Fixed points subgroups by two involutive automorphisms σ, γ of compact exceptional Lie groups F_4, E_6 and E_7

By

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Introduction

For simply connected compact exceptional Lie groups $G = F_4, E_6$ and E_7, we consider two involutions σ, γ and determine the group structure of subgroups $G^{\sigma,\gamma}$ of G which are the intersection $G^\sigma \cap G^\gamma$ of the fixed points subgroups of G^σ and G^γ. The motivation is as follows. In [1], we determine the group structure of $(F_4)^{\sigma,\sigma'}, (E_6)^{\sigma,\sigma'}$ and $(E_7)^{\sigma,\sigma'}$, and in [2], we also determine the group structure of $(G_2)^{\gamma,\gamma'}, (F_4)^{\gamma,\gamma'}$ and $(E_6)^{\gamma,\gamma'}$. So, in this paper, we try to determine the type of groups $(F_4)^{\sigma,\gamma}, (E_6)^{\sigma,\gamma}$ and $(E_7)^{\sigma,\gamma}$. Our results are the following second columns. The first columns are already known in [3],[4] or [5] and these play an important role to obtain our results. In Table 1, the results of the group structure of $G^{\sigma,\gamma}$ are obtained by the result of G^γ and in Table 2, ones are obtained by the result of G^σ. In this paper, we show the proof of the results of the first and the second line of Table 1 and the third line of Table 2.

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Table 1

G	G^γ	$G^{\sigma,\gamma}$
F_4	$(Sp(1) \times Sp(3))/\boldsymbol{Z}_2$	$(Sp(1) \times Sp(1) \times Sp(2))/\boldsymbol{Z}_2$
E_6	$(Sp(1) \times SU(6))/\boldsymbol{Z}_2$	$(Sp(1) \times S(U(2) \times U(4)))/\boldsymbol{Z}_2$
E_7	$(SU(2) \times Spin(12))/\boldsymbol{Z}_2$	$(SU(2) \times Spin(4) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$

Table 2

G	G^σ	$G^{\sigma,\gamma}$
F_4	$Spin(9)$	$(Spin(4) \times Spin(5))/\boldsymbol{Z}_2$
E_6	$(U(1) \times Spin(10))/\boldsymbol{Z}_4$	$(U(1) \times Spin(4) \times Spin(6))/\boldsymbol{Z}_2$
E_7	$(SU(2) \times Spin(12))/\boldsymbol{Z}_2$	$(SU(2) \times Spin(4) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$

As for the group $(E_8)^{\sigma,\gamma}$, we can not realize explicitly, however we conjecture

$$(E_8)^{\sigma,\gamma} \cong (Spin(4) \times Spin(12))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$$

REMARK. In E_7, since γ is conjugate to $-\sigma$, we have $(E_7)^\gamma \cong (E_7)^\sigma$. (In detail, see [4].) Note that the results of Table 1 and Table 2 are the same as a set, however they are different as realizations.

Notation

(1) For a group G and an element s of G, we denote $\{g \in G \,|\, sg = gs\}$ by G^s.

(2) For a transformation group G of a space M, the isotropy subgroup of G at $m_1, \cdots, m_k \in M$ is denoted by $G_{m_1,\cdots,m_k} = \{g \in G \mid gm_1 = m_1, \cdots, gm_k = m_k\}$.

(3) For a $\boldsymbol{R}$-vector space V, its complexification $\{u + iv \mid u, v \in V\}$ is denoted by V^C. The complex conjugation in V^C is denoted by $\tau : \tau(u + iv) = u - iv$. In particular, the complexification of $\boldsymbol{R}$ is briefly denoted by $C : \boldsymbol{R}^C = C$.

(4) For a Lie group G, the Lie algebra of G is denoted by the corresponding German small letter $\mathfrak{g}$. For example, $\mathfrak{so}(n)$ is the Lie algebra of the group $SO(n)$.

(5) Although we will give all definitions used in the following Sections, if in case of insufficiency, refer to [3],[4] or [5].

1. Group F_4

We use the same notation as in [1], [2] or [5] (however, some will be rewritten). For example, the Cayley algebra $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$,

the exceptional Jordan algebra $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \,|\, X^* = X\}$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the elements $E_1, E_2, E_3 \in \mathfrak{J}$,

the group $F_4 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$.

We define $\boldsymbol{R}$-linear transformations σ and γ of $\mathfrak{J}$ by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \ \gamma X = \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x}_2 \\ \overline{\gamma x}_3 & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x}_1 & \xi_3 \end{pmatrix},$$

respectively, where $\gamma x_k = \gamma(m_k + a_k e_4) = m_k - a_k e_4,\ x_k = m_k + a_k e_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}$. Then, $\sigma, \gamma \in F_4$ and $\sigma^2 = \gamma^2 = 1$. σ and γ are commutative. From $\sigma\gamma = \gamma\sigma$, we have

$$(F_4)^\sigma \cap (F_4)^\gamma = ((F_4)^\sigma)^\gamma = ((F_4)^\gamma)^\sigma.$$

Hence, this group will be denoted briefly by $(F_4)^{\sigma,\gamma}$.

PROPOSITION 1.1. $(F_4)^\gamma \cong (Sp(1) \times Sp(3))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

PROOF. The isomorphism is induced by the homomorphism $\varphi : Sp(1) \times Sp(3) \to (F_4)^\gamma,\ \varphi(p, A)(M + \boldsymbol{a}) = AMA^* + p\boldsymbol{a}A^*,\ M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3 = \mathfrak{J}$. (In detail, see [3], [5].)

LEMMA 1.2. *$\varphi : Sp(1) \times Sp(3) \to (F_4)^\gamma$ of Proposition 1.1 satisfies $\sigma\varphi(p, A)\sigma = \varphi(p, I_1AI_1)$, where $I_1 = \mathrm{diag}(-1, 1, 1)$.*

PROOF. From $\sigma = \varphi(-1, I_1)$, we have the required one.

