, \lambda^{a_n} z_n).$$

In general $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ will have singularities where the action of \mathbb{C}^* on $\mathbb{C}^{n+1} \setminus \{0\}$ is not free. The stabiliser groups of these points are finite and so we can treat $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as an orbifold. We can also treat $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as a singular algebraic variety by considering $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as Proj of a graded ring. This will be a useful viewpoint for us because it shows the similarities between weighted projective spaces and the usual straight projective space.

Let R be the graded ring $\mathbb{C}[z_0, \dots, z_n]$ where z_i has weight a_i . R has a direct sum decomposition $R = \bigoplus_d R_d$ into its graded pieces. Elements of R_d will be called *weighted homogeneous polynomials of degree d* but we will soon drop the term weighted and leave it as understood.

We can treat $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as a variety as Proj(R). From generalities on taking Proj of graded rings, see [13, Prop. 5.11], we have that a finitely generated graded R -module determines a coherent sheaf of $\mathcal{O}_{\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n}$ -modules. In particular the module $R(m)$ determines a sheaf, which we will denote $\mathcal{O}(m)$ for brevity.

We should be careful when distinguishing between the two viewpoints. For example we have the following result from [15, Cor. 5.9], which holds only when considering $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as a variety.

Lemma 4.2. *Let a_0, \dots, a_n be positive integers with $\text{gcd}(a_0, \dots, a_n) = 1$. Let $q = \text{gcd}(a_1, \dots, a_n)$. Then $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n \simeq \mathbb{C}\mathbb{P}_{a_0, a_1/q, \dots, a_n/q}^n$ as varieties.*

We have the following corollary.

Corollary 4.3. *Let a_0, \dots, a_n be positive integers with $\gcd(a_0, \dots, a_n) = 1$. Then $\mathbb{CP}_{a_0, \dots, a_n}^n \simeq \mathbb{CP}_{b_0, \dots, b_n}^n$ as varieties for some weights b_0, \dots, b_n such that $\gcd(b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_n) = 1$ for each i .*

This motivates the following definition.

Definition 4.4. We say $\mathbb{CP}_{a_0, \dots, a_n}^n$ is *well-formed* if

$$\gcd(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = 1 \text{ for each } i.$$

The condition of being well-formed is related to the structure of the singularities of $\mathbb{CP}_{a_0, \dots, a_n}^n$. The singularities of $\mathbb{CP}_{a_0, \dots, a_n}^n$ are all cyclic quotient singularities. We say a cyclic quotient singularity $\mathbb{C}^n/\mathbb{Z}_m$ is of type $\frac{1}{m}(a_1, \dots, a_n)$ if \mathbb{Z}_m acts on \mathbb{C}^n as

$$(z_1, \dots, z_n) \xrightarrow{\xi} (\xi^{a_1} z_1, \dots, \xi^{a_n} z_n)$$

where $\xi^m = 1$.

For any subset $I \subset \{0, \dots, n\}$ we define $S_I = \{[z_0, \dots, z_n] : z_j = 0; \forall j \notin I\} \subset \mathbb{CP}_{a_0, \dots, a_n}^n$. Now suppose $\gcd(a_{i_0}, \dots, a_{i_k}) = m \neq 1$, then a generic point $p \in S_{i_0, \dots, i_k}$ is an orbifold point modelled on the singularity $\mathbb{C}^k \times \mathbb{C}^{n-k}/\mathbb{Z}_m$. If we extend the sequence (i_0, \dots, i_k) to be a permutation (i_0, \dots, i_n) of the sequence $(0, \dots, n)$ then the singularity $\mathbb{C}^{n-k}/\mathbb{Z}_m$ is of type $\frac{1}{m}(a_{i_{k+1}}, \dots, a_{i_n})$.

Example 4.5. Consider the weighted projective space $\mathbb{CP}_{1,2,3,6}^3$. The singular locus consists of the union of the two curves $S_{1,3} \cup S_{2,3}$, which intersect at the singular point S_3 . The singularity at a generic point of $S_{1,3}$ is modelled on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, where \mathbb{Z}_2 acts as $(z_0, z_2) \mapsto (-z_0, -z_2)$. The singularity at a generic point of $S_{1,3}$ is modelled on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$, where \mathbb{Z}_3 acts as $(z_0, z_1) \mapsto (\xi z_0, \xi^{-1} z_1)$ and $\xi^3 = 1$. Finally we have a nonisolated singular point at S_3 , which is modelled on $\mathbb{C}^3/\mathbb{Z}_6$, where \mathbb{Z}_6 acts as $(z_0, z_1, z_2) \mapsto (\zeta z_0, \zeta^2 z_1, \zeta^3 z_2)$ and $\zeta^6 = 1$.

From the description of singularities above we see that the condition of being well-formed is equivalent to $\mathbb{CP}_{a_0, \dots, a_n}^n$ having only singularities in complex codimension greater than 1.

4.1. Hypersurfaces. A section $f \in \Gamma(\mathbb{CP}_{a_0, \dots, a_n}^n, \mathcal{O}(d)) = R_d$ determines a hypersurface in $\mathbb{CP}_{a_0, \dots, a_n}^n$, by definition of degree d . It can be shown that any hypersurface is determined by such a section [7, Th. 3.7].

4.1.1. Quasismoothness. Recall that a projective variety is smooth if the affine cone is smooth away from the origin. We shall define quasismoothness in a similar way. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \mapsto \mathbb{CP}_{a_0, \dots, a_n}^n$ be the projection.

Definition 4.6. Let $Y \subset \mathbb{CP}_{a_0, \dots, a_n}^n$ be an algebraic variety. We say Y is *quasismooth* if $\pi^{-1}(Y)$ is smooth.

An algebraic variety Y is quasismooth if Y only has singularities coming from the orbifold singularities of $\mathbb{CP}_{a_0, \dots, a_n}^n$. We will restrict our attention to quasismooth hypersurfaces because we can understand their singularities easily in terms of those of the ambient weighted projective space. Regarding $\mathbb{CP}_{a_0, \dots, a_n}^n$ as an orbifold, quasismoothness of Y is equivalent to being a sub-orbifold, see [2, Prop. 3.5].

Note that a generic hypersurface of a fixed degree in a particular weighted projective space is not necessarily quasismooth.

Example 4.7. Consider the graded ring $R = \mathbb{C}[z_0, z_1, z_2]$ where the weights are 1, 2, 2 respectively, then $\text{Proj}(R) = \mathbb{CP}_{1,2,2}^2$. A generic weighted homogeneous polynomial of degree 3 is of the form $f = \lambda_1 z_0 z_1 + \lambda_2 z_0 z_2 + \lambda_3 z_0^3$. The hypersurface $Y_3 = V(f)$ is not quasismooth since the affine variety defined by f is singular along the set $\{(z_0, z_1, z_2) \in \mathbb{C}^3 : z_0 = 0 \text{ and } \lambda_1 z_1 + \lambda_2 z_2 = 0\}$.

