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ON STABLY FREE MODULES OVER LAURENT POLYNOMIAL RINGS

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Abstract. We prove constructively that for any finite-dimensional commutative ring R and $n \geq \dim(R)+2$, the group $\mathrm{E}_n(R[X,X^{-1}])$ acts transitively on $\mathrm{Um}_n(R[X,X^{-1}])$. In particular, we obtain that for any finite-dimensional ring R, every finitely generated stably free module over $R[X,X^{-1}]$ of rank $> \dim R$ is free, i.e., $R[X,X^{-1}]$ is $(\dim R)$-Hermite.

1. Introduction

We denote by R a commutative ring with unity and $\mathbb{N}$ the set of non-negative integers. $\mathrm{Um}_n(R)$ is the set of unimodular rows of length n over R, that is all $(x_0,\ldots,x_{n-1}) \in R^n$ such that $x_0R+\cdots+x_{n-1}R = R$. If $u, v \in \mathrm{Um}_n(R)$ and G is a subgroup of $\mathrm{GL}_n(R)$, we write $u \sim_{\mathrm{G}} v$ if there exists g in G such that $v = ug$. Recall that $\mathrm{E}_n(R)$ denotes the subgroup of $\mathrm{GL}_n(R)$, generated by all $\mathrm{E}_{ij}(a) := I_n + ae_{ij}$ (where $i \neq j$, $a \in R$ and e_{ij} denotes the $n \times n$- matrix whose only non-zero entry is 1 on the (i,j)- th place). We abbreviate the notation $u \sim_{\mathrm{E}_n(R)} v$ to $u \sim_{\mathrm{E}} v$. We say that a ring R is Hermite (resp. d-Hermite) if any finitely generated stably free R-module (resp., any finitely generated stably free R-module of rank $> d$) is free.

In [6], A.A.Suslin proved:

Theorem 1.1. (A.A.Suslin)
If R is a Noetherian ring and

$$A = R[X_1^{\pm 1},\ldots,X_k^{\pm 1},X_{k+1},\ldots,X_n].$$

Then for $n \geq \max(3,\dim(R)+2)$ the group $\mathrm{E}_n(A)$ acts transitively on $\mathrm{Um}_n(A)$.

In particular, we obtain that $\mathrm{E}_n(R[X,X^{-1}])$ acts transitively on $\mathrm{Um}_n(R[X,X^{-1}])$ for any Noetherian ring R, where $n \geq \max(3,\dim(R)+2)$. In [7], I.Yengui proved:

Theorem 1.2. (I.Yengui)
Let R be a ring of dimension d, $n \geq d+1$, and let $f \in \mathrm{Um}_{n+1}(R[X])$. Then there exists $E \in \mathrm{E}_{n+1}(R[X])$ such that $f \cdot E = e_1$.

In this article we generalize by proving:

Theorem 1.3. *For any finite-dimensional ring R, $\mathrm{E}_n(R[X,X^{-1}])$ acts transitively on $\mathrm{Um}_n(R[X,X^{-1}])$, where $n \geq \dim(R)+2$.*

This gives a positive answer to Yengui's question (Question 9 of [7]). The proof we give is a close adaptation of Yengui's proof to the Laurent case.

Key words and phrases. Stably free modules, Hermite rings, Unimodular rows, Laurent polynomial rings, Constructive Mathematics.

2. Preliminary results on unimodular rows

A.A.Suslin proved in [6], that if $f = (f_0, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X])$, where f_1 is unitary and $n \geq 1$, then there exists $w \in \mathrm{SL}_2(R[X]) \cdot \mathrm{E}_{n+1}(R[X])$ such that $f \cdot w = e_1$. In fact, this theorem is a crucial point in his proof of Serre's conjecture. R.A.Rao generalized in [[4], Corollary 2.5] by proving:

Theorem 2.1. (R.A.Rao, [4])
Let $f = (f_0, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X])$, where $n \geq 2$. If some f_i is unitary, then f is completable to a matrix in $\mathrm{E}_n(R[X])$.

Recall that the boundary ideal of an element a of a ring R is the ideal $\mathcal{I}(a)$ of R generated by a and all $y \in R$ such that ay is nilpotent. Moreover, $\dim R \leq d \Leftrightarrow \dim(R/\mathcal{I}(a)) \leq d-1$ for all $a \in R$ [3].

