ON INVARIANT GIBBS MEASURES CONDITIONED ON MASS AND MOMENTUM

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ABSTRACT. We construct a Gibbs measure for the nonlinear Schrödinger equation (NLS) on the circle, conditioned on prescribed mass and momentum:

$$d\mu_{a,b} = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}}|u|^2=a\}}\mathbf{1}_{\{i\int_{\mathbb{T}}u\overline{u}_x=b\}}e^{\pm\frac{1}{p}\int_{\mathbb{T}}|u|^p-\frac{1}{2}\int_{\mathbb{T}}|u|^2}dP$$

for $a\in\mathbb{R}^+$ and $b\in\mathbb{R}$, where P is the complex-valued Wiener measure on the circle. We also show that $\mu_{a,b}$ is invariant under the flow of NLS. We note that $i\int_{\mathbb{T}}u\overline{u}_x$ is the Lévy stochastic area, and in particular that this is invariant under the flow of NLS.

CONTENTS

1. INTRODUCTION

We consider the periodic nonlinear Schrödinger equation (NLS) on the circle:

$$iu_t+u_{xx}=\pm|u|^{p-2}u,\qquad (x,t)\in\mathbb{T}\times\mathbb{R}\tag{1.1}$$

where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Recall that (1.1) is a Hamiltonian PDE with Hamiltonian:

$$H(u)=\frac{1}{2}\int_{\mathbb{T}}|u_x|^2\pm\frac{1}{p}\int_{\mathbb{T}}|u|^p.\tag{1.2}$$

Indeed, (1.1) can be written as

$$u_t=i\frac{\partial H}{\partial\bar{u}}.\tag{1.3}$$

Recall that (1.1) also conserves the mass $M(u)=\int|u|^2$ and the momentum $P(u)=i\int u\overline{u}_x$. Moreover, the cubic NLS ($p=4$) is known to be completely integrable [ZS, GKP] in the sense that it enjoys the Lax pair structure and thus there exist infinitely many conservation laws for (1.1). For general $p\neq 4$, the mass M, the momentum P, and the Hamiltonian H are the only known conservation laws. Our main goal in this paper is to construct an invariant Gibbs measure conditioned on mass and momentum.

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First, consider a Hamiltonian flow on $\mathbb{R}^{2n}$:

$$\dot{p}_i = \tfrac{\partial H}{\partial q_j}, \quad \dot{q}_i = -\tfrac{\partial H}{\partial p_j} \tag{1.4}$$

with Hamiltonian $H(p,q) = H(p_1,\dots,p_n,q_1,\dots,q_n)$. Then, Liouville's theorem states that the Lebesgue measure $\prod_{j=1}^n dp_j dq_j$ on $\mathbb{R}^{2n}$ is invariant under the flow. Then, it follows from the conservation of the Hamiltonian H that the Gibbs measure $e^{-H(p,q)}\prod_{j=1}^n dp_j dq_j$ is invariant under the flow of (1.4). Now note that if $F(p,q)$ is any (reasonable) function that is conserved under the flow of (1.4), then the measure $d\mu_F = F(p,q)e^{-H(p,q)}\prod_{j=1}^n dp_j dq_j$ is also invariant.

By viewing (1.1) as an infinite dimensional Hamiltonian system, one can consider the issue of invariant Gibbs measures for (1.1). Lebowitz-Rose-Speer [LRS] constructed Gibbs measures of the form

$$d\mu = Z^{-1}e^{-H(u)}\prod_{x\in\mathbb{T}} du(x) = Z^{-1}e^{\mp\frac{1}{p}\int_{\mathbb{T}}|u|^p}\underbrace{e^{-\frac{1}{2}\int_{\mathbb{T}}|u_x|^2}\prod_{x\in\mathbb{T}} du(x)}_{=\text{ Wiener measure } P} \tag{1.5}$$