The conditions for the generic hypersurface of degree d to be quasismooth are described below, taken from [15, Th. 8.1].

Theorem 4.8. *The generic hypersurface Y_d of degree d in $\mathbb{CP}_{a_0, \dots, a_n}^n$ is quasismooth if and only if either $a_i = d$ for some i , i.e. Y_d is a linear cone, or for every nonempty subset $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$ either*

- (i) *there exists a monomial $z_{i_0}^{d_0} \dots z_{i_k}^{d_k}$ of degree d ; or*
- (ii) *for $j = 0, \dots, k$ there exist monomials $z_{i_0}^{d_{0,j}} \dots z_{i_k}^{d_{k,j}}$ of degree d , where the $e_j \notin \{i_0, \dots, i_k\}$ are distinct.*

4.1.2. *Canonical Sheaf of a Hypersurface.* Recall that if $Y_d \subset \mathbb{CP}^n$ is a smooth hypersurface of degree d , i.e. defined by a homogeneous polynomial of degree d , then the adjunction formula gives us that $K_{Y_d} = \mathcal{O}(d - n - 1)|_{Y_d}$. We would like a similar result for weighted projective spaces so that we could test the triviality of the canonical sheaf easily. Fortunately we have such a result for a large class of hypersurfaces, namely those which are quasismooth and well-formed.

Definition 4.9. Let $Y \subset \mathbb{CP}_{a_0, \dots, a_n}^n$ be a hypersurface. We say Y is *well-formed* if $\mathbb{CP}_{a_0, \dots, a_n}^n$ is well-formed and Y does not contain a codimension 2 singular set of $\mathbb{CP}_{a_0, \dots, a_n}^n$.

We have the following criterion for well-formedness for generic hypersurfaces from [15, Prop. 6.10].

Proposition 4.10. *The generic hypersurface of degree d in $\mathbb{CP}_{a_0, \dots, a_n}^n$ is well-formed if and only if*

- (i) $\gcd(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n) \mid d$ for all i, j and
- (ii) $\gcd(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = 1$ for all i .

Proposition 4.11. *Let $Y_d \subset \mathbb{CP}_{a_0, \dots, a_n}^n$ be a well-formed quasismooth hypersurface of degree d . Then the canonical sheaf is $K_{Y_d} = \mathcal{O}(d - n - 1)|_{Y_d}$.*

Hence for $n > 1$ we have that the canonical bundle of a degree d well-formed quasismooth hypersurface is trivial if $d = n + 1$.

5. ANTIHOLOMORPHIC INVOLUTIONS

Let $Y \subset \mathbb{CP}_{a_0, \dots, a_5}^5$ be a well-formed quasismooth hypersurface with trivial canonical bundle. We now wish to consider antiholomorphic involutions on Y . We will consider only antiholomorphic involutions which arise as restrictions of antiholomorphic involutions on $\mathbb{CP}_{a_0, \dots, a_5}^5$. The main result of this section is the classification of antiholomorphic involutions of weighted projective spaces, Proposition 5.3.

It is shown in [23] that for standard projective space $\mathbb{C}\mathbb{P}^n$ the number of conjugacy classes of antiholomorphic involutions is either 1 or 2 depending on whether n is odd or even respectively. If n is odd, then the only antiholomorphic involution of $\mathbb{C}\mathbb{P}^n$ up to conjugacy is the standard one:

$$[z_0, \dots, z_n] \mapsto [\bar{z}_0, \dots, \bar{z}_n].$$

If n is even we also have the involution

$$[z_0, \dots, z_n] \mapsto [\bar{z}_1, -\bar{z}_0, \dots, \bar{z}_n, -\bar{z}_{n-1}].$$

We will consider antiholomorphic involutions up to conjugation by automorphisms of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$. Therefore we should first describe $\text{Aut}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$.

5.1. Automorphisms of weighted projective spaces. Consider the action of \mathbb{C}^* on \mathbb{C}^{n+1} defining a weighted projective space with weights a_0, \dots, a_n . We want to decompose \mathbb{C}^{n+1} by the action of \mathbb{C}^* . We relabel the collection of weights $w_1 < \dots < w_m$ and let k_i be the number of times w_i appears in the sequence a_0, \dots, a_n . We decompose $\mathbb{C}^{n+1} = \bigoplus_{i \in I} W_i$ where \mathbb{C}^* acts on W_i with weight w_i , $\dim(W_i) = k_i$ and $I = \{1, \dots, m\}$.

Any automorphism of $\mathbb{C}^{n+1} \setminus \{0\}$ extends to an automorphism of \mathbb{C}^{n+1} for $n > 0$ by Hartogs' Theorem. Let $\text{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ denote the \mathbb{C}^* -equivariant automorphisms of \mathbb{C}^{n+1} . We can also describe $\text{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ as the centralizer of \mathbb{C}^* in $\text{Aut}(\mathbb{C}^{n+1})$. Any \mathbb{C}^* -equivariant morphism of \mathbb{C}^{n+1} descends to an automorphism of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ and the converse can be shown to hold.

A \mathbb{C}^* -equivariant morphism $F : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is determined by a collection of polynomials $(F_{i,j})$ where $i \in I$, $1 \leq j \leq k_i$ and $F_{i,j}$ is of degree w_i . Each polynomial $F_{i,j}$ can be decomposed

$$F_{i,j} = A_{i,j} + f_{i,j}$$

into a linear part and a non-linear, i.e. weighted homogenous quadratic and higher, part.

Example 5.1. Consider the graded ring $R = \mathbb{C}[z_0, z_1, z_2]$ where z_0, z_1, z_2 have weights 1, 1, 2 respectively. \mathbb{C}^3 splits as $\mathbb{C}^3 = W_1 \oplus W_2$ as representations of \mathbb{C}^* where \mathbb{C}^* acts on W_1 with weight 1 and on W_2 with weight 2.

A \mathbb{C}^* -equivariant morphism $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ is determined by a collection $F_{1,1}, F_{1,2}, F_2$ where $F_{1,1}, F_{1,2}$ are linear functions of z_0, z_1 and F_2 is a sum of a linear multiple of z_2 and a homogeneous quadratic polynomial in z_0, z_1 .

For each $i \in I$ we define A_i to be the matrix formed from the rows $(A_{i,j})_{1 \leq j \leq k_i}$. The morphism F is invertible on W_1 if A_1 is invertible since there are no polynomials with degree less than w_1 . An inductive argument gives us that F is invertible if and only if each A_i is.