Now, we shall determine the group structure of $(F_4)^{\sigma,\gamma} = ((F_4)^\gamma)^\sigma = ((F_4)^\sigma)^\gamma = (F_4)^\sigma \cap (F_4)^\gamma$.

THEOREM 1.3. $(F_4)^{\sigma,\gamma} \cong (Sp(1) \times Sp(1) \times Sp(2))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1, 1, E), (-1, -1, -E)\}$.

PROOF. We define a map $\varphi_4 : Sp(1) \times Sp(1) \times Sp(2) \to (F_4)^{\sigma,\gamma}$ by

$$\varphi_4(p, q, B)(M + \boldsymbol{a}) = \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{array}\right) M \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{array}\right)^* + p\boldsymbol{a} \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{array}\right)^*,$$

$M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3 = \mathfrak{J}$, as the restriction of Proposition 1.1. By Lemma 1.2, φ_4 is well-defined and a homomorphism. We shall show that φ_4 is onto. Let $\alpha \in (F_4)^{\sigma,\gamma}$. Since $(F_4)^{\sigma,\gamma} \subset (F_4)^\gamma$, there exist $p \in Sp(1)$ and $A \in Sp(3)$ such that $\alpha = \varphi(p, A)$ (Proposition 1.1). From $\sigma\alpha\sigma = \alpha$, we have $\varphi(p, I_1AI_1) = \varphi(p, A)$ (Lemma 1.2). Hence,

$$\left\{\begin{array}{l} p = p \\ I_1AI_1 = A \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} p = -p \\ I_1AI_1 = -A \end{array}\right.$$

The latter case is impossible because $p = 0$ is false. In the former case, from $I_1AI_1 = A$, we have

$A = \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{array}\right),\ q \in Sp(1),\ B \in Sp(2)$. Hence,

$$\alpha = \varphi(\ q, \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{array}\right)) = \varphi_4(p, q, B),$$

that is, φ_4 is onto. And $\mathrm{Ker}\varphi_4 = \{(1, 1, E), (-1,\ -1, -E)\} = \boldsymbol{Z}_2$. Thus, we have the required isomorphism $(Sp(1) \times Sp(1) \times Sp(2))/\boldsymbol{Z}_2 \cong (F_4)^{\sigma,\gamma}$.

2. Group E_6

We use the same notation as in [1], [2] or [5] (however, some will be rewritten). For example, the complex exceptional Jordan algebra $\mathfrak{J}^C = \{X \in M(3, \mathfrak{C}^C) \,|\, X^* = X\}$, the Freudenthal multiplication $X \times Y$ and the Hermitian inner product $\langle X, Y \rangle$,

the group $E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$, and the natural inclusion $F_4 \subset E_6$.

Proposition 2.1. $(E_6)^\gamma \cong (Sp(1) \times SU(6))/\boldsymbol{Z}_2, \ \boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

Proof. The isomorphism is induced by the homomorphism $\varphi : Sp(1) \times SU(6) \to (E_6)^\gamma$, $\varphi(p, A)(M + \boldsymbol{a}) = {k_J}^{-1}(Ak_J(M)^t A) + p\boldsymbol{a}k^{-1}(A^*)$, $M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H})^C \oplus (\boldsymbol{H}^3)^C = \mathfrak{J}^C$. (In detail, see [3], [5].)

Lemma 2.2. *$\varphi : Sp(1) \times SU(6) \to (E_6)^\gamma$ of Proposition 2.1 satisfies $\sigma\varphi(p, A)\sigma = \varphi(p, I_2AI_2)$, where $I_2 = \mathrm{diag}(-1, -1, 1, 1, 1, 1)$.*

Proof. From $\sigma = \varphi(-1, I_2)$, we have the required one.

Now, we shall determine the group structure of $(E_6)^{\sigma,\gamma} = ((E_6)^\gamma)^\sigma = ((E_6)^\sigma)^\gamma = (E_6)^\sigma \cap (E_6)^\gamma$.

Theorem 2.3. $(E_6)^{\sigma,\gamma} \cong (Sp(1) \times S(U(2) \times U(4)))/\boldsymbol{Z}_2, \ \boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

Proof. We define a map $\varphi_6 : Sp(1) \times S(U(2) \times U(4)) \to (E_6)^{\sigma,\gamma}$ by

$$\varphi_6(p, A)(M + \boldsymbol{a}) = {k_J}^{-1}(Ak_J(M)^t A) + p\boldsymbol{a}k^{-1}(A^*),$$

$M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H})^C \oplus (\boldsymbol{H}^3)^C = \mathfrak{J}^C$, as the restriction of φ of Proposition 2.1. By Lemma 2.2, φ_6 is well-defined and a homomorphism. We shall show that φ_6 is onto. Let $\alpha \in (E_6)^{\sigma,\gamma}$. Since $(E_6)^{\sigma,\gamma} \subset (E_6)^\gamma$, there exist $p \in Sp(1)$ and $A \in SU(6)$ such that $\alpha = \varphi(p, A)$ (Proposition 2.1). From $\sigma\alpha\sigma = \alpha$, we have $\varphi(p, I_2AI_2) = \varphi(p, A)$ (Lemma 2.2). Hence,

$$\begin{cases} p = p \\ I_2AI_2 = A \end{cases} \quad \text{or} \quad \begin{cases} p = -p \\ I_2AI_2 = -A \end{cases}$$

The latter case is impossible because $p = 0$ is false. In the former case, we have $A \in S(U(2) \times U(4))$. Therefore, φ_6 is onto. $\mathrm{Ker}\varphi_6 = \{(1, E), (-1, -E)\} = \boldsymbol{Z}_2$. Thus, we have the required isomorphism $(Sp(1) \times S(U(2) \times U(4)))/\boldsymbol{Z}_2 \cong (E_6)^{\sigma,\gamma}$.