Theorem 2.2. [[2], Theorem 2.4]
Let R be a ring of dimension $\leq d$ and $a = (a_0, \dots, a_n) \in \mathrm{Um}_{n+1}(R)$ where $n \geq d+1$, then there exist $b_1, \dots, b_n \in R$ such that

$$\langle a_1 + b_1 a_0, \dots, a_n + b_n a_0 \rangle = R$$

In fact, we can obtain a stronger result if $f \in \mathrm{Um}_{n+1}(R_S)$, where S is a multiplicative subset of R:

Proposition 2.3. *Let S be a multiplicative subset of R such that $S^{-1}R$ has dimension d. Let $(a_0, \dots, a_n) \in M_{n+1}(R)$ be a row such that $(\frac{a_0}{1}, \dots, \frac{a_n}{1}) \in \mathrm{Um}_{n+1}(S^{-1}R)$, where $n > d$. Then there exist $b_1, \dots, b_n \in R$ and $s \in S$ such that*

$$s \in (a_1 + b_1 a_0)R + \cdots + (a_n + b_n a_0)R.$$

Proof. By induction on d, if $d = 0$ then $R_S/\mathcal{I}(\frac{a_n}{1}) \cong (R/J)_{\overline{S}}$ is trivial, where $\overline{S} = \{s + J \mid s \in S\}$, $J = i^{-1}(\mathcal{I}(\frac{a_n}{1}))$, and $i : R \to R_S$ is the natural homomorphism. So $1 \in \langle \frac{a_n}{1}, \frac{b_n}{1} \rangle$ in R_S, where $b_n \in R$ and $\frac{a_n b_n}{1}$ is nilpotent. Since $1 \in \langle \frac{a_1}{1}, \dots, \frac{a_{n-1}}{1}, \frac{a_n}{1}, \frac{b_n a_0}{1} \rangle$, so by [[2], Lemma 2.3], $1 \in \langle \frac{a_1}{1}, \dots, \frac{a_{n-1}}{1}, \frac{a_n + b_n a_0}{1} \rangle$, i.e., there exist $s \in S$ such that $s \in a_1 R + \cdots + a_{n-1}R + (a_n + b_n a_0)R$.

Assume now $d > 0$. By the induction assumption with respect to the ring $R_S/\mathcal{I}(\frac{a_n}{1}) \cong (R/J)_{\overline{S}}$ we can find $\bar{b}_1, \dots, \bar{b}_{n-1} \in R/J$ such that

$$\langle \frac{\bar{a}_1 + \bar{b}_1 \bar{a}_0}{\bar{1}}, \dots, \frac{\bar{a}_{n-1} + \bar{b}_{n-1} \bar{a}_0}{\bar{1}} \rangle = (R/J)_{\overline{S}}.$$

So $\langle \overline{\frac{a_1 + b_1 a_0}{1}}, \dots, \overline{\frac{a_{n-1} + b_{n-1} a_0}{1}} \rangle = R_S/\mathcal{I}(\frac{a_n}{1})$, this means that

$$\langle \frac{a_1 + b_1 a_0}{1}, \dots, \frac{a_{n-1} + b_{n-1} a_0}{1}, \frac{a_n}{1}, \frac{b_n}{1} \rangle = R_S$$

where $\frac{a_n b_n}{1}$ is nilpotent. So by [[2], Lemma 2.3]

$$\langle \frac{a_1 + b_1 a_0}{1}, \dots, \frac{a_{n-1} + b_{n-1} a_0}{1}, \frac{a_n + b_n a_0}{1} \rangle = R_S.$$

□

Let $f \in \mathrm{Um}_{n+1}(R[X])$, where $n \geq \frac{d}{2} + 1$, with R a local ring of dimension d. M.Roitman's argument in [[5], Theorem 5], shows how one could decrease the degree of all but one (special) co-ordinate of f. In the absence of a monic polynomial as a co-ordinate of f he uses a Euclid's algorithm and this is achieved via,

Lemma 2.4. (M.Roitman, [[5], Lemma 1])
Let $(x_0,\dots,x_n)\in \mathrm{Um}_{n+1}(R)$, $n\geq 2$, *and let* t *be an element of* R *which is invertible* $\mathrm{mod}(Rx_0+\cdots+Rx_{n-2})$. *Then*

$$(x_0,\dots,x_n)\sim_{\mathrm{E}_{n+1}(R)}(x_0,\dots,t^2x_n)\sim_{\mathrm{E}_{n+1}(R)}(x_0,\dots,tx_{n-1},tx_n).$$

3. The main results

Definitions 3.1. Let $f\in R[X,X^{-1}]$ be a nonzero Laurent polynomial. We denote $\deg(f)=\mathrm{hdeg}(f)-\mathrm{ldeg}(f)$, where $\mathrm{hdeg}(f)$ and $\mathrm{ldeg}(f)$ denote respectively the highest and the lowest degree of f.