The map that sends a morphism F to the corresponding collection of linear maps A_i is a surjective homomorphism

$$\text{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \longrightarrow \prod_{i \in I} \text{GL}(W_i).$$

The kernel of this homomorphism is the set of morphisms of the form $F_{i,j} = z_{i,j} + f_{i,j}$ where $(z_{i,j})_{1 \leq j \leq k_i}$ are coordinates on W_i . Let us denote this

kernel by H . We have an inclusion of groups $\prod_{i \in I} \mathrm{GL}(W_i) \hookrightarrow \mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ and hence the short exact sequence

$$0 \longrightarrow H \longrightarrow \mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \longrightarrow \prod_{i \in I} \mathrm{GL}(W_i) \longrightarrow 0$$

is right split and we have $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \simeq H \rtimes \prod_{i \in I} \mathrm{GL}(W_i)$.

Each element of $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ determines an automorphism of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$, but in order to determine an automorphism uniquely we must take a quotient by the diagonal action of \mathbb{C}^* on $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$. More explicitly, consider the homomorphism

$$\begin{aligned} \mathbb{C}^* &\hookrightarrow \mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_m) \\ \lambda &\mapsto (\lambda^{w_1}, \lambda^{w_2}, \dots, \lambda^{w_m}) \end{aligned}$$

which is an embedding since $\mathrm{gcd}(w_1, \dots, w_m) = 1$. Then the quotient of $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ by this subgroup is isomorphic to $\mathrm{Aut}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$.

The linear structure on polynomials gives a linear structure to the group H . The group $\prod_{i \in I} \mathrm{GL}(W_i)$ acts on H via the adjoint action, which we will denote $\mathrm{Ad}_A : H \rightarrow H$ for $A \in \prod_{i \in I} \mathrm{GL}(W_i)$. With respect to this linear structure the adjoint action of $\prod_{i \in I} \mathrm{GL}(W_i)$ on H is linear. H decomposes as a vector space as $H = \bigoplus_{i \in I} H_i$ where H_i consists of the morphisms in H with $f_{i',j} = 0$ for $i' \neq i$.

We can describe some of the group structure on H using the order on I given by $i < i'$ if $w_i < w_{i'}$. For $f, g \in H$, let i be such that $f_{i',j} = 0 \forall i' < i$, then $(gf)_{i,j} = g_{i,j} + f_{i,j}$. In particular $(f^{-1})_{i,j} = -f_{i,j}$.

5.2. Classification of Antiholomorphic Involutions. Any invertible antiholomorphic map can be written as the composition of the standard antiholomorphic involution coming from complex conjugation on \mathbb{C}^{n+1} , which we will denote by c , followed by an automorphism of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ so let us write $\widetilde{\mathrm{Aut}}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$ for $\mathrm{Aut}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n) \rtimes \mathbb{Z}_2$ where \mathbb{Z}_2 is generated by c .

By the discussion above we can write any antiholomorphic involution of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ as a composition (f, A, c) where $f \in H$ and $A \in \prod_{i \in I} \mathrm{GL}(W_i)$. Up to conjugation by elements of $\mathrm{Aut}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$ we can ignore H due to the following lemma.

Lemma 5.2. *Let $\tau = (f, A, c) \in \widetilde{\mathrm{Aut}}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$ be an antiholomorphic involution. Then τ is conjugate to $(0, A, c)$.*

Proof. Since τ is an involution we have $f \mathrm{Ad}_A(\bar{f}) = 1$. Let $f = f_1 + \cdots + f_m$ be the direct sum decomposition of f and let i be minimal such that $f_i \neq 0$. Since i is minimal and the action of Ad_A on H fixes each H_j we have $(f \mathrm{Ad}_A \bar{f})_i = f_i + \mathrm{Ad}_A(\bar{f}_i) = 0$.

Let $h \in H$ be such that $h_i = -\frac{1}{2}f_i$ and $h_j = 0$ for $j < i$. Then $(h\tau h^{-1})_j = 0$ for $j < i$ and

$$\begin{aligned} (h\tau h^{-1})_i &= h_i + f_i - \text{Ad}_A(\bar{h}_i) \\ &= -\frac{1}{2}f_i + f_i + \frac{1}{2}\text{Ad}_A(\bar{f}_i) \\ &= \frac{1}{2}(f_i + \text{Ad}_A(\bar{f}_i)) \\ &= 0, \end{aligned}$$

where we have used the linearity of the action of $\prod_{i \in I} \text{GL}(W_i)$ on H and the discussion of the group structure of H . Now by induction on i we find that $(0, A, c)$ is in the conjugacy class of τ . \square

Proposition 5.3. *The number of conjugacy classes of antiholomorphic involutions of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ is either 1 or 2. Let k_j, w_j be defined in terms of a_0, \dots, a_n as above. Then $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ admits a non-standard antiholomorphic involution if and only if $w_i k_i$ is even for each i .*

Proof. Let $\tau \in \widetilde{\text{Aut}}(\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n)$ be an antiholomorphic involution. By Lemma 5.2 we have that τ is conjugate to $A \circ c$ so we will assume $\tau = A \circ c$. Now $A \in \prod_{i \in I} \text{GL}(W_i)$ and for τ to be an involution we must have

$$(2) \quad A_j \bar{A}_j = \lambda^{w_j}$$

for some $\lambda \in \mathbb{C}^*$. Taking trace shows that λ^{w_j} is real for each j so λ is real. By an action of the diagonal \mathbb{C}^* we can ensure that $|\lambda| = 1$. Taking determinants of Eq. (2) then implies that $\lambda^{w_j k_j} = 1$.

We know that there are exactly two antiholomorphic involutions of $\mathbb{C}\mathbb{P}^{k_j}$ for k_j even and one for k_j odd up to conjugation and scale. For the standard antiholomorphic involution we have $A_j \bar{A}_j = 1$ and for the non-standard involution we have $A_j \bar{A}_j = -1$.

For A_j to be non-standard we must have $\lambda^{w_j} = -1$ so w_j must be odd and k_j must be even. In this case $\lambda = -1$ and since $(-1)^{w_i k_i} = 1$ we must have $w_i k_i$ even for each i . \square

The standard antiholomorphic involution has a fixed point locus of (real) dimension n so the fixed point locus of the involution restricted to a hypersurface will never consist of isolated points. We therefore must consider only weighted projective spaces which admit non-standard involutions.

Let τ be a non-standard antiholomorphic involution and let w_j, k_j be defined as before. The fixed point locus of $\tau \circ c$ is of (complex) dimension

$$\sum_{j: w_j \in 2\mathbb{Z}} k_j - 1.$$

Since we want τ to have isolated fixed points when acting on a generic hypersurface we therefore require that $\sum_{j: w_j \in 2\mathbb{Z}} k_j = 2$.

For the case we are interested in, namely $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^5$ the discussion above imposes conditions on the allowed sets of weights. In order for $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^5$ to admit an antiholomorphic involution whose fixed locus has (real) dimension

1 we must have, without loss of generality, $a_0 = a_1$ and $a_2 = a_3$, all of which are odd, and a_4, a_5 both even. The action of τ on $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ can be given as

$$(3) \quad \tau : [z_0, z_1, z_2, z_3, z_4, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

From now on we will assume τ is of this form.