as a weighted Wiener measure on $\mathbb{T}$. In the focusing case, i.e. with the plus sign in (1.5), the result only holds for $p<6$ with an L^2-cutoff $\mathbf{1}_{\{\int|u|^2\leq B\}}$ for any $B>0$, and for $p=6$ with sufficiently small B. By analogy with the finite dimensional case, we expect such a Gibbs measure μ is invariant under the flow of (1.1). (Recall that the L^2-norm is conserved.) In addressing the question of invariance of μ, we need to have a well-defined flow on the support of μ. However, as a weighted Winer measure, the regularity of μ is inherited from that of the Wiener measure. i.e. μ is supported on $H^s(\mathbb{T})\setminus H^{\frac{1}{2}}(\mathbb{T})$, $s<\frac{1}{2}$. In [B1], Bourgain proved local well-posedness of (1.1)

- in $L^2(\mathbb{T})$ for (sub-) cubic NLS ($p\leq 4$),
- in $H^s(\mathbb{T})$, $s>0$, for (sub-) quintic NLS ($4<p\leq 6$),
- in $H^s(\mathbb{T})$, $s>\frac{1}{2}-\frac{1}{p}$, for $p>6$.

Using the Fourier analytic approach, he [B2] continued the study of Gibbs measures and proved the invariance of μ under the flow of NLS.

Once the invariance of the Gibbs measure μ is established, we can regard the flow map of (1.1) as a measure-preserving transformation on an (infinite-dimensional) phase space, say $H^{\frac{1}{2}-\epsilon}$, equipped with the Gibbs measure μ. Then, it follows from Poincaré recurrence theorem that almost all the points of the phase space are stable according to Poisson [Z], i.e. if $\mathcal{S}_t$ denotes a flow map of (1.1): $u_0\mapsto u(t)=\mathcal{S}_t u_0$, then for almost all u_0, there exists a sequence $\{t_n\}$ tending to ∞ such that $\mathcal{S}_{t_n}u_0\to u_0$. Moreover, such dynamics is also multiply recurrent in view of Furstenberg's multiple recurrence theorem [F]: let A be any measurable set with $\mu(A)>0$. Then, for any integer $k>1$, there exists $n\neq 0$ such that $\mu(A\cap\mathcal{S}_nA\cap\mathcal{S}_{2n}A\cap\cdots\cap\mathcal{S}_{(k-1)n}A)>0$. Note that this recurrence property is known to hold only in the support of the Gibbs measure, i.e. not for smooth functions.

Then, one of the natural questions, posed by Lebowitz-Rose-Speer [LRS] and Bourgain [B4], is the ergodicity of the invariant Gibbs measure μ. i.e. is the phase space irreducible under the dynamics, or can it be decomposed into disjoint subsets, where the dynamics is recurrent within each disjoint component? In order to ask such a question, one needs to prescribe the L^2-norm since it is an integral of motion for (1.1). It is not difficult to see that the momentum is also finite almost surely on the support of the Gibbs measure. Indeed, if

u is distributed according to the Wiener measure, then it can be represented as[1]

$$u(x) = \sum_{n \neq 0} \frac{g_n}{2\pi n} e^{2\pi i n x}, \tag{1.6}$$

where $\{g_n\}_{n\neq 0}$ is a family of independent standard complex-valued Gaussian random variables, i.e. its real and imaginary parts are independent Gaussian random variables with mean zero and variance 1. Then, we can write the momentum as

$$P(u) = i \int u \overline{u}_x = \sum_{n \neq 0} \frac{|g_n(\omega)|^2}{2\pi n} = \sum_{n \geq 1} \frac{|g_n(\omega)|^2 - |g_{-n}(\omega)|^2}{2\pi n}.$$

Thus, we have $\mathbb{E}\big[\big(P(u)\big)^2\big] \lesssim \sum_{n\geq 1} n^{-2} < \infty$.[2] Hence, $|P(u)| < \infty$ a.s. In the following, we construct invariant Gibbs measures with prescribed L^2-norm and momentum as the first step in studying finer dynamical properties of the NLS flow equipped with the invariant Gibbs measure, viewed as an infinite-dimensional dynamical system with a measure-preserving transformation.