Let $\mathrm{hc}(f)$ and $\mathrm{lc}(f)$ denote respectively the coefficients of the highest and the lowest degree term of f. An element $f\in R[X,X^{-1}]$ is called a doubly unitary if $\mathrm{hc}(f),\mathrm{lc}(f)\in U(R)$.

For example, $\deg(X^{-3}+X^2)=5$.

Lemma 3.2. *Let* $f_1,\dots,f_n\in R[X,X^{-1}]$ *such that* $\mathrm{hdeg}(f_i)\leq k-1$, $\mathrm{ldeg}(f_i)\geq -m$ *for all* $1\leq i\leq n$. *Let* $f\in R[X,X^{-1}]$ *with* $\mathrm{hdeg}(f)=k$, $\mathrm{ldeg}(f)\geq -m$, *where* $k,m\in\mathbb{N}$. *Assume that* $\mathrm{hc}(f)\in U(R)$ *and the coefficients of* $f_1,\dots,f_n$ *generate the ideal* (1) *of* R, *then* $I=\langle f_1,\dots,f_n,f\rangle$ *contains a polynomial* h *of* $\mathrm{hdeg}(h)=k-1$, $\mathrm{ldeg}(h)\geq -m$ *and* $\mathrm{hc}(h)\in U(R)$.

Proof. Since $X^mf_1,\dots,X^mf_n,X^mf\in R[X]$, by [[1], §4, Lemma 1(b)], I contains a polynomial $h_1\in R[X]$ of degree $m+k-1$ which is unitary. So $h=X^{-m}h_1\in I$ of $\mathrm{hdeg}(h)=k-1$, $\mathrm{ldeg}(h)\geq -m$ and $\mathrm{hc}(h)\in U(R)$. □

Proposition 3.3. *Let* $I\trianglelefteq R[X,X^{-1}]$ *be an ideal,* $J\trianglelefteq R$, *such that* I *contains a doubly unitary polynomial. If* $I+J[X,X^{-1}]=R[X,X^{-1}]$ *then* $(I\cap R)+J=R$.

Proof. Let us denote by h_1 a doubly unitary polynomial in I. Since $I+J[X,X^{-1}]=R[X,X^{-1}]$, there exist $h_2\in I$ and $h_3\in J[X,X^{-1}]$ such that $h_2+h_3=1$. Let $g_i=X^{-\mathrm{ldeg}(h_i)}h_i$, for $i=1,2,3$. Since $X^l\in\sum_{i=1}^3 g_iR[X]$, for some $l\geq 0$, and $g_1\equiv u \bmod XR[X]$, where $u\in U(R)$, we obtain that $\langle g_1,g_2,g_3\rangle=\langle 1\rangle$ in $R[X]$. By [[8], Lemma 2], we obtain $(\langle g_1,g_2\rangle\cap R)+J=R$. So $(I\cap R)+J=R$. □

Theorem 3.4. *Let* $f=(f_0,\dots,f_n)\in\mathrm{Um}_{n+1}(R[X,X^{-1}])$, *where* $n\geq 2$. *Assume that* f_0 *is a doubly unitary polynomial, then*

$$f\sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])}(1,0,\dots,0).$$

Proof. By (2.4), $f\sim_{\mathrm{E}}(X^{-\mathrm{ldeg}(f_0)}f_0,X^{-\mathrm{ldeg}(f_0)}f_1,f_2,\dots,f_n)\sim_{\mathrm{E}}$ $(X^{-\mathrm{ldeg}(f_0)}f_0,X^{-\mathrm{ldeg}(f_0)+2k}f_1,X^{2k}f_2,\dots,X^{2k}f_n)=(g_0,\dots,g_n)$ where $k\in\mathbb{N}$. For sufficiently big k, we obtain that $g_0,\dots,g_n\in R[X]$. Clearly, $X^l\in\sum_{i=0}^n g_iR[X]$ for some $l\geq 0$. But $g_0\equiv u \bmod XR[X]$, where $u\in U(R)$, then $X^lR[X]+g_0R[X]=R[X]$, so $g\in\mathrm{Um}_n(R[X])$. By (2.1), $g\sim_{\mathrm{E}}e_1$. □

Remark 3.5. Let $a=(a_1,\dots,a_n)\in\mathrm{Um}_{n+1}(R)$, where $n\geq 2$. If

$$a\sim_{\mathrm{E}_n(R/\mathrm{Nil}(R))}e_1$$

then $a\sim_{\mathrm{E}_n(R)}e_1$.