6. SINGULARITIES

Recall that we require the Calabi–Yau 4-orbifold to have singularities of the type $\frac{1}{4}(1, 1, 1, 1)$, which are fixed by τ . The fixed point locus of τ is contained in $S_{4,5}$ so we should find hypersurfaces with singularities of the correct type in $S_{4,5}$.

Suppose the weights a_0, \dots, a_n have been chosen so that a generic hypersurface of degree $d = \sum_i a_i$ is well-formed and quasismooth. The isolated singularities of type $\frac{1}{4}(1, 1, 1, 1)$ can occur in two ways, either Y_d transversely intersects the singular locus $S_{4,5}$ at a generic point and $\gcd(a_4, a_5) = 4$ or Y_d contains a point S_4 or S_5 with $a_4 = 4$ or $a_5 = 4$ respectively.

For Y_d to intersect a generic point of $S_{4,5}$ transversely we must have that there exist at least two monomials $z_4^{d_4} z_5^{d_5}$ of degree d . These singularities are of the type $\frac{1}{4}(1, 1, 1, 1)$ if $a_k \equiv 1 \pmod{4}$ for $k \neq 4, 5$.

Y_d contains the point S_4 if $a_4 \nmid d$ so that there does not exist a monomial $z_4^{d_4}$ of degree d . For Y_d to be quasismooth we must have that $a_4 \mid d - a_j$ for some j so there exists a monomial $z_4^{d_4} z_j$ of degree d . As before in order for this singularity to be of the type $\frac{1}{4}(1, 1, 1, 1)$ we must have $a_k \equiv 1 \pmod{4}$ for $k \neq 4, j$, however a_4 and a_5 are both even so therefore we must have $j = 5$.

If $\gcd(a_4, a_5) = 2$ then for Y_d to be quasismooth we must have that either Y_d intersects $S_{4,5}$ transversely at generic points with singularities modelled on $\mathbb{C}^4/\mathbb{Z}_2$ or we have a monomial $z_4 z_5$ of degree d . We eliminate the first possibility because the singularity of type $\frac{1}{2}(1, 1, 1, 1)$ does not admit a crepant resolution and we eliminate the second because $d = \sum_i a_i$. Hence $\gcd(a_4, a_5) = 4$ and Y_d does not contain S_4 or S_5 . We summarize our results in the following proposition.

Proposition 6.1. *Suppose the generic hypersurface of degree $d = \sum_i a_i$ in $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ has isolated singularities of type $\frac{1}{4}(1, 1, 1, 1)$ and $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ admits an antiholomorphic involution, which fixes only these points in Y_d , then without loss of generality the weights a_0, \dots, a_5 satisfy*

- (i) $a_0 = a_1$ and $a_2 = a_3$, and
- (ii) $\gcd(a_4, a_5) = 4$, and
- (iii) $a_i \equiv 1 \pmod{4}$ for $0 \leq i \leq 3$, and
- (iv) $a_4 \mid d$ and $a_5 \mid d$.

6.1. Resolving Undesired Singularities. Y_d may have other singularities, which we first need to resolve. We will use methods from [8] to determine whether a given cyclic quotient singularity of dimension 4 admits a crepant resolution.

We will assume that the reader is familiar with the basic definitions of toric geometry [10]. Consider a cyclic quotient singularity of the type $\frac{1}{m}(a_1, a_2, a_3, a_4)$. We can describe this as an affine toric variety. Let $N =$

$\mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{m}(a_1, a_2, a_3, a_4)$ be a lattice and $\sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ the cone spanned by the unit vectors $e_1 = (1, 0, 0, 0), \dots, e_4 = (0, 0, 0, 1)$. The affine toric variety associated to the cone σ is isomorphic to the cyclic quotient singularity of type $\frac{1}{m}(a_1, a_2, a_3, a_4)$.

The set of elements of age i , σ_i , is defined to be the convex hull in N of the elements $\{ie_1, ie_2, ie_4, ie_4\} \in N$. The following theorem, from [8, Th. 6.1], gives a necessary condition for the cyclic quotient singularity to admit a crepant resolution.

Theorem 6.2. *Let \mathbb{C}^n/G be a quotient singularity, where $G \subset \mathrm{SL}(n, \mathbb{C})$ is a finite abelian group. If \mathbb{C}^n/G admits a crepant resolution, then the set of elements of age 1, σ_1 , is a minimal generating set for σ over \mathbb{Z} .*

Theorem 6.2 gives us a necessary condition for a given cyclic quotient singularity to admit a crepant resolution. This condition is sufficient for all singularities of codimension 4 where the cyclic group has order less than 39 and is sufficient in all but 10 cases for quotient singularities of codimension 4 with cyclic group of order less than 100 [8].

Example 6.3. Consider the isolated cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1, 1)$. The elements of age 1 are e_1, \dots, e_4 . The element $\frac{1}{2}(1, 1, 1, 1) \in \sigma$ cannot be written as a sum of e_1, \dots, e_4 with integer coefficients. Therefore the elements of age 1 do not form a generating set for σ over \mathbb{Z} and hence the singularity of type $\frac{1}{2}(1, 1, 1, 1)$ does not admit a crepant resolution.

Example 6.4. The generic hypersurface, Y_{84} , of degree 84 in $\mathbb{C}\mathbb{P}_{1,1,21,21,12,28}^5$ is a well-formed quasismooth Calabi–Yau hypersurface. The singularities of Y_{84} consist of the curves $Y_{84} \cap S_{2,3,4}$ and $Y_{84} \cap S_{2,3,5}$, which intersect in the 4 points $\{p_1, \dots, p_4\} = Y_{84} \cap S_{2,3}$. The singularities of Y_{84} at each of the p_i are of type $\frac{1}{21}(1, 1, 7, 12)$.

Let $N = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{21}(1, 1, 7, 12)$ and $\sigma \subset N_{\mathbb{Q}}$ the cone spanned by $e_1 = (1, 0, 0, 0), \dots, e_4 = (0, 0, 0, 1)$. The elements of age 1, which are listed in Table 1, are a minimal generating set for σ and hence the singularity $\mathbb{C}^4/\mathbb{Z}_{21}$ admits a crepant resolution.

$$\begin{array}{cccc} (1, 0, 0, 0), & (0, 1, 0, 0), & (0, 0, 1, 0), & (0, 0, 0, 1), \\ \frac{1}{21}(1, 1, 7, 12), & \frac{1}{21}(2, 2, 14, 3), & \frac{1}{21}(3, 3, 0, 15), & \frac{1}{21}(4, 4, 7, 6), \\ \frac{1}{21}(6, 6, 0, 9), & \frac{1}{21}(7, 7, 7, 0), & \frac{1}{21}(9, 9, 0, 3), & \end{array}$$

TABLE 1. Elements of age 1 in the cone σ , which defines the cyclic quotient singularity of type $\frac{1}{21}(1, 1, 7, 12)$.