3. Group E_7

We use the same notation as in [1],[4] or [5] (however, some will be rewritten). For example, the Freudenthal C-vector space $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$, the Hermitian inner product $\langle P, Q\rangle$, the C-linear map $P \times Q : \mathfrak{P}^C \to \mathfrak{P}^C (P, Q \in \mathfrak{P}^C)$,

the group $E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(X \times Y)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$, the natural inclusion $E_6 \subset E_7$ and elements $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$, $\lambda \in E_7$.

We shall consider the following subgroup of F_4.

$$((F_4)^{\sigma,\gamma})_{F_1(h)} = \{\alpha \in (F_4)^{\sigma,\gamma} \,|\, \alpha F_1(h) = F_1(h) \text{ for all } h \in \boldsymbol{H}\}.$$

Proposition 3.1. $\quad ((F_4)^{\sigma,\gamma})_{F_1(h)} \cong Sp(1) \times Sp(1) (= Spin(4))$.

Proof. We define a map $\varphi : Sp(1) \times Sp(1) \to ((F_4)^{\sigma,\gamma})_{F_1(h)}$ by

$$\varphi(p,q)(M + \boldsymbol{a}) = \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{E} \end{array}\right) M \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{E} \end{array}\right)^{*} + p\boldsymbol{a} \left(\begin{array}{c|cc} q & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{E} \end{array}\right)^{*},$$

as the restriction of φ_4 of Theorem 1.3. By $F_1(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & \overline{h} & 0 \end{pmatrix} + \varnothing$, φ is well-defined and homomorphism. We shall show that φ is onto. Let $\alpha \in ((F_4)^{\sigma,\gamma})_{F_1(h)}$. Since $((F_4)^{\sigma,\gamma})_{F_1(h)} \subset (F_4)^{\sigma,\gamma}$, there exist $p, q \in Sp(1)$ and $B \in Sp(2)$ such that $\alpha = \varphi_4(p, q, B)$(Theorem 1.3). From $\alpha F_1(h) = F_1(h)$, we have $B \begin{pmatrix} 0 & h \\ \overline{h} & 0 \end{pmatrix} B^* = \begin{pmatrix} 0 & h \\ \overline{h} & 0 \end{pmatrix}$, so that

$$\alpha = \varphi_4(p, q, E) \quad \text{or} \quad \alpha = \varphi_4(p, q, -E).$$

In the former case, we have $\alpha = \varphi_4(p, q, E) = \varphi(p, q)$. In the latter case, we have

$$\begin{aligned} \alpha &= \varphi_4(p, q, -E) = \varphi_4(-p, -q, E)\varphi_4(-1, -1, -E) \\ &= \varphi_4(-p, -q, E)1 = \varphi(-p, -q). \end{aligned}$$

Hence, φ is onto. $\mathrm{Ker}\varphi = \{(1, 1)\}$. Thus, we have the required isomorphism $Sp(1) \times Sp(1) \cong ((F_4)^{\sigma,\gamma})_{F_1(h)}$.

Hereafter, in $\mathfrak{P}^C$, we use the following notations.

$$\begin{aligned} &(F_1(h), 0, 0, 0) = \dot{F}_1(h), \quad (0, E_1, 0, 1) = \tilde{E}_1, \\ &(0, E_1, 0, -1) = \tilde{E}_{-1}, \quad (E_2 + E_3, 0, 0, 0) = \dot{E}_{23}. \end{aligned}$$

We shall consider a subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h), \tilde{E}_1, \tilde{E}_{-1}, \dot{E}_{23}}$ of E_7.

LEMMA 3.2. *The Lie algebra* $(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$ *of the group* $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$ *is given by*

$$
\begin{aligned}
&(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}} \\
&= \left\{ \varPhi(\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D'_4 \end{array}\right), 0,0,0) \,\middle|\, \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D'_4 \end{array}\right) \in \mathfrak{so}(8),\ D'_4 \in \mathfrak{so}(4) \right\}.
\end{aligned}
$$

In particular, we have

$$\dim((((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}) = 6.$$

Hereafter, $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D'_4 \end{array}\right)$ will be denoted by D'_4, and also $\varPhi(D'_4, 0, 0, 0)$ will be denoted by $\varPhi_4$.

PROPOSITION 3.3. $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}} = ((F_4)^{\sigma,\gamma})_{F_1(h)}$.

PROOF. Let $\alpha \in ((F_4)^{\sigma,\gamma})_{F_1(h)}$. Since $((F_4)^{\sigma,\gamma})_{F_1(h)} \subset (F_4)^{\sigma} = (F_4)_{E_1}$ (as for $(F_4)^{\sigma} = (F_4)_{E_1}$, see [3], [5]), we see $\alpha E_1 = E_1$. As a result, because κ and μ are defined using by E_1 (see [1], [4] or [5]), we see that $\kappa\alpha = \alpha\kappa$ and $\mu\alpha = \alpha\mu$. From $\alpha E = E$ (see [3], [5]), we have $\alpha(E_2 + E_3) = E_2 + E_3$. Hence, $\alpha\dot{E}_{23} = \dot{E}_{23}$. Moreover, from $\alpha(0,0,0,1) = (0,0,0,1)$ (see [4], [5]), we have $\alpha\tilde{E}_1 = \tilde{E}_1$ and $\alpha\tilde{E}_{-1} = \tilde{E}_{-1}$. Obviously $\alpha\dot{F}_1(h) = \dot{F}_1(h)$. Thus, $\alpha \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$. Conversely, let $\alpha \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$. From $\alpha\tilde{E}_1 = \tilde{E}_1$ and $\alpha\tilde{E}_{-1} = \tilde{E}_{-1}$, we have $\alpha(0, E_1, 0, 0) = (0, E_1, 0, 0)$ and $\alpha(0,0,0,1) = (0,0,0,1)$. Hence, $\alpha \in ((E_6)^{\gamma})_{F_1(h),E_1,E_2+E_3}$(see [4], [5]). Thus, $(((F_4)_{E_1})^{\gamma})_{F_1(h)} = ((F_4)^{\sigma,\gamma})_{F_1(h)}$. Therefore, the proof of this proposition is completed.