Proposition 3.6. *If* R *is a zero-dimensional ring and* $f=(f_0,\dots,f_n)$ $\in\mathrm{Um}_{n+1}(R[X,X^{-1}])$, *where* $n\geq 1$. *Then*

$$f\sim_{\mathrm{E}}e_1.$$

Proof. We prove by induction on $\deg f_0+\deg f_1$. We may assume that R is reduced ring. Let $a=\mathrm{hc}(f_0)$ and $b=\mathrm{lc}(f_0)$. Assume that $ab\in U(R)$, then by elementary transformations of the form

$$f_1 - X^{\mathrm{ldeg}(f_1)-\mathrm{ldeg}(f_0)}b^{-1}\,\mathrm{lc}(f_1)f_0$$

we obtain that $f\sim_E (f_0,h_1,f_2,\dots,f_n)$, where $\mathrm{ldeg}(h_1)>\mathrm{ldeg}(f_0)$. By elementary transformations of the form

$$f_1 - X^{\mathrm{hdeg}(f_1)-\mathrm{hdeg}(f_0)}a^{-1}\,\mathrm{hc}(f_1)f_0$$

we obtain that $f\sim_E (f_0,g_1,f_2,\dots,f_n)$, where $\mathrm{ldeg}(g_1)\geq \mathrm{ldeg}(f_0)$ and $\mathrm{hdeg}(g_1)<\mathrm{hdeg}(f_0)$. So we may assume that $\deg f_0\leq \deg f_1$ and $ab\notin U(R)$. Assume that $a\notin U(R)$. We have $Ra=Re$ for some idempotent e. Let $c=\mathrm{hc}(f_1)$. Since $e\in Ra$, we may assume that $c\neq 0$ and that $c\in R(1-e)$. Note that

$$(1-e)f=(f_0(1-e),\dots,f_n(1-e))\in \mathrm{Um}_{n+1}(R(1-e)[X,X^{-1}]) \text{ and}$$
$$ef=(f_0e,\dots,f_ne)\in \mathrm{Um}_{n+1}(Re[X,X^{-1}]).$$

By the inductive assumption, there are matrices

$$A\in \mathrm{E}_{n+1}(R(1-e)[X,X^{-1}]),\ B\in \mathrm{E}_{n+1}(Re[X,X^{-1}])$$

so that $(1-e)fA=(1-e,0,\dots,0)$ and $efB=(e,0,\dots,0)$. Let

$$A=\prod_{s=1}^{k}\mathrm{E}_{ij}(h_s),\ B=\prod_{s=1}^{t}\mathrm{E}_{ij}(g_s)$$

where

$$\mathrm{E}_{ij}(h_s)=(1-e)I_{n+1}+h_se_{ij},\ \mathrm{E}_{ij}(g_s)=eI_{n+1}+g_se_{ij}$$

and $i\neq j\in\{1,\dots,n+1\},\ h_s\in R(1-e)[X,X^{-1}],\ g_s\in Re[X,X^{-1}]$. Let

$$A'=\prod_{s=1}^{k}(I_{n+1}+h_se_{ij}),\ B'=\prod_{s=1}^{t}(I_{n+1}+g_se_{ij}).$$

Clearly, $(1-e)A'=A$, $eB'=B$ and $A',B'\in \mathrm{E}_{n+1}(R[X,X^{-1}])$. Let $C=A'B'$, then $C\in \mathrm{E}_{n+1}(R[X,X^{-1}])$ and

$$(1-e)C=(1-e)A'(1-e)B'=A(1-e)I_{n+1}=(1-e)A'=A.$$

Similarly, we have $eC=B$. Let $fC=(g_0,\dots,g_n)=g$. Thus

$$g_0(1-e)=1-e \text{ and } g_1e=e.$$

So

$$f\sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])}(g_0,\dots,g_n)\sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])}(g_0+e,\dots,g_n)=$$
$$(1+g_0e,\dots,g_n)\sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])}(1+g_0e,-g_0e,\dots,g_n)\sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])}e_1.$$

Similarly, if $b\notin U(R)$, then $f\sim_{\mathrm{E}} e_1$. □

Proposition 3.7. *If R is a zero-dimensional ring, then*

$$\mathrm{SL}_n(R[X,X^{-1}])=\mathrm{E}_n(R[X,X^{-1}])$$

for all $n\geq 2$.