We are now in a position to determine whether a particular weighted projective space contains a suitable Calabi–Yau 4-orbifold as a well-formed quasismooth hypersurface.

Proposition 6.5. *The weights a_0, \dots, a_5 such that*

- (i) *The generic hypersurface of degree $d = \sum_i a_i$ in $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ is well-formed and quasismooth;*

- (ii) Y_d has isolated singularities of the type $\frac{1}{4}(1, 1, 1, 1)$;
- (iii) $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ admits an antiholomorphic involution whose fixed point locus intersects Y_d at the isolated singularities of type $\frac{1}{4}(1, 1, 1, 1)$;
- (iv) Any other singularities of Y_d admit crepant resolutions;

are listed in Table 2.

Proof. Lynker et al. [22] determined the complete set of weighted projective spaces of dimension 5 such that the generic hypersurface of degree $d = \sum_i a_i$ is quasismooth. The list of weights can be found at <http://thp.uni-bonn.de/Supplements/cy.html>

Propositions 4.10 and 6.1 translate conditions (i)–(iii) into numerical conditions on the weights a_0, \dots, a_5 . We use a computer programme to search the list of 1,100,055 sets of weights given by Lynker to get a list of 18 sets of weights, for which conditions (i)–(iii) apply.

Then we use Theorem 6.2 to test whether any undesired singularities of the generic hypersurface admit crepant resolutions. This test eliminates the weights which are not listed in Table 2. \square

| $\{a_0, \dots, a_5\}$ | | | | | | | | | | | |
|-----------------------|----|----|----|---|----|---|---|----|----|----|----|
| 1 | 1 | 1 | 1 | 4 | 4 | 1 | 1 | 1 | 1 | 4 | 8 |
| 1 | 1 | 1 | 1 | 8 | 12 | 1 | 1 | 5 | 5 | 8 | 20 |
| 1 | 1 | 9 | 9 | 4 | 4 | 5 | 5 | 13 | 13 | 4 | 4 |
| 1 | 1 | 13 | 13 | 4 | 8 | 1 | 1 | 21 | 21 | 4 | 16 |
| 5 | 5 | 25 | 25 | 4 | 16 | 1 | 1 | 21 | 21 | 12 | 28 |
| 1 | 1 | 37 | 37 | 8 | 28 | 1 | 1 | 53 | 53 | 20 | 32 |
| 21 | 21 | 49 | 49 | 4 | 24 | 1 | 1 | 69 | 69 | 16 | 52 |

TABLE 2. The admissible weights of the ambient weighted projective spaces of Calabi–Yau 4-orbifolds. The weighted projective spaces with weights listed in the first two rows appear as ambient spaces for Calabi–Yau 4-orbifolds in [17].

7. DETERMINING BETTI NUMBERS OF M

Let us now suppose that Y is a Calabi–Yau 4-orbifold contained in one of the weighted projective spaces we have determined in Proposition 6.5. In general Y will have singularities, which we first need to resolve. We will denote the resolution of Y by \hat{Y} and let us assume that τ lifts to \hat{Y} so that \hat{Y} satisfies Condition 3.4.

For the moment let us assume that we can determine the Hodge numbers of \hat{Y} . We can determine the Betti numbers of $Z = \hat{Y}/\langle\tau\rangle$ and the resulting Spin(7)-manifold M as follows.

Proposition 7.1. *Let \hat{Y} , Z , M be as above. Suppose Z has k singularities modelled on \mathbb{R}^8/G as in Section 3.3. Then the Betti numbers of M are*

$$\begin{aligned}
 b^2(M) &= b^2(Z), & b_+^4(M) &= \frac{1}{2}(h^{2,2}(\hat{Y}) + k) - b^2(Z) + 1, \\
 b^3(M) &= \frac{1}{2}b^3(\hat{Y}), & b_-^4(M) &= h^{3,1}(\hat{Y}) + b^2(\hat{Y}) - b^2(Z) + k - 1.
 \end{aligned}$$

Proof. Let $h_\tau^{p,p}(\hat{Y})$ be the dimension of the τ -invariant part of $H^{p,p}(\hat{Y})$. Noting that in all of the cases we will discuss we have $h^{2,0}(\hat{Y}) = h^{3,0}(\hat{Y}) = 0$ and $h^{4,0}(\hat{Y}) = 1$ since \hat{Y} has $\text{Hol}(\hat{Y}) = SU(4)$, the Betti numbers of Z can then be expressed as

$$\begin{aligned} b^2(Z) &= h_\tau^{1,1}(\hat{Y}), & b_+^4(Z) &= h_\tau^{2,2}(\hat{Y}) - h^{1,1}(\hat{Y}) + h_\tau^{1,1}(\hat{Y}) + 2, \\ b^3(Z) &= h^{2,1}(\hat{Y}), & b_-^4(Z) &= h^{3,1}(\hat{Y}) + h^{1,1}(\hat{Y}) - h_\tau^{1,1}(\hat{Y}) - 1. \end{aligned}$$

Applying the Lefschetz fixed point theorem we find that

$$k = 2 + 4h_\tau^{1,1}(\hat{Y}) - 2h^{1,1}(\hat{Y}) + 2h_\tau^{2,2}(\hat{Y}) - h^{2,2}(\hat{Y}),$$

which we use to eliminate $h_\tau^{2,2}(\hat{Y})$ from the expressions for the Betti numbers of Z .

The ALE Spin(7) manifolds that we use to resolve the quotient singularities of Z have Betti numbers $b^1 = b^2 = b^3 = b_+^4 = 0$ and $b_-^4 = 1$ hence the Betti numbers of M satisfy

$$\begin{aligned} b^j(M) &= b^j(Z) \text{ for } j = 1, 2, 3 \\ b_+^4(M) &= b_+^4(Z) \text{ and } b_-^4(M) = b_-^4(Z) + k. \end{aligned}$$

Combining these facts gives us the result. \square

From Proposition 7.1 we see that to determine the Betti numbers of M it suffices to know the Hodge numbers of the orbifold \hat{Y} and to understand how τ acts on $H^{1,1}(\hat{Y})$. We will use techniques from toric geometry both to determine the Hodge numbers and to understand the action of τ .

The rest of this section will almost entirely be material from [1], and we direct the reader to this paper for more details.

7.1. Lattice Polytopes. We will now change our viewpoint from weighted projective spaces to toric varieties associated to reflexive polytopes. With this change we will find the Hodge numbers of the resolved hypersurface \hat{Y} , show that the antiholomorphic involution τ lifts to \hat{Y} , and determine the dimension of $H_\tau^2(\hat{Y})$.