Remark 1.1. Recall that the cubic NLS ($p = 4$) is completely integrable. Hence, it makes sense to pose a question of ergodicity only for $p \neq 4$. See [LRS].

There are infinitely many conservation laws for the cubic NLS, with the leading term of the form $\int_{\mathbb{T}} |\partial_x^k u|^2 dx$, roughly corresponding to the H^k-norm, and of the form $\int_{\mathbb{T}} u\, \partial_x^{2k+1} \overline{u}\, dx$, $k \in \mathbb{N} \cup \{0\}$. See [FT, ZM]. By (1.6), we can easily see that all these conservation laws, except for the L^2-norm and momentum, are almost surely divergent under the Gibbs measure. Thus, it may seem that the L^2-norm and momentum are the only conserved quantities which are finite a.s. in the support of the Gibbs measure. However, from a different perspective, we have a different set of infinitely many conserved quantities for (1.1), namely the spectrum of the Zakharov-Shabat operator L (also called the Dirac operator) appearing in the Lax pair formulation of (1.1): $\partial_t L = [B, L]$ (with some appropriate B.) These are finite under the Gibbs measure. Expressing the flow of (1.1) in the Liouville coordinates (or rather in the Birkhoff coordinates) with actions and angles (which are determined in terms of the spectral data), the flow basically becomes trivial. See [GKP].

In constructing a Gibbs measure conditioned on mass and momentum, we first condition the Wiener measure on mass and momentum. Recall that if u is distributed according to the Wiener measure P given by[3]

$$dP = Z^{-1} e^{-\frac{1}{2}\int_{\mathbb{T}} |u|^2 - \frac{1}{2}\int_{\mathbb{T}} |u_x|^2} \prod_{x \in \mathbb{T}} du(x), \tag{1.7}$$

then it can be represented as

$$u(x) = \sum_{n \neq 0} \frac{g_n}{\sqrt{1 + 4\pi^2 n^2}} e^{2\pi i n x}. \tag{1.8}$$

[1] We ignore the zero-frequency issue here. See (1.8) below.

[2] We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some $C > 0$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$.

[3] The mass is added to take care of the zeroth frequency. We still refer to P in (1.7) and u in (1.8) as the Wiener measure and the Brownian motion, respectively.

Note that (1.8) is basically the Fourier-Wiener series for the Brownian motion (except for the zeroth mode.) Given $a > 0$ and $b \in \mathbb{R}$, define the conditional Wiener measures $P_\varepsilon = P_{\varepsilon,a,b}$, $\varepsilon > 0$, as follows. Given a measurable set E, we define $P_\varepsilon(E)$ by

$$P_\varepsilon(E) = P\bigg(E\,\Big|\,\int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a),\, i\int_{\mathbb{T}} u\overline{u}_x \in B_\varepsilon(b)\bigg), \tag{1.9}$$

where $A_\varepsilon(a)$ and $B_\varepsilon(b)$ are neighborhoods shrinking nicely[4] to a and b as $\varepsilon \to 0$. Here $P(C \mid D) = P(C \cap D)/P(D)$ is the standard, naive, conditional probability given by Bayes' rule. In terms of the density, we have

$$dP_\varepsilon = \hat{Z}_\varepsilon^{-1}\mathbf{1}_{\{\int_{\mathbb{T}}|u|^2 \in A_\varepsilon(a)\}}\mathbf{1}_{\{i\int_{\mathbb{T}} u\overline{u}_x \in B_\varepsilon(b)\}}dP. \tag{1.10}$$

Now, we would like to define the conditioned measure

$$P_0(E) = P_{0,a,b}(E) = P\bigg(E\,\Big|\,\int_{\mathbb{T}} |u|^2 = a,\, i\int_{\mathbb{T}} u\overline{u}_x = b\bigg)$$

by $P_0 = \lim_{\varepsilon\to 0} P_\varepsilon$. Namely, we define P_0 by

$$P_0(E) := \lim_{\varepsilon \to 0} P\bigg(E\,\Big|\,\int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a),\, i\int_{\mathbb{T}} u\overline{u}_x \in B_\varepsilon(b)\bigg). \tag{1.11}$$

Note that the normalization constant $\hat{Z}_\varepsilon$ in (1.10) tends to 0 as $\varepsilon \to 0$. Hence, some care is needed. We discuss details in Subsection 2.1.