Proof. Clearly, $\mathrm{E}_n(R[X,X^{-1}]) \subseteq \mathrm{SL}_n(R[X,X^{-1}])$. Let $M \in \mathrm{SL}_n(R[X,X^{-1}])$. By (3.6), we can perform suitable elementary transformations to bring M to M_1 with first row $(1,0,\ldots,0)$. Now a sequence of row transformations bring M_1 to

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix}$$

where $M' \in \mathrm{SL}_{n-1}(R[X,X^{-1}])$. The proof now proceeds by induction on n. □

Lemma 3.8. *Let $(f_0,\ldots,f_n) \in \mathrm{Um}_{n+1}(R[X,X^{-1}])$, where $n \geq 2$. Assume that $\mathrm{hc}(f_0)$ is invertible modulo f_0. Then*

$$f \sim_{\mathrm{E}} (f_0, g_1, \ldots, g_n)$$

where $\mathrm{hdeg}(g_i) < \mathrm{hdeg}(f_0), \mathrm{ldeg}(g_i) \geq \mathrm{ldeg}(f_0)$, for all $1 \leq i \leq n$.

Proof. By (2.4), $f \sim_{\mathrm{E}} (f_0, X^{2k}f_1, \ldots, X^{2k}f_n)$ for all $k \in \mathbb{Z}$. So we may assume that $\mathrm{ldeg}(f_i) > \mathrm{ldeg}(f_0)$. Let $a = \mathrm{hc}(f_0)$. By (2.4) we have

$$f \sim_{\mathrm{E}} (f_0, a^2 f_1, \ldots, a^2 f_n).$$

Using elementary transformations of the form

$$a^2 f_i - aX^{\mathrm{hdeg}(f_i)-\mathrm{hdeg}(f_0)}\,\mathrm{hc}(f_i) f_0$$

we lower the degrees of f_i, for all $1 \leq i \leq n$, and obtain the required row. □

Lemma 3.9. *Let R be a ring of dimension $d > 0$ and*

$$f = (r, f_1, \ldots, f_n) \in \mathrm{Um}_{n+1}(R[X,X^{-1}])$$

where $r \in R,\ n \geq d+1$. Assume that for every ring T of dimension $<\ d$ and $n \geq \dim(T)+1$, the group $\mathrm{E}_{n+1}(T[X,X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(T[X,X^{-1}])$. Then $f \sim_{\mathrm{E}(R[X,X^{-1}])} e_1$.

Proof. Since $\dim(R/\mathcal{I}(r)) < \dim(R)$ so over $R/\mathcal{I}(r)$, we can complete $(f_1,\ldots,f_n)$ to a matrix in $\mathrm{E}_n(R/\mathcal{I}(r)[X,X^{-1}])$. If we lift this matrix, we obtain that

$$(r, f_1, \ldots, f_n) \sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])} (r, 1+rw_1+h_1, \ldots, rw_n+h_n) \sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])} (r, 1+h_1, \ldots, h_n)$$

where $h_i,\ w_i \in R[X,X^{-1}]$ and $rh_i = 0$ for all $1 \leq i \leq n$. Then

$$f \sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])} (r - r(1+h_1), 1+h_1, \ldots, h_n) \sim_{\mathrm{E}_{n+1}(R[X,X^{-1}])} e_1.$$

□

Lemma 3.10. *Let R be a ring of dimension $d > 0$ and*

$$f = (f_0, \ldots, f_n) \in \mathrm{Um}_{n+1}(R[X,X^{-1}])$$

such that $n \geq d+1,\ f_0 = ag$ and $a^t = \mathrm{hc}(f_0)$, where $a \in R \setminus U(R),\ 0 \neq t \in \mathbb{N}$. Assume that for every ring T of dimension $<\ d$ and $n \geq \dim(T)+1$, the group $\mathrm{E}_{n+1}(T[X,X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(T[X,X^{-1}])$. Then $f \sim_{\mathrm{E}(R[X,X^{-1}])} e_1$.