Batyrev and Cox [1, 3] have determined the Hodge numbers of the crepant resolutions of Calabi–Yau hypersurfaces in toric varieties associated to reflexive polytopes. We will give the definition of reflexive polytope and show how to associate a toric variety to such a polytope.

In this subsection N will denote a lattice and $M = \text{Hom}(N, \mathbb{Z})$ its dual. We will denote a fan by Σ and a rational strongly convex cone by σ . Let $\Delta \subset M$ be an n -dimensional lattice polytope, i.e. a polytope with vertices in M , and suppose Δ contains the origin. We associate a toric variety to Δ by taking cones over the maximal faces of Δ as described in the following proposition.

Proposition 7.2. *For every k -dimensional face $\Theta \subset \Delta$ let $\check{\sigma}(\Theta) \subset M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$ be the cone over Θ , $\check{\sigma}(\Theta) = \{\lambda x \in M_\mathbb{Q} : x \in \Theta \text{ and } \lambda \in \mathbb{Q}\}$, and let $\sigma(\Theta) \subset N_\mathbb{Q}$ be the $(n - k)$ -dimensional dual cone. Then the collection of dual cones $\Sigma(\Delta) = \{\sigma(\Theta) : \Theta \subset \Delta\}$ is a fan and hence determines a toric variety P_Δ .*

Example 7.3. We recall how weighted projective space can be constructed as a toric variety. Let a_0, \dots, a_n be positive integers with $\gcd(a_0, \dots, a_n) = 1$. Let \overline{N} be generated by e_0, \dots, e_n and let $N = \overline{N}/\mathbb{Z} \cdot (a_0e_0 + \dots + a_ne_n)$. N is a lattice since $\gcd(a_0, \dots, a_n) = 1$. $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ is the toric variety associated to the fan whose n dimensional cones are given by $\text{span}(e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n)$ for $i = 0, \dots, n$ in $N_{\mathbb{Q}}$.

Example 7.4. Let \overline{N} be generated by e_0, \dots, e_n and let \overline{M} be the dual lattice. Suppose $x \in \mathbb{Z}_{>0}e_0 + \dots + \mathbb{Z}_{>0}e_n$ is a primitive element in \overline{N} , i.e. $x = a_0e_0 + \dots + a_ne_n$ where $\gcd(a_0, \dots, a_n) = 1$ and $a_i > 0$ for each i . Then the set $\Delta = \{y \in \overline{M} : \langle y, x \rangle = 0 \text{ and } \langle y, e_i \rangle \geq -1\}$ is an n -dimensional lattice polytope in the lattice $M = \{y \in \overline{M} : \langle y, x \rangle = 0\}$.

The dual lattice $N = \text{Hom}(M, \mathbb{Z})$ can be identified with the quotient $\overline{N}/\mathbb{Z} \cdot (a_0e_0 + \dots + a_ne_n)$. If $\gcd(a_0, \dots, \widehat{a}_i, \dots, a_n) = 1$ for each i so that $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ is well-formed, then the fan $\Sigma(\Delta)$ is a refinement of the fan of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$ so the toric variety P_{Δ} is a partial resolution of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$.

The previous example shows that to any weighted projective space we can associate a lattice polytope, and after applying the toric construction to the polytope we get a toric variety, which is a partial resolution of the original weighted projective space.

A polytope Δ determines not only a toric variety P_{Δ} but also a choice of ample invertible sheaf $\mathcal{O}_{\Delta}(1)$ on P_{Δ} by [1, Prop. 2.1.5].

Definition 7.5. Let M be a lattice and $\Delta \subset M$ a lattice polytope of the same dimension as M containing the origin. We define the *dual polytope* $\Delta^* \subset N_{\mathbb{Q}}$ as the set

$$\Delta^* = \{x \in N_{\mathbb{Q}} : \langle y, x \rangle \geq -1 \text{ for all } y \in \Delta\}$$

We say a lattice polytope Δ is *reflexive* if Δ^* is also a lattice polytope, i.e. if the vertices of Δ^* lie in N and not just $N_{\mathbb{Q}}$.

The relevance of reflexive polytopes to Calabi–Yau orbifolds is described in the following result from [1, Th. 4.1.9].

Theorem 7.6. *Let Δ be an integral polytope and P_{Δ} the corresponding projective toric variety. The following conditions are equivalent.*

- (i) *the ample invertible sheaf $\mathcal{O}_{\Delta}(1)$ on P_{Δ} is anticanonical;*
- (ii) *Δ is reflexive.*

Now suppose Δ is reflexive. Then a generic section of the sheaf $\mathcal{O}_{\Delta}(1)$ determines a quasismooth Calabi–Yau hypersurface, by an application of the adjunction formula.

The lattice polytopes associated to weighted projective spaces as we have described above are not always reflexive. For example the weights 1, 1, 1, 1, 1, 2 do not determine a reflexive polytope. Fortunately the lattice polytopes associated to the weights described in Proposition 6.5 are all reflexive.

The Hodge numbers of the resolved hypersurface \hat{Y} are given in terms of combinatorial properties of the lattice polytope Δ defined by the weights a_0, \dots, a_5 . We will not give the formulae here but direct the reader to [1, 2].

7.2. τ -Equivariant resolutions of singularities. If $Y \subset P_\Delta$ is a Calabi–Yau hypersurface then a crepant resolution of the singularities of P_Δ will induce a crepant resolution of the singularities of Y . We will resolve the singularities of P_Δ and in the process desingularize all of the Calabi–Yau hypersurfaces.

A subdivision of the polytope Δ^* will determine a refinement of the fan of P_Δ and hence a partial resolution of P_Δ . A subdivision of Δ^* is called a (*maximal*) *triangulation* if every lattice point in Δ^* is the vertex of some simplex in the subdivision. A triangulation of Δ^* will determine a maximal partial crepant resolution of P_Δ and hence Y . A triangulation is called *projective* if the associated resolution is. The existence of projective triangulations is guaranteed by [11, Prop. 4] and the resulting resolution will, very importantly, be crepant by [1, Th. 2.2.24].

However we require that τ lifts to the resolution of P_Δ , which we will denote by \hat{P}_Δ . The antiholomorphic involution τ can be decomposed into three parts

$$\tau = t \cdot \tau_m \cdot c,$$

where c denotes the standard antiholomorphic involution, τ_m is an element of the torus in P_Δ , and t is a morphism of P_Δ induced by an involution of the lattice N , which fixes the polytope Δ^* (and Δ).

The involution τ will lift to the resolution if the triangulation of Δ^* is invariant under the action of t , by which we mean that t sends a simplex in the triangulation to another simplex in the triangulation. Unfortunately we cannot always find such an invariant triangulation as the following example shows.