Next, we shall consider the following subgroup of F_4.

$$((F_4)^{\sigma,\gamma})_{F_1(he_4)} = \{\alpha \in (F_4)^{\sigma,\gamma} \,|\, \alpha F_1(he_4) = F_1(he_4) \text{ for all } h \in \boldsymbol{H}\}.$$

PROPOSITION 3.4. $((F_4)^{\sigma,\gamma})_{F_1(he_4)} \cong Sp(2)(= Spin(5))$.

PROOF. We define a map $\varphi : Sp(2) \to ((F_4)^{\sigma,\gamma})_{F_1(he_4)}$ by

$$\varphi(B)(M + \boldsymbol{a}) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right) M \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right)^{*} + \boldsymbol{a} \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right)^{*},$$

as the restriction of φ_4 of Theorem 1.3. Obviously φ is well-defined and homomorphism. We shall show that φ is onto. Let $\alpha \in ((F_4)^{\sigma,\gamma})_{F_1(he_4)}$. Since $((F_4)^{\sigma,\gamma})_{F_1(he_4)} \subset (F_4)^{\sigma,\gamma}$, there exist $p, q \in Sp(1)$ and $B \in Sp(2)$ such that

$\alpha = \varphi_4(p,q,B)$(Theorem 1.3). From $\alpha F_1(he_4) = F_1(he_4)(= O + (h,0,0))$, we have $ph\overline{q} = h (h \in \boldsymbol{H})$, so that

$$\alpha = \varphi_4(1,1,B) \quad \text{or} \quad \alpha = \varphi_4(-1,-1,B).$$

In the former case, we have $\alpha = \varphi_4(1,1,B) = \varphi(B)$. In the latter case, we have

$$\begin{aligned}\alpha &= \varphi_4(-1,-1,B) = \varphi_4(1,1,-B)\varphi_4(-1,-1,-E)\\ &= \varphi_4(1,1,-B)1 = \varphi(-B).\end{aligned}$$

Hence, φ is onto. $\mathrm{Ker}\varphi = \{E\}$. Thus, we have the required isomorphism $Sp(2) \cong ((F_4)^{\sigma,\gamma})_{F_1(he_4)}$.

Then, we have the following proposition.

PROPOSITION 3.5. $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}} = ((F_4)^{\sigma,\gamma})_{F_1(he_4)}$.

PROOF. This proof is in the way similar to Proposition 3.3.

We shall consider the subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ of E_7.

LEMMA 3.6. *The Lie algebra* $(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ *of the group* $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ *is given by*

$$\begin{aligned}&(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}\\ &= \Big\{\varPhi\Big(\left(\begin{array}{c|c} D_4 & 0 \\ \hline 0 & 0\end{array}\right) + \widetilde{A}_1(p) + i\begin{pmatrix}0 & 0 & 0\\ 0 & \epsilon & q\\ 0 & \overline{q} & -\epsilon\end{pmatrix}^{\sim}, 0,0,0\Big) \,\Big|\, \left(\begin{array}{c|c} D_4 & 0 \\ \hline 0 & 0\end{array}\right) \in \mathfrak{so}(8),\\ &\qquad\qquad D_4 \in \mathfrak{so}(4),\ \epsilon \in \boldsymbol{R},\ p,q \in \boldsymbol{H}\Big\}.\end{aligned}$$

In particular, we have

$$\dim((((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}) = 15.$$

Hereafter, $\left(\begin{array}{c|c} D_4 & 0 \\ \hline 0 & 0\end{array}\right)$ will be denoted by D_4.

LEMMA 3.7. (1) *For* $a \in \boldsymbol{H}$, *we define a map* $\widetilde{\alpha}_1(a)$ *of* $\mathfrak{J}^C$ *by*

$$\begin{cases}\xi_1' = \xi_1\\ \xi_2' = \dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2}\cos|a| + i\dfrac{(a,x_1)}{|a|}\sin|a|\\ \xi_3' = -\dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2}\cos|a| + i\dfrac{(a,x_1)}{|a|}\sin|a|\end{cases}$$

$$\begin{cases} x_1' = x_1 + i\dfrac{(\xi_2+\xi_3)a}{|a|}\sin|a| - \dfrac{2(a,x_1)a}{|a|^2}(\sin\dfrac{|a|}{2})^2 \\ x_2' = x_2\cos\dfrac{|a|}{2} + i\dfrac{\overline{x_3a}}{|a|}\sin\dfrac{|a|}{2} \\ x_3' = x_3\cos\dfrac{|a|}{2} + i\dfrac{\overline{ax_2}}{|a|}\sin\dfrac{|a|}{2} \end{cases}$$

Then, $\widetilde{\alpha}_1(a) \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$.

(2) *For* $t \in \boldsymbol{R}$*, we define a map* $\widetilde{\alpha}_{23}(t)$ *of* $\mathfrak{J}^C$ *by*

$$\widetilde{\alpha}_{23}(t)\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & e^{it/2}x_3 & e^{-it/2}\overline{x}_2 \\ e^{it/2}\overline{x}_3 & e^{it}\xi_2 & x_1 \\ e^{-it/2}x_2 & \overline{x}_1 & e^{-it}\xi_3 \end{pmatrix}.$$

Then, $\widetilde{\alpha}_{23}(t) \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$.