Proof. We prove by induction on the number M of non-zero coefficients of the polynomial f_0, that $f \sim_{\mathrm{E}} e_1$. If $M = 1$, so $f_0 = rX^m$ where $r \in R, m \in \mathbb{Z}$. By (2.4), $f \sim_{\mathrm{E}} (r, X^{-m}f_1, f_2, \ldots, f_n)$. So by (3.9), we obtain that $f \sim_{\mathrm{E}} e_1$. Assume now that $M > 1$. Let S be the multiplicative subset of R generated by a, b, where $b = \mathrm{lc}(g)$, i.e., $S = \left\{ a^{k_1}b^{k_2} \,\middle|\, k_1, k_2 \in \mathbb{N} \right\}$. By the inductive step, with respect to the ring R/abR, we obtain from f a row $\equiv (1,0,\ldots,0) \bmod abR[X,X^{-1}]$, also we can perform such transformation so that at every stage the row contains a doubly unitary polynomial in $R_S[X,X^{-1}]$, indeed, if we have to perform, e.g., the elementary transformation

$$(g_0,\dots,g_n)\to(g_0,g_1+hg_0,\dots,g_n)$$

and g_1 is a doubly unitary polynomial in $R_S[X,X^{-1}]$, then we replace this elementary transformation by the two transformations:

$$(g_0,\dots,g_n)\to(g_0+abX^mg_1+abX^kg_1,g_1,\dots,g_n)\to$$
$$(g_0+abX^mg_1+abX^kg_1,g_1+h(g_0+abX^mg_1+abX^{-k}g_1),\dots,g_n)$$

where $m>\mathrm{hdeg}(g_0),\ k<\mathrm{ldeg}(g_0)$. So we may assume that

$$(f_0,\dots,f_n)\equiv(1,0,\dots,0)\bmod abR[X,X^{-1}]$$

and f_0 is a doubly unitary polynomial in $R_S[X,X^{-1}]$. By (3.8), we may assume that $\mathrm{hdeg}(f_i)<\mathrm{hdeg}(f_0),\ \mathrm{ldeg}(f_i)\ge\mathrm{ldeg}(f_0)$.

We prove that f can be transformed by elementary transformation into a row with one constant entry. We use an argument similar to that in the proof of [[5], Theorem 5].

Assume that the number of the coefficients of $f_2,\dots,f_n$ is $\ge 2(n-1)$. Since $d>0$, we obtain that $2(n-1)\ge d+1$. Let $a_1,\dots,a_t$ be the coefficients of $f_2,\dots,f_n$ and $J=\frac{a_1}{1}R_S+\cdots+\frac{a_t}{1}R_S$. Let $I=R_S[X,X^{-1}]f_0+R_S[X,X^{-1}]f_1$. Since $I+J[X,X^{-1}]=R_S[X,X^{-1}]$ and f_0 is a doubly unitary in $R_S[X,X^{-1}]$, by (3.3), we obtain that $(I\cap R_S)+J=R_S$. So $(\frac{f_0h_0+f_1h_1}{s})+\frac{r_1}{s_1}\frac{a_1}{1}+\cdots+\frac{r_t}{s_t}\frac{a_t}{1}=\frac{1}{1}$,where $h_0,h_1\in R[X,X^{-1}]$ and $r_i\in R,\ s,s_i\in S$ for all $1\le i\le t$. This means that $(\frac{f_0h_0+f_1h_1}{1},\frac{a_1}{1},\dots,\frac{a_t}{1})\in\mathrm{Um}_{t+1}(R_S)$. By (2.3), there exist $s\in S$ and $b_1,\dots,b_t\in R$, such that

$$s\in(a_1+b_1(f_0h_0+f_1h_1))R+\cdots+(a_t+b_t(f_0h_0+f_1h_1))R.$$

Using elementary transformations, we may assume that $J=R_S$. By (3.2), the ideal $\langle f_0,f_2,\dots,f_n\rangle$ contains a polynomial h such that $a^{k_1}b^{k_2}=\mathrm{hc}(h)$ and $\mathrm{hdeg}(h)=\mathrm{hdeg}(f_0)-1,\mathrm{ldeg}(h)\ge\mathrm{ldeg}(f_0)$ where $k_1,k_2\in\mathbb{N}$. Let $r=\mathrm{hc}(f_1)$, So

$$f\sim_{\mathrm{E}}(f_0,a^{2k_1}b^{2k_2}f_1,f_2,\dots,f_n)\sim_{\mathrm{E}}(f_0,a^{2k_1}b^{2k_2}f_1+(1-a^{k_1}b^{k_2}r)h,f_2,\dots,f_n).$$

Then we may assume that $a^{k_1}b^{k_2}=\mathrm{hc}(f_1)$. By the proof of Lemma (3.8), we can decrease the $\mathrm{hdeg}(f_i)$ for all $2\le i\le n$.