Example 7.7. Let $e_0 = (0, 0, -2)$, $e_1 = (1, 1, 1)$, $e_2 = (1, -1, 1)$, $e_3 = (-1, -1, 1)$, and $e_4 = (-1, 1, 1)$ be points in \mathbb{R}^3 . Let N be the lattice generated by e_1, e_2, e_3 and let Δ^* denote the reflexive polytope given by the convex hull of the set $\{e_0, \dots, e_4\}$. The polytope Δ^* can be pictured as a cone over a square with vertices e_1, e_2, e_3, e_4 . Let t be the lattice isomorphism defined by $t(e_1) = e_2$, $t(e_2) = e_1$ and $t(e_3) = e_4$, which is reflection in a plane in \mathbb{R}^3 .

There are two triangulations of the polytope Δ^* . One contains the edge joining e_1 and e_3 and the other contains the edge joining e_2 and e_4 . Neither of these triangulations is invariant under the action of t and we see that the map induced by t on the toric variety does not lift to any resolution. It is interesting to note that the polytope is not simplicial and hence the associated toric variety is not an orbifold. We have not found an example of a simplicial reflexive polytope with an involution t , which does not admit a t -invariant projective triangulation.

Projective triangulations of the lattice polytope, Δ^* , are in one-to-one correspondence with the faces of an associated polytope, known as the secondary polytope [12, Ch. 7]. For Δ^* to admit a t -invariant triangulation, we require that t must fix a face of the secondary polytope, or equivalently the fixed point set of t intersects the interior of a face of the secondary polytope. The secondary polytope sits inside the vector space $A_{n-1}(\hat{P}_\Delta) \otimes \mathbb{Q}$, where \hat{P}_Δ is any maximal partial crepant resolution of P_Δ . Recall that $A_{n-1}(\hat{P}_\Delta)$

is determined by an exact sequence [10]

$$0 \longrightarrow M \longrightarrow \sum_{\rho \in \Delta^* \setminus \{0\}} \mathbb{Z} \cdot e_\rho \longrightarrow A_{n-1}(\hat{P}_\Delta) \longrightarrow 0,$$

where $m \in M \mapsto \sum_{\rho \in \Delta^* \setminus \{0\}} \langle m, \rho \rangle e_\rho$. The lattice isomorphism t acts on M and Δ^* and hence on $A_{n-1}(\hat{P}_\Delta)$. If we tensor this sequence with \mathbb{Q} we can check that for all of the cases we are interested in, t in fact fixes the whole secondary polytope. This means that τ will lift to any maximal crepant resolution of P_Δ .

7.3. Hodge Numbers of Y . The Hodge numbers of the resolved Calabi–Yau hypersurface, \hat{Y} , are given by formulae presented in [1, 3]. To understand how τ acts on $H^{1,1}(\hat{Y})$ we will describe in more detail how $h^{1,1}(\hat{Y})$ is calculated.

The basic idea is that $h^{1,1}(\hat{Y})$ counts the components of the intersection of the resolution with the union of all irreducible toric divisors in the desingularization of $\mathbb{C}\mathbb{P}_{a_0, \dots, a_n}^n$. If the resolution intersects a divisor with dimension > 0 , then it is irreducible, so the action of τ on this component can be determined by whether τ fixes the toric divisor or swaps it with another.

We denote the torus in \hat{P}_Δ by T and let X denote the intersection of \hat{Y} with the union of all irreducible T -invariant divisors in \hat{P}_Δ . We have a short exact sequence of cohomology groups

$$0 \longrightarrow H_c^6(\hat{Y}) \longrightarrow H_c^6(X) \longrightarrow H_c^7(\hat{Y} \setminus X) \longrightarrow 0,$$

which is natural under τ . By Poincaré duality we have that $b^2(\hat{Y}) = \dim(H_c^6(\hat{Y}))$ and hence $b_\tau^2(\hat{Y})$ is the difference of the dimension of the τ -invariant parts of $H_c^6(X)$ and $H_c^7(\hat{Y} \setminus X)$. Using a Lefschetz-type theorem [9] for affine hypersurfaces in algebraic tori we have that $H_c^7(\hat{Y} \setminus X) \simeq H_c^9(T)$ is an isomorphism. Given the description of τ as in Eq. (3), it is easy to check that the dimension of the τ -invariant part of $H_c^9(T)$ is 3.

The irreducible T -invariant divisors in \hat{P}_Δ are indexed by the set $\Delta^* \setminus \{0\}$. Let $\rho \in \Delta^* \setminus \{0\}$ and D_ρ the corresponding T -invariant divisor. If ρ is contained in the interior of a face of codimension ≥ 3 , then the intersection $D_\rho \cap \hat{Y}$ is irreducible, while if ρ is contained in the interior of a face of codimension 2, then D_ρ intersects \hat{Y} in isolated points, the number of which is determined by the polytope.

If $D_\rho \cap \hat{Y}$ is irreducible, then the action of τ is determined by whether τ fixes $\rho \in \Delta^* \setminus \{0\}$ or not. If τ fixes ρ , then ρ does not contribute to $b_\tau^2(\hat{Y})$ and a pair ρ_1, ρ_2 of T -invariant divisors, which are swapped by τ , contribute 1 to $b_\tau^2(\hat{Y})$.

If D_ρ intersects \hat{Y} in d points, then for a generic Y , τ will swap $d/2$ pair of points if d is even and $(d-1)/2$ pairs of points if d is odd. However we can choose Y so that τ swaps k pairs of points where $0 \leq k \leq d/2$, in which case this contributes k to $b_\tau^2(\hat{Y})$. In this way we see that we can get Spin(7)-manifolds with different topological invariants arising from the same family of Calabi–Yau 4-orbifolds.

We have used the software PALP [20] to find the toric divisors and to determine the toric divisors fixed by τ .

Example 7.8. Consider the reflexive polytope with weights $1, 1, 9, 9, 4, 4$ $\Delta \subset M$. Let $N = M^*$ and Δ^* the dual polytope of Δ . The points of $\Delta^* \setminus \{0\}$ correspond to toric divisors in P_Δ . There are exactly 11 of these, which are listed in Table 3 with respect to a particular basis of N . The antiholomorphic involution swaps the elements in the first column in pairs and leaves the other 7 invariant.

$$\begin{array}{lll} (1, 0, 0, 0, 0), & (0, 0, 0, 1, 0), & (0, -5, -5, -2, -2), \\ (0, 1, 0, 0, 0), & (0, 0, 0, 0, 1), & (0, -3, -3, -1, -1), \\ (0, 0, 1, 0, 0), & (0, -7, -7, -3, -3), & (0, -2, -2, -1, -1), \\ (-1, -9, -9, -4, -4), & (0, -1, -1, 0, 0), & \end{array}$$

TABLE 3. T -invariant divisors in a maximal partial crepant resolution of the reflexive polytope with weights $1, 1, 9, 9, 4, 4$.