Proof.(1) For $a \in \boldsymbol{H}$, we have $i\widetilde{F}_1(a) \in (((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$(Lemma 3.6). Hence, $\widetilde{\alpha}_1(a) = \exp i\widetilde{F}_1(a) \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$.

(2) For $t \in \boldsymbol{R}$, we have $it(E_2-E_3)^{\sim} \in (((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$(Lemma 3.6). Hence, $\widetilde{\alpha}_{23}(t) = \exp it(E_2-E_3)^{\sim} \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$.

We define a 6 dimensional $\boldsymbol{R}$-vector space V^6 by

$$\begin{aligned} V^6 &= \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \mu\tau\lambda P = P, \gamma P = P, \langle P, \tilde{E}_1\rangle = 0, \langle P, \tilde{E}_{-1}\rangle = 0\} \\ &= \left\{P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, 0, 0, 0\right) \,\middle|\, \xi \in C,\ h \in \boldsymbol{H}\right\} \end{aligned}$$

with the norm (see [5] for the definition of $\{\quad,\quad\}$'s)

$$(P,P)_\mu = \frac{1}{2}\{\mu P, P\} = \frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + \overline{h}h.$$

Then, $S^5 = \{P \in V^6 \,|\, (P,P)_\mu = 1\}$ is a 5 dimensional sphere.

Lemma 3.8 $\quad (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}/Spin(5) \simeq S^5$.
In particular, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ *is connected.*

Proof. Since E_7 is commutative with $\tau\lambda$, the group $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ acts on S^5. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^5$ can be transformed to $(i(E_2+E_3),0,0,0) \in S^5$ under the action of $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$. Now, for a given

$$P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, 0, 0, 0\right) \in S^5,$$

choose $t \in \boldsymbol{R}$ such that $e^{it}\xi \in \boldsymbol{R}$. For this $t \in \boldsymbol{R}$, operate $\widetilde{\alpha}_{23}(t)$ (Lemma 3.7(2)) $\in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ on P. Then, we have

$$\widetilde{\alpha}_{23}(t)P = \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & r & h \\ 0 & \overline{h} & -r \end{pmatrix}, 0, 0, 0 \Big) = P_1, \ r \in \boldsymbol{R}.$$

In the case of $h \neq 0$, operate $\widetilde{\alpha}_1(\pi h/2|h|)$ (Lemma 3.7(1)) $\in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ on P_1. Then, we have

$$\widetilde{\alpha}_1(\frac{\pi h}{2|h|})P_1 = \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi' & 0 \\ 0 & 0 & -\tau\xi' \end{pmatrix}, 0, 0, 0 \Big) = P_2 \in S^5, \ \xi' \in C.$$

Here, from $(\tau\xi')\xi' = 1$, $\xi' \in C$, we can put $\xi' = e^{i\theta}, 0 \leq \theta < 2\pi$. Operate $\widetilde{\alpha}_{23}(-\theta)$ on P_2. Then,

$$\widetilde{\alpha}_{23}(-\theta)P_2 = (E_2 - E_3, 0, 0, 0) = P_3.$$

Moreover, operate $\widetilde{\alpha}_{23}(\pi/2)$ on P_3,

$$\widetilde{\alpha}_{23}(\frac{\pi}{2})P_3 = (i(E_2 + E_3), 0, 0, 0) = i\dot{E}_{23}.$$

This shows the transitivity. The isotropy subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ at $\dot{E}_{23}$ is $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}} = Sp(2)$ (Propositions 3.4, 3.5) $= Spin(5)$. Therefore, we have the homeomorphism
$(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}/Spin(5) \simeq S^5$.

Proposition 3.9. $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}} \cong Spin(6)$.

Proof. Since $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ is connected (Lemma 3.8), we can define a homormorphism $\pi : (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}} \to SO(6) = SO(V^6)$ by

$$\pi(\alpha) = \alpha|V^6.$$

It is not difficult to see that $\mathrm{Ker}\varphi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since
$\dim((((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}) = 15$ (Lemma 3.6) $= \dim(\mathfrak{so}(6))$, π is onto. Hence, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}/\boldsymbol{Z}_2 \cong SO(6)$. Therefore, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ is isomorphism to $Spin(6)$ as a double covering group of $SO(6)$.

We shall consider a subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{F_1(\dot{h}e_4),\tilde{E}_1}$ of E_7.

Lemma 3.10. *The Lie algebra* $(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ *of the group* $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ *is given by*

$$(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$$
$$= \Big\{ \varPhi\Big(D_4 + \widetilde{A}_1(p) + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & q \\ 0 & \overline{q} & -\epsilon \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & ix \\ 0 & \overline{ix} & \tau\alpha \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & ix \\ 0 & \overline{ix} & \tau\alpha \end{pmatrix}, 0 \Big)$$

$$\Big|\, D_4 \in \mathfrak{so}(4) \subset \mathfrak{so}(8), \epsilon \in \boldsymbol{R}, \alpha \in C, p, q, x \in \boldsymbol{H} \Big\}.$$

In particular, we have

$$\dim((((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}) = 21.$$

Lemma 3.11. *For $a \in \boldsymbol{R}$, we define maps $\alpha_k(a), k = 2, 3$ of $\mathfrak{P}^C$ by*

$$\alpha_k(a)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (1 + (\cos a - 1)p_k)X - 2(\sin a)E_k \times Y + \eta(\sin a)E_k \\ 2(\sin a)E_k \times X + (1 + (\cos a - 1)p_k)Y - \xi(\sin a)E_k \\ ((\sin a)E_k, Y) + (\cos a)\xi \\ (-(\sin a)E_k, X) + (\cos a)\eta \end{pmatrix},$$

where $p_k : \mathfrak{J}^C \to \mathfrak{J}^C$ is defined by

$$p_k(X) = (X, E_k)E_k + 4E_k \times (E_k \times X), \quad X \in \mathfrak{J}^C.$$

Then, $\alpha_k \in E_7$ and $\alpha_2(a), \alpha_3(b) (a, b \in \boldsymbol{R})$ commute with each other.