Finally, we define the conditioned Gibbs measure $\mu_0 = \mu_{a,b}$ in terms of the Wiener measure $P_0 = P_{0,a,b}$ conditioned on mass and momentum, by setting

$$d\mu_0 = Z_0^{-1} e^{\mp\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_0. \tag{1.12}$$

In the defocusing case, this clearly defines a probability measure since $e^{-\frac{1}{p}\int_{\mathbb{T}}|u|^p} \leq 1$. In the focusing case, we need to show that

$$e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p} \in L^1(dP_0). \tag{1.13}$$

Lebowitz-Rose-Speer [LRS] and Bourgain [B2] proved a similar integrability result of the weight $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}$ with respect to the (unconditioned) Wiener measure P defined in (1.7). Bourgain's argument was based on dyadic pigeonhole principle and a large deviation estimate (see Lemma 4.2 in [OQV].) In Subsection 2.2, we follow Bourgain's argument and prove (1.13) by dyadic pigeonhole principle and a large deviation estimate for P_0. This large deviation estimate for P_0 is by no means automatic, and we need to deduce it by establishing a *uniform* large deviation estimate for the conditioned Wiener measures P_ε, $\varepsilon > 0$ (see Lemma 2.1 below.) As a result, we obtain the uniform integrability result

$$\mathbb{E}_{P_\varepsilon}\Big[e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}\Big] \leq C_p < \infty$$

for all sufficiently small $\varepsilon \geq 0$. We point out that the proof of Lemma 2.1 (and hence the argument in Subsection 2.1) is the heart of this paper.

We state the main theorem. The proof is presented in in the next section.

[4]See Subsection 2.1 for the definition.

Theorem 1. *Let $a > 0$ and $b \in \mathbb{R}$. For $p > 2$, let μ_0 be the Gibbs measure $\mu_0 = \mu_{a,b}$ conditioned on mass and momentum defined in* (1.12). *Also, assume that $p \leq 6$ in the focusing case. Then, μ_0 is a well-defined probability measure (with sufficiently small mass a when $p = 6$ in the focusing case), absolutely continuous to the conditioned Wiener measure P_0. Moreover, μ_ε converges weakly to μ_0 as $\varepsilon \to 0$, where μ_ε is defined by*

$$d\mu_\varepsilon := Z_\varepsilon^{-1} e^{\mp \frac{1}{p}\int_{\mathbb{T}} |u|^p} dP_\varepsilon. \tag{1.14}$$

It follows from invariance of the Gibbs measure μ in (1.5) (with an L^2-cutoff in the focusing case) the conservation of mass and momentum that μ_ε is invariant under the flow of (1.1) for each *fixed* $\varepsilon > 0$. As a corollary to Theorem 1, we obtain invariance of the conditioned Gibbs measure μ_0.

Theorem 2. *Let $a > 0$, $b \in \mathbb{R}$, and $p > 2$ be as in Theorem 1. Then, the conditioned Gibbs measure $\mu_0 = \mu_{a,b}$ defined in* (1.12) *is invariant under the flow of NLS* (1.1).

We conclude this introduction with several remarks. The first is about conditional probabilities.