Repeating the argument above, we obtain that

$$f\sim_{\mathrm{E}}(rX^m,g_1,\dots,g_n)\sim_{\mathrm{E}}(r,g_1X^{-m},g_2,\dots,g_n)$$

where $r\in R,m\in\mathbb{Z},\ g_1,\dots,g_n\in R[X,X^{-1}]$. By (3.9), $f\sim_{\mathrm{E}}e_1$. □

Lemma 3.11. *Let R be a ring of dimension $d>0$ and*

$$f=(f_0,\dots,f_n)\in\mathrm{Um}_{n+1}(R[X,X^{-1}])$$

such that $n\ge d+1,\ f_0=cg$ and $c^t=\mathrm{lc}(f_0)$, where $c\in R\setminus U(R),\ 0\ne t\in\mathbb{N}$. Assume that for every ring T of dimension $<\ d$ and $n\ge\dim(T)+1$, the group $\mathrm{E}_{n+1}(T[X,X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(T[X,X^{-1}])$. Then $f\sim_{\mathrm{E}(R[X,X^{-1}])}e_1$.

Proof. By making the change of variable: $X\to X^{-1}$ and Proposition (3.10), we obtain that $f\sim_{\mathrm{E}(R[X,X^{-1}])}e_1$. □

Theorem 3.12. *Let R be a ring of dimension d and $n \geq d+1$, then $\mathrm{E}_{n+1}(R[X, X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(R[X, X^{-1}])$.*

Proof. Let $f = (f_0, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X, X^{-1})$. We prove the theorem by induction on d, we may assume that R is reduced ring. If $d = 0$, by (3.6), we are done. Assume that the theorem is true for the dimensions $0, 1, \dots, d-1$, where $d > 0$. We prove by induction on the number N of nonzero coefficients of the polynomials $f_0, \dots, f_n$, that $f \sim_{\mathrm{E}} e_1$ if $\dim R = d$. Starting with $N = 1$. Let $N > 1$. Let $a = \mathrm{hc}(f_0)$ and $c = \mathrm{lc}(f_0)$, if $ac \in U(R)$ then by (3.4), we are done. Otherwise, assume that $a \notin U(R)$, by the inductive step, with respect to the ring R/aR, we obtain from f a row $\equiv (1, 0, \dots, 0) \bmod aR[X, X^{-1}]$ using elementary transformations. We can perform such transformations so that at every stage the row contains a polynomial $g \in R[X, X^{-1}]$ such that $\mathrm{hc}(g) = a^t$, where $t \in \mathbb{N}$. Indeed, if we have to perform, e.g., the elementary transformation

$$(g_0, \dots, g_n) \to (g_0, g_1 + hg_0, \dots, g_n)$$

and $\mathrm{hc}(g_1) \in U(R_a)$, then we replace this elementary transformation by the two transformations:

$$(g_0, \dots, g_n) \to (g_0 + aX^m g_1, g_1, \dots, g_n) \to (g_0 + aX^m g_1, g_1 + h(g_0 + aX^m g_1), \dots, g_n)$$

where $m > \mathrm{hdeg}(g_0)$.

So we have $f_0 = ag$, and $a^t = \mathrm{hc}(f_0)$, where $0 \neq t \in \mathbb{N}$. By (3.10), $f \sim_{\mathrm{E}} e_1$. Similarly, if $c \notin U(R)$, by (3.11) we obtain that $f \sim_{\mathrm{E}} e_1$. □

Corollary 3.13. *For any ring R with Krull dimension $\leq d$, all finitely generated stably free modules over $R[X, X^{-1}]$ of rank $> d$ are free.*

The following conjecture is the analogue of Conjecture 8 of [7] in the Laurent case:

Conjecture 3.14. *For any ring R with Krull dimension $\leq d$, all finitely generated stably free modules over $R[X_1^{\pm 1}, \dots, X_k^{\pm 1}, X_{k+1}, \dots, X_n]$ of rank $> d$ are free.*

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