Proof. For $\Phi_k(a) = \Phi(0, aE_k, -aE_k, 0) \in \mathfrak{e}_7$, we have $\alpha_k(a) = \exp\Phi_k(a) \in E_7$. Since $[\Phi_2(a), \Phi_3(b)] = 0, \alpha_2(a)$ and $\alpha_3(b)$ are commutative.

We define a 7 dimensional $\boldsymbol{R}$-vector space V^7 by

$$\begin{aligned} V^7 &= \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \mu\tau\lambda P = P, \gamma P = P, \langle P, \tilde{E}_1 \rangle = 0\} \\ &= \Big\{ P = \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, \begin{pmatrix} i\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, -i\eta \Big) \,\Big|\, \xi \in C,\ h \in \boldsymbol{H},\ \eta \in \boldsymbol{R} \Big\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + \overline{h}h + \eta^2.$$

Then, $S^6 = \{P \in V^7 \,|\, (P, P)_\mu = 1\}$ is a 6 dimensional sphere.

Lemma 3.12 $\quad (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}/Spin(6) \simeq S^6$.
In particular, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ is connected.

Proof. The group $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ acts on S^6. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^6$ can be transformed to $(0, -iE_1, 0, i) \in S^6$ under the action of $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$. Now, for a given

$$P = \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, \begin{pmatrix} i\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, -i\eta \Big) \in S^6,$$

choose $a \in \boldsymbol{R}, 0 \le a < \pi/2$ such that $\tan 2a = \dfrac{i2\eta}{\tau\xi - \xi}$(if $\tau\xi - \xi = 0$, then let $a = \pi/4$). Operate $\alpha_{23}(a) := \alpha_2(a)\alpha_3(a) = \exp(\varPhi(0, a(E_2 + E_3), -a(E_2 + E_3), 0))$ (Lemma 3.11) $\in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$(Lemma 3.10) on P. Then, the $\eta-$term of $\alpha_{23}(a)P$ is $(1/2)(\xi - \tau\xi)\sin 2a + i\eta\cos 2a = 0$. Hence,

$$\alpha_{23}(a)P = \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta & m \\ 0 & \overline{m} & -\tau\zeta \end{pmatrix}, 0, 0, 0\Big) = P_1 \in S^5 \subset S^6.$$

Since $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}} (\subset (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1})$ acts transitivity on S^5 (Lemma 3.8), there exist $\beta \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}}$ such that $\beta P_1 = (i(E_2 + E_3), 0, 0, 0) = P_2 \in S^5 \subset S^6$. Moreover, operate $\alpha_{23}(-\pi/4)$ on P_2,

$$\alpha_{23}\big(-\frac{\pi}{4}\big)P_2 = (0, -iE_1, 0, i) = -i\tilde{E}_{-1}.$$

This shows the transitivity. The isotropy subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ at $\tilde{E}_{-1}$ is $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1,\tilde{E}_{-1}} = Spin(6)$(Proposition 3.9). Thus, we have the homeomorphism $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}/Spin(6) \simeq S^6$.

Proposition 3.13. $\quad (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1} \cong Spin(7)$.

Proof. Since $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ is connected (Lemma 3.12), we can define a homormorphism $\pi : (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1} \to SO(7) = SO(V^7)$ by

$$\pi(\alpha) = \alpha|V^7.$$

It is not difficult to see that $\mathrm{Ker}\varphi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim((((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}) = 21$ (Lemma 3.10) $= \dim(\mathfrak{so}(7))$, π is onto. Hence, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}/\boldsymbol{Z}_2 \cong SO(7)$. Therefore, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4),\tilde{E}_1}$ is isomorphism to $Spin(7)$ as a double covering group of $SO(7)$.

We shall consider the subgroup $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ of E_7.

Lemma 3.14. *The Lie algebra* $(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ *of the group* $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ *is given by*

$$\begin{aligned}&(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)} \\ &= \Big\{\varPhi\Big(D_4 + \widetilde{A}_1(p) + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & q \\ 0 & \overline{q} & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix}, \\ &-\frac{3}{2}i\epsilon_1\Big) \,\Big|\, D_4 \in \mathfrak{so}(4) \subset \mathfrak{so}(8), \alpha_k \in C, p, q \in \boldsymbol{H}, x \in \boldsymbol{H}^C, \epsilon_k \in \boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\Big\}.\end{aligned}$$

In particular, we have

$$\dim((((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}) = 28.$$

Hereafter, any element of the Lie algebra $(((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ will be denoted by $\varPhi_8$.

LEMMA 3.15. *For $t \in \boldsymbol{R}$, we define a map $\alpha(t)$ of $\mathfrak{P}^C$ by*

$$\alpha(t)(X, Y, \xi, \eta) = \left(\begin{pmatrix} e^{2it}\xi_1 & e^{it}x_3 & e^{it}\overline{x}_2 \\ e^{it}\overline{x}_3 & \xi_2 & x_1 \\ e^{it}x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\overline{y}_2 \\ e^{-it}\overline{y}_3 & \eta_2 & y_1 \\ e^{-it}y_2 & \overline{y}_1 & \eta_3 \end{pmatrix}, e^{-2it}\xi, e^{2it}\eta \right).$$

Then, $\alpha(t) \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$.