Remark 1.2. A natural way to proceed with this construction is to start with the (unconditioned) Gibbs measure μ in (1.5) on the space Ω, which is the space of continuous complex-valued functions on the circle, with the topology of uniform convergence and the Borel σ-field $\mathcal{F}$. This is a complete separable metric space. Let $\mathcal{G}$ be the sub σ-field generated by the measurable maps $\int_{\mathbb{T}} |u|^2$ and $i\int_{\mathbb{T}} u\overline{u}_x$. There is a general theorem which guarantees the existence of a conditional probability, i.e. a family of measures μ_u, $u \in \Omega$ such that (i) for any $A \in \mathcal{F}$, $\mu_u(A)$ is measurable with respect to $\mathcal{G}$ as a function of u; (ii) for any $A \in \mathcal{G}$ and $B \in \mathcal{F}$, $\mu(A \cap B) = E_\mu[\mathbf{1}_A \mu_u(B)]$. It follows from (i) and (ii) that given $B \in \mathcal{F}$, we have

$$\mu_u(B) = \mu_{\int_{\mathbb{T}} |u|^2,\, i\int_{\mathbb{T}} u\overline{u}_x}(B) \tag{1.15}$$

for μ-almost every u. The sets of measure zero, on which (1.15) fails, depend on $B \in \mathcal{F}$, and thus their union could be a set of nontrivial measure. Hence, one needs some regularity. The best that can be said in such a general context is that if $\mathcal{G}$ is countably generated (and one can check ours is), then μ_u is a *regular* conditional probability in the sense that (iii) $\mu_u(A) = \mathbf{1}_A(u)$ for $A \in \mathcal{G}$. In our context, this reassures us that our conditioned Gibbs measure $\mu_0 = \mu_{a,b}$ gives mass one to u with $\int_{\mathbb{T}} |u|^2 = a$ and $i\int_{\mathbb{T}} u\overline{u}_x = b$. However, we only know that this property holds for *almost every* a and b, and there is no soft way out to obtain the same for *all* a and b. (Another way to think of this is that applying the Lebesgue differentiation theorem to (ii) gives Theorem 1 for almost every a and b.) Since we want our conditioned measures to be defined for every value of a and b, we have to define them directly. For the conditioned Wiener measure P_0, which is just a Gaussian measure, this is straightforward. In this case, we can even use the fact that the distributions of a and b are basically explicit. However, for the Gibbs measure $\mu_{a,b}$, it requires hard analysis.

Remark 1.3. Consider the (generalized) Korteweg-de Vries equation (gKdV):

$$u_t + u_{xxx} = \pm u^{p-2} u_x. \tag{1.16}$$

For an integer $p \geq 3$, (1.1) is a Hamiltonian PDE with Hamiltonian:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} u_x^2 \pm \frac{1}{p}\int_{\mathbb{T}} u^p, \tag{1.17}$$

and (1.16) can be written as $u_t = \partial_x \frac{dH}{du}$. Also recall that (1.16) preserves the mean $\int_{\mathbb{T}} u$ and the L^2-norm. Bourgain [B2] constructed Gibbs measures of the form (1.5) (with an appropriate L^2-cutoff $\mathbf{1}_{\{\int |u|^2 \leq B\}}$ unless it is defocusing when p is even) for (1.16), and proved its invariance under the flow for $p = 3, 4$. Recently, Richards [R] established invariance of the Gibbs measure for (1.16) when $p = 5$. In an attempt to study more dynamical properties of (1.16), one can construct Gibbs measure conditioned on mass by an argument similar to Theorem 1. In this case, an analogue of Theorem 2 holds for all (even) p when (1.16) is defocusing but only for $p \leq 5$ when it is non-defocusing, due to lack of a result on invariance of the Gibbs measure for focusing quintic ($p = 6$) KdV. Note that KdV ($p = 3$) and mKdV ($p = 4$) are completely integrable. Hence, a question of ergodicity can be posed only for $p \geq 5$. See Remark 1.1.

Remark 1.4. An interesting but straightforward comment is that the momentum $P(u)$ is nothing but the Lévy stochastic area of the planar loop $(\operatorname{Re} u(x), \operatorname{Im} u(x))$, $0 \leq x < 2\pi$,

$$P(u) = i \int_{\mathbb{T}} u \overline{u}_x = \int_{\mathbb{T}} (\operatorname{Re} u)\, d(\operatorname{Im} u) - (\operatorname{Im} u)\, d(\operatorname{Re} u). \tag{1.18}$$