There is one divisor contained in an interior of a face with codimension 2, which corresponds to the singularities of our Calabi–Yau hypersurface of type $\frac{1}{4}(1, 1, 1, 1)$. A generic hypersurface intersects the divisor in 7 points. We can choose the hypersurface so that τ swaps k pairs of points where $0 \leq k \leq 3$ and fixes the remaining singular points. After taking the quotient by τ and resolving the quotient singularities in the usual way, we can find Spin(7) manifolds with $0 \leq b^2(M) \leq 3$.

8. RESULTS AND FURTHER WORK

Table 4 lists the examples of Spin(7)-manifolds constructed from well-formed quasmooth hypersurfaces in weighted projective spaces. We include the examples already given in [19, Ch. 15] for the sake of completeness, which appear as the first four rows.

The sets of Betti numbers realised by manifolds constructed in this thesis are all distinct from those of compact manifolds with holonomy Spin(7) already known. Also it should be noted that the example with $b^4 = 15118$ and $b_4^- = 5031$ has the largest known value of b^4 or b_4^- for a compact manifold with holonomy Spin(7).

For a fixed weighted projective space if we consider the family of Calabi–Yau 4-folds, which are fixed by the antiholomorphic involution τ , it is interesting to note that the resulting Spin(7)-manifolds coming from resolutions of different Calabi–Yau 4-folds in the family all have $b_4^- + b^2 + 1$ constant, which is the dimension of the Conformal Field Theory moduli space [24].

The next step for looking for Calabi–Yau orbifolds satisfying Condition 3.4 would be to look for hypersurfaces in toric varieties coming from reflexive polytopes, which is invariant under an involution of the lattice. One could also look for orbifolds with more general types of singularities, which admit resolutions with holonomy Spin(7), for example the singularities in [18, Sec. 4.3]. The methods we have described for calculating the Betti numbers would immediately apply.

It should be noted that there is a notion of mirror symmetry for Calabi–Yau hypersurfaces in toric varieties determined by reflexive polytopes. The mirror polytope will also admit a non-standard antiholomorphic involution

| $\{a_0, \dots, a_5\}$ | | | | | | b^2 | b^3 | b_+^4 | b_-^4 |
|-----------------------|----|----|----|----|----|-------------------|-------|------------|------------|
| 1 | 1 | 1 | 1 | 4 | 4 | $0 \leq k \leq 1$ | 0 | $1639 - k$ | $807 - k$ |
| 1 | 1 | 1 | 1 | 4 | 8 | 0 | 0 | 3175 | 1575 |
| 1 | 1 | 1 | 1 | 8 | 12 | 0 | 0 | 7784 | 3879 |
| 1 | 1 | 5 | 5 | 8 | 20 | 0 | 6 | 2493 | 1237 |
| 1 | 1 | 9 | 9 | 4 | 4 | $0 \leq k \leq 3$ | 0 | $1415 - k$ | $695 - k$ |
| 5 | 5 | 13 | 13 | 4 | 4 | $0 \leq k \leq 5$ | 0 | $295 - k$ | $135 - k$ |
| 1 | 1 | 13 | 13 | 4 | 8 | $0 \leq k \leq 2$ | 0 | $983 - k$ | $1991 - k$ |
| 1 | 1 | 21 | 21 | 4 | 16 | $0 \leq k \leq 1$ | 0 | $3927 - k$ | $1951 - k$ |
| 5 | 5 | 25 | 25 | 4 | 16 | $0 \leq k \leq 2$ | 0 | $487 - k$ | $231 - k$ |
| 1 | 1 | 21 | 21 | 12 | 28 | $0 \leq k \leq 2$ | 0 | $2983 - k$ | $1479 - k$ |
| 1 | 1 | 37 | 37 | 8 | 28 | 0 | 0 | 5911 | 2943 |
| 1 | 1 | 53 | 53 | 20 | 32 | 0 | 0 | 6055 | 3015 |
| 21 | 21 | 49 | 49 | 4 | 24 | $0 \leq k \leq 9$ | 0 | $263 - k$ | $119 - k$ |
| 1 | 1 | 69 | 69 | 16 | 52 | 0 | 0 | 10087 | 5031 |

TABLE 4. The weights of the ambient weighted projective spaces of the Calabi–Yau 4-folds and Betti numbers of the resulting Spin(7)-manifolds.

but there will be a choice involved. Also the mirror Calabi–Yau will, in general, not have the type of singularities we described in Section 3.3.

REFERENCES

- [1] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Algebraic Geom.*, 3:493–535, 1994.
- [2] V. V. Batyrev and D. A. Cox. On the Hodge structure of projective hypersurfaces in toric varieties. *Duke Math. J.*, 75:293–338, 1994.
- [3] V. V. Batyrev and D. I. Dais. Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry. *Topology*, 35:901–929, 1996.
- [4] M. Berger. Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.
- [5] R. L. Bryant. Metrics with exceptional holonomy. *Ann. of Math.*, 126:525–576, 1987.
- [6] R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58:829–850, 1989.
- [7] D. A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.
- [8] D. I. Dais, M. Henk, and G. M. Ziegler. On the existence of crepant resolutions of Gorenstein abelian quotient singularities in dimensions ≥ 4 . In *Algebraic and geometric combinatorics*, volume 423 of *Contemp. Math.*, pages 125–193. Amer. Math. Soc., Providence, RI, 2006.
- [9] V. I. Danilov and A. G. Khovanskii. Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(5):925–945, 1986.
- [10] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [11] I. Gelfand, M. Kapranov, and A. Zelevinsky. Hypergeometric functions and toric varieties. *Funct. Anal. Appl.*, 23:94–106, 1989.
- [12] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

- [13] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
 - [14] R. Harvey and H. B. Lawson, Jr. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.
 - [15] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.
 - [16] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy G_2 . I, II. *J. Differential Geom.*, 43:291–328, 329–375, 1996.
 - [17] D. D. Joyce. Compact 8-manifolds with holonomy Spin(7). *Invent. Math.*, 123:507–552, 1996.
 - [18] D. D. Joyce. A new construction of compact 8-manifolds with holonomy Spin(7). *J. Differential Geom.*, 53:89–130, 1999.
 - [19] D. D. Joyce. *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
 - [20] M. Kreuzer and H. Skarke. PALP: a package for analysing lattice polytopes with applications to toric geometry. *Comput. Phys. Comm.*, 157:87–106, 2004.
 - [21] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
 - [22] M. Lynker, R. Schimmrigk, and A. Wißkirchen. Landau-Ginzburg vacua of string, M- and F-theory at $c = 12$. *Nuclear Phys. B*, 550:123–150, 1999.
 - [23] H. Partouche and B. Pioline. Rolling among G_2 vacua. *J. High Energy Phys.*, pages Paper 5, 25, 2001.
 - [24] S. L. Shatashvili and C. Vafa. Superstrings and manifolds of exceptional holonomy. *Selecta Math. (N.S.)*, 1:347–381, 1995.
- E-mail address:* `clancy@maths.ox.ac.uk`