PROOF. For $\varPhi = \varPhi(2itE_1 \vee E_1, 0, 0, -2it) \in (((\mathfrak{e}_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ (Lemma 3.14), we have $\alpha(t) = \exp\varPhi \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ by $E_1 \vee E_1 = (1/3)(2E_1 - E_2 - E_3)^{\sim}$.

We define an 8 dimensional $\boldsymbol{R}$-vector space V^8 by

$$\begin{aligned} V^8 &= \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \mu\tau\lambda P = P, \gamma P = P\} \\ &= \left\{ P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \,\middle|\, \xi, \eta \in C,\ h \in \boldsymbol{H} \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + \overline{h}h + (\tau\eta)\eta.$$

Then, $S^7 = \{P \in V^8 \,|\, (P, P)_\mu = 1\}$ is a 7 dimensional sphere.

LEMMA 3.16. $\quad (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}/Spin(7) \simeq S^7$.
In particular, $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ is connected.

PROOF. The group $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ acts on S^7. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^7$ can be transformed to $(0, E_1, 0, 1) \in S^7$ under the action of $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$. Now, for a given

$$P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \in S^7,$$

choose $t \in \boldsymbol{R}$ such that $e^{-2it}\eta \in i\boldsymbol{R}$. Operate $\alpha(t)$ (Lemma 3.15) $\in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)}$ on P. Then,

$$\alpha(t)P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & h \\ 0 & \overline{h} & -\tau\xi \end{pmatrix}, \begin{pmatrix} i\eta' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, -i\eta' \right) = P_1 \in S^6 \subset S^7,\ \eta' \in \boldsymbol{R}$$

Since $(((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4), \tilde{E}_1} (\subset (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4)})$ acts transitivity on S^6 (Lemma 3.12), there exists $\beta \in (((E_7)^{\kappa,\mu})^{\gamma})_{\dot{F}_1(he_4), \tilde{E}_1}$ such that $\beta P_1 = (0, -iE_1, 0, i) = P_2 \in$

$S^6 \subset S^7$. Moreover, operate $\alpha(-\pi/4)$ (Lemma 3.15) on P_2,

$$\alpha\big(-\frac{\pi}{4}\big)P_2 = (0, E_1, 0, 1) = \tilde{E}_1.$$

This shows the transitivity. The isotropy subgroup $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}$ at $\tilde{E}_1$ is $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4),\tilde{E}_1} = Spin(7)$ (Proposition 3.12). Thus, we have the homeomorphism $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}/Spin(7) \simeq S^7$.

Proposition 3.17. $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)} \cong Spin(8)$.

Proof. Since $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}$ is connected (Lemma 3.16), we can define a homormorphism $\pi : (((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)} \to SO(8) = SO(V^8)$ by

$$\pi(\alpha) = \alpha|V^8.$$

It is not difficult to see that $\mathrm{Ker}\varphi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim((((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}) = 28$ (Lemma 3.14) $= \dim(\mathfrak{so}(8))$, π is onto. Hence, $(((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}/\boldsymbol{Z}_2 \cong SO(8)$. Therefore, $((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}$ is isomorphism to $Spin(8)$ as a double covering group of $SO(8)$.

We shall determine the group structre of $((E_7)^{\kappa,\mu})^\gamma$.

Lemma 3.18. *The Lie algebra $((\mathfrak{e}_7)^{\kappa,\mu})^\gamma$ of the group $((E_7)^{\kappa,\mu})^\gamma$ is given by*

$$\begin{aligned}
&((\mathfrak{e}_7)^{\kappa,\mu})^\gamma \\
&= \Big\{\Phi\Big(D_4 + D_4' + \widetilde{A}_1(p) + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & q \\ 0 & \bar{q} & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix} \\
&-\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix}, -\frac{3}{2}i\epsilon_1\Big) \,\Big|\, D_4, D_4' \in \mathfrak{so}(4) \subset \mathfrak{so}(8), \alpha_k \in C, p, q \in \boldsymbol{H}, \\
&\qquad\qquad x \in \boldsymbol{H}^C, \epsilon_k \in \boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\Big\}.
\end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{e}_7)^{\kappa,\mu})^\gamma) = 34.$$

Proposition 3.19. $((E_7)^{\kappa,\mu})^\gamma \cong (Spin(4) \times Spin(8))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,1), (-1, -1)\}$.

Proof. For $Spin(4) = Sp(1) \times Sp(1) = (((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$ (Propositions 3.1, 3.3) and $Spin(8) = (((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}$ (Proposition 3.17), we define a map $\phi_1 : Spin(4) \times Spin(8) \to ((E_7)^{\kappa,\mu})^\gamma$ by

$$\phi_1(\alpha, \beta) = \alpha\beta.$$

Then, ϕ_1 is well-defined. For $\Phi_4 \in \mathfrak{spin}(4)$ (Lemma 3.2) and $\Phi_8 \in \mathfrak{spin}(8)$ (Lemma 3.14), since $[\Phi_4, \Phi_8] = 0$, we have $\alpha\beta = \beta\alpha$. Hence, ϕ_1 is a homomorphism. It is not difficult to see that $\mathrm{Ker}\phi_1 = \{(1,1), (-1,-1)\} = \boldsymbol{Z}_2$. Since $((E_7)^{\kappa,\mu})^\gamma (\cong (Spin(12))^\gamma$ (see [4],[5])) is connected and $\dim(((E_7)^{\kappa,\mu})^\gamma) = 34$ (Lemma 3.18)$= 6+28 = \dim(\mathfrak{spin}(4) \oplus \mathfrak{spin}(8))$, ϕ_1 is onto. Thus, we have the required isomorphism $(Spin(4) \times Spin(8))/\boldsymbol{Z}_2 \cong ((E_7)^{\kappa,\mu})^\gamma$.

Now, we shall determine the group structure of $(E_7)^{\sigma,\gamma}$.