Note that this is not the actual area enclosed by the loop, but a signed version. A Brownian loop has infinitely many self-intersections. Regularizing the Brownian loop gives a loop with finitely many self-intersections. The 'area' is then computed through the path integral above, with each subregion bounded by non-intersecting part of the loop having area counted positive or negative depending on whether the boundary is traversed in the counterclockwise or clockwise direction, respectively. This includes the fact that the areas inside internal loops are multiply counted. Removing the regularization gives the Lévy stochastic area. Remarkably, unlike other stochastic integrals, the limit does not depend on the regularization procedure. For example, one can check directly that the Itô (left endpoint rule in the Riemann sum) and Stratonovich (midpoint rule) versions of (1.18) give the same result. The stochastic area has attracted a great deal of attention. Lévy [L] found the exact expression $\frac{1}{4}(\cosh(x/2))^{-2}$ for its density under the standard Brownian motion measure. Our base Gaussian measure (1.7) is almost the same as the standard Brownian motion, and the analogous computation can be performed (see Section 2.1.) Our Gibbs measures $\mu_0 = \mu_{a,b}$ are absolutely continuous with respect to the base Brownian motion, so most of the results about the stochastic area continue to hold, though, of course, there are no longer any exact formulas. The Lévy area is basically the only new element when one moves from the Wiener-Itô chaos of order one to order two. Therefore, it is a natural object to supplement the Brownian path itself, and this is the basis of the rough path theory [LQ]. It seems a remarkable fact that the flow of NLS preserves the Lévy area.

2. Proof of Theorem 1: Construction of the conditioned Gibbs measures

2.1. **Wiener measure conditioned on mass and momentum.** In this subsection, we construct the Wiener measure P_0 conditioned on mass a and momentum b for any *fixed* $a > 0$ and $b \in \mathbb{R}$. Given P_ε as in (1.10), we define P_0 as a limit of P_ε by (1.11), where E is an arbitrary set in the σ-field $\mathcal{F}$. In the following, we show that (1.11) indeed defines a probability measure. For this purpose, we can simply take E to be in some generating family of $\mathcal{F}$. Let us choose the increasing family $\mathcal{F}_N = \sigma(g_n, ; |n| \leq N)$ as such a generating family of $\mathcal{F}$.

Fix a nonnegative integer N and a Borel set F in $\mathbb{C}^{2N+1}$. Let $E = \{\omega : (g_n; |n| \leq N) \in F\}$. Then, by (1.9), we have

$$P_\varepsilon(E) = P\bigg((g_n; |n| \leq N) \in F \,\bigg|\, \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a),\, i\int_{\mathbb{T}} u\overline{u}_x \in B_\varepsilon(b)\bigg),$$

where $A_\varepsilon(a)$ and $B_\varepsilon(b)$ are neighborhoods shrinking nicely to a and b as $\varepsilon \to 0$. That is,

(a) For each $\varepsilon > 0$, we have

$$A_\varepsilon(a) \subset (a-\varepsilon, a+\varepsilon) \quad \text{and} \quad B_\varepsilon(b) \subset (b-\varepsilon, b+\varepsilon).$$

(b) There exists $\alpha > 0$, independent of ε, such that

$$|A_\varepsilon(a)| > \alpha\varepsilon \quad \text{and} \quad |B_\varepsilon(b)| > \alpha\varepsilon.$$

By (1.8) and independence of $\{g_n\}_{|n|\leq N}$ and $\{g_n\}_{|n|>N}$, we have

$$P_\varepsilon(E) = \int_F \frac{P\Big(\sum_{|n|\geq N+1} \langle\widetilde{n}\rangle^{-2}|g_n|^2 \in A_\varepsilon(\widetilde{a}),\, \sum_{|n|\geq N+1} \langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2 \in B_\varepsilon(\widetilde{b})\Big)}{P\Big(\sum_n \langle\widetilde{n}\rangle^{-2}|g_n|^2 \in A_\varepsilon(a),\, \sum_n \langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2 \in B_\varepsilon(b)\Big)} \times \frac{e^{-\frac{1}{2}\sum_{|n|\leq N}|g_n|^2}}{(2\pi)^{2N+1}} \prod_{|n|\leq N} dg_n, \tag{2.1}$$

where $\widetilde{n} = 2\pi n$, $\langle\widetilde{n}\rangle = \sqrt{1+\widetilde{n}^2}$, and $A_\varepsilon(\widetilde{a})$ and $B_\varepsilon(\widetilde{b})$ are translates of $A_\varepsilon(a)$ and $B_\varepsilon(b)$ centered at

$$\widetilde{a} = a - \sum_{|n|\leq N} \langle\widetilde{n}\rangle^{-2}|g_n|^2, \quad \text{and} \quad \widetilde{b} = b - \sum_{|n|\leq N} \langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2, \tag{2.2}$$

respectively.

Now, define the density $f_N(a,b)$ by

$$f_N(a,b)\, dadb = P\bigg(\sum_{|n|\geq N} \langle\widetilde{n}\rangle^{-2}|g_n|^2 \in da,\, \sum_{|n|\geq N} \langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2 \in db\bigg).$$

By computing the characteristic function of f_N, we have

$$\begin{aligned}
\hat{f}_N(s,t) &= \mathbb{E}\bigg[\exp\Big(is\sum_{|n|\geq N} \langle\widetilde{n}\rangle^{-2}|g_n|^2 + it\sum_{|n|\geq N} \langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\Big)\bigg] \\
&= \prod_{|n|\geq N} \mathbb{E}\Big[e^{i(s\langle\widetilde{n}\rangle^{-2} + t\langle\widetilde{n}\rangle^{-2}\widetilde{n})|g_n|^2}\Big] \\
&= \prod_{|n|\geq N} \frac{1}{1-2i(s\langle\widetilde{n}\rangle^{-2} + t\,\langle\widetilde{n}\rangle^{-2}\widetilde{n})} \qquad (2.3)\\
&= \prod_{n\geq N} \frac{1}{\big(1-2i(s\langle\widetilde{n}\rangle^{-2} + t\,\langle\widetilde{n}\rangle^{-2}\widetilde{n})\big)\big(1-2i(s\langle\widetilde{n}\rangle^{-2} - t\,\langle\widetilde{n}\rangle^{-2}\widetilde{n})\big)} \\
&= \prod_{n\geq N} \frac{1}{1+4\langle\widetilde{n}\rangle^{-4}(\widetilde{n}^2t^2 - s^2) - 4is\langle\widetilde{n}\rangle^{-2}}. \qquad (2.4)
\end{aligned}$$

It follows from (2.3) that $|\hat{f}_N(s,t)| \leq 1$. Thus, we have

$$\int_{|s|\leq 1,\, |t|\leq 1} |\hat{f}_N(s,t)|\, dsdt \leq C_1 < \infty. \tag{2.5}$$

Next, we consider the contribution from $\{|t| > 1\}$. Fix s and $|t| > 1$. In this case, if $|1 + 4\langle\widetilde{n}\rangle^{-4}(\widetilde{n}^2t^2 - s^2)| \ll t^2$, then we have $c_n s^2 < t^2 < c_n' s^2$. Thus, we have

$$|1 + 4\langle\widetilde{n}\rangle^{-4}(\widetilde{n}^2t^2 - s^2) - 4is\langle\widetilde{n}\rangle^{-2}| \gtrsim \max(\langle s\rangle, t^2, |t|), \tag{2.6}$$

where the implicit constant depends on n. It follows from (2.3) that each factor in (2.4) is less than or equal to 1. Thus, from (2.4) and (2.6), we have

$$|\hat{f}_N(s,t)| \leq \prod_{n=N}^{N+3} \frac{1}{1 + 4\langle\widetilde{n}\rangle^{-4}(\widetilde{n}^2t^2 - s^2) - 4is\langle\widetilde{n}\rangle^{-2}} \leq C_2(N)\frac{1}{\langle s\rangle^2}\frac{1}{t^2}$$

for $|t| > 1$, where $C_2(N)$ is at most a power of N. Hence, we have

$$\int_{|t|>1} |\hat{f}_N(s,t)|\, dsdt \leq C_2(N)\bigg(\int \langle s\rangle^{-2} ds\bigg)\bigg(\int_{|t|>1} t^{-2}dt\bigg) < \infty. \tag{2.7}$$