Lemma 3.20. *The Lie algebra $(\mathfrak{e}_7)^{\sigma,\gamma}$ of the group $(E_7)^{\sigma,\gamma}$ is given by*

$$(\mathfrak{e}_7)^{\sigma,\gamma}$$
$$= \Big\{ \Phi\Big(D_4 + D_4' + \widetilde{A}_1(p) + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & q \\ 0 & \bar{q} & \epsilon_3 \end{pmatrix}^\sim, \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix},$$
$$-\tau \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & x \\ 0 & \overline{x} & \alpha_3 \end{pmatrix}, \nu \Big) \Big| \, D_4, D_4' \in \mathfrak{so}(4) \subset \mathfrak{so}(8), \alpha_k \in C, p, q \in \boldsymbol{H}, x \in \boldsymbol{H}^C,$$
$$\epsilon_k \in \boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \nu \in i\boldsymbol{R} \Big\}.$$

In particular, we have

$$\dim((\mathfrak{e}_7)^{\sigma,\gamma}) = 37.$$

Proposition 3.21. *For $A \in SU(2) = \{A \in M(2,C) \,|\, (\tau\, {}^t\!A)A = E, \det A = 1\}$, we define C-linear transformations $\phi(A)$ of $\mathfrak{P}^C$ by*

$$\phi(A)\Big(\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y}_2 \\ \overline{y}_3 & \eta_2 & y_1 \\ y_2 & \overline{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \Big)$$
$$= \Big(\begin{pmatrix} \xi_1' & x_3' & \overline{x}_2{}' \\ \overline{x}_3{}' & \xi_2' & x_1' \\ x_2' & \overline{x}_1{}' & \xi_3' \end{pmatrix}, \begin{pmatrix} \eta_1' & y_3' & \overline{y}_2{}' \\ \overline{y}_3{}' & \eta_2' & y_1' \\ y_2' & \overline{y}_1{}' & \eta_3' \end{pmatrix}, \xi', \eta' \Big),$$

$$\begin{pmatrix} \xi_1{}' \\ \eta' \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = A \begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix},$$
$$\begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = A \begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix}, \quad \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = (\tau A) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$
$$\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

Then, $\phi(A) \in (E_7)^{\sigma,\gamma}$.

PROOF. Let $\Phi = \Phi(2\nu E_1 \vee E_1, aE_1, -\tau a E_1, \nu), a \in C, \nu \in i\boldsymbol{R}$. Then, $\Phi \in (\mathfrak{e}_7)^{\sigma,\gamma}$(Lemma 3.20). Therefore, for $A = \exp\begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \in SU(2)$, we have $\phi(A) = \exp\Phi \in (E_7)^{\sigma,\gamma}$.

PROPOSITION 3.22. $(E_7)^\sigma \cong (SU(2) \times Spin(12))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, -\sigma)\}$.

PROOF. The isomorphism is induced by the homomorphism $\varphi_1 : SU(2) \times Spin(12) \to (E_7)^\sigma$ by $\varphi_1(A, \delta) = \phi(A)\delta$. (In detail, see [4], [5].)

THEOREM 3.23. $(E_7)^{\sigma,\gamma} \cong (SU(2) \times Spin(4) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$, $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \{(E, 1, 1), (E, \sigma, \sigma)\} \times \{(E, 1, 1), (-E, \gamma, -\sigma\gamma)\}$.

PROOF. For $SU(2)$ (Proposition 3.21), $Spin(4) = (((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(h),\tilde{E}_1,\tilde{E}_{-1},\dot{E}_{23}}$ (Propositions 3.1, 3.3) and $Spin(8) = (((E_7)^{\kappa,\mu})^\gamma)_{\dot{F}_1(he_4)}$(Proposition 3.17), we define a map $\varphi : SU(2) \times Spin(4) \times Spin(8) \to (E_7)^{\sigma,\gamma}$ by

$$\varphi(A, \alpha, \beta) = \phi(A)\alpha\beta.$$

Then, φ is well-defined. From Propositions 3.19, 3.22, φ is a homomorphim. We shall show that φ is onto. Let $\rho \in (E_7)^{\sigma,\gamma}$. Since $(E_7)^{\sigma,\gamma} \subset (E_7)^\sigma$, there exist $A \in SU(2)$ and $\delta \in Spin(12)$ such that $\rho = \varphi_1(A, \delta)$ (Proposition 3.22). Now, From $\gamma\rho\gamma = \rho$, we have $\phi(A)(\gamma\delta\gamma) = \phi(A)\delta$. Hence,

$$\left\{\begin{array}{l} A = A \\ \gamma\delta\gamma = \delta \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} A = -A \\ \gamma\delta\gamma = -\sigma\delta \end{array}\right.$$

The latter case is impossible because $A = 0$ is false. In the former case, from Proposition 3.19, there exist $\alpha \in Spin(4)$ and $\beta \in Spin(8)$ such that $\delta = \phi_1(\alpha, \beta)$. Hence, we have

$$\begin{aligned} \rho &= \varphi_1(A, \delta) = \phi(A)\delta = \phi(A)\phi_1(\alpha, \beta) \\ &= \phi(A)\alpha\beta = \varphi(A, \alpha, \beta). \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} \mathrm{Ker}\varphi &= \{(E, 1, 1), (E, \sigma, \sigma), (-E, \gamma, -\sigma\gamma), (-E, \sigma\gamma, -\gamma)\} \\ &= \{(E, 1, 1), (E, \sigma, \sigma)\} \times \{(E, 1, 1), (-E, \gamma, -\sigma\gamma)\} \\ &= \boldsymbol{Z}_2 \times \boldsymbol{Z}_2. \end{aligned}$$

Thus, we have the required isomorphism $(SU(2) \times Spin(4) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2) \cong (E_7)^{\sigma,\gamma}$.

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