Lastly, we consider the contribution from $\{|s| > 1, |t| \leq 1\}$. As before, it follows from (2.4) that $|\hat{f}_N(s,t)| \leq C_3(N)s^{-2}$, where $C_3(N)$ is at most a power of N. Hence, we have

$$\int_{|s|>1,\,|t|\leq 1} |\hat{f}_N(s,t)|\, dsdt \leq C_3'(N)\int_{|s|>1} s^{-2}ds < \infty. \tag{2.8}$$

Therefore, from (2.5), (2.7), and (2.8), we have

$$\|\hat{f}_N\|_{L^1_{s,t}} < C(N) < \infty.$$

Note that $C(N)$ is at most a power of N. This shows that $\hat{f}_N \in L^1(\mathbb{R}^2)$. Hence, f_N is bounded and uniformly continuous. In particular, we have, for any $N \geq 0$,

$$\begin{aligned}\frac{P\Big(\sum_{|n|\geq N}\langle n\rangle^{-2}|g_n|^2 \in A_\varepsilon(\widetilde{a}),\, \sum_{|n|\geq N}\langle n\rangle^{-2}n|g_n|^2 \in B_\varepsilon(\widetilde{b})\Big)}{|A_\varepsilon(\widetilde{a}) \times B_\varepsilon(\widetilde{b})|} \\ = \frac{1}{|A_\varepsilon(\widetilde{a}) \times B_\varepsilon(\widetilde{b})|}\int_{A_\varepsilon(\widetilde{a})\times B_\varepsilon(\widetilde{b})} f_N(a',b')da'db' \longrightarrow f_N(\widetilde{a},\widetilde{b}),\end{aligned} \tag{2.9}$$

as $\varepsilon \to 0$. By the uniform continuity of f_N, this convergence is uniform in $\widetilde{a}$ and $\widetilde{b}$.

In taking the limit of (2.1) as $\varepsilon \to 0$, the expression $f_0(a,b)$ appears in the denominator. Hence, we need to show that $f_0 > 0$ everywhere. First, write f_0 as $f_0 = f_1 *_a \chi_2^2$, where χ_2^2 is the density for the (rescaled) chi square distribution with two degrees of freedom, corresponding to $|g_0|^2 = (\operatorname{Re} g_0)^2 + (\operatorname{Im} g_0)^2$, and $*_a$ denotes the convolution only in the first variable of f_1.

Now, suppose that $f_0(a^*, b^*) = 0$ for some a^* and b^*. Then, from

$$0 = f_0(a^*,b^*) = \int f_1(a^* - x, b^*)\chi_2^2(x)dx$$

and the positivity of χ_2^2, we must have $f_1(a, b^*) = 0$ for any a. Write f_1 as $f_1 = f_2 * g$, where $g(a,b) = \delta_{\frac{b}{2\pi}}(a) \otimes \chi_4^2(b)$ and χ_4^2 is the density for the (rescaled) chi square distribution with four degrees of freedom, corresponding to

$$(1+4\pi^2)^{-2}2\pi(|g_1|^2 + |g_{-1}|^2) = (1+4\pi^2)^{-2}2\pi\big((\operatorname{Re} g_1)^2 + (\operatorname{Im} g_1)^2 + (\operatorname{Re} g_{-1})^2 + (\operatorname{Im} g_{-1})^2\big).$$

Then, we have

$$\begin{aligned} 0 = f_1(a, b^*) &= \int f_2(a - x, b^* - y)\delta_{\frac{y}{2\pi}}(x) \otimes \chi_4^2(y)dxdy \\ &= \int f_2(a - \frac{y}{2\pi}, b^* - y)\chi_4^2(y)dy \quad \text{for any } a. \end{aligned}$$

From the positivity of χ_4^2 and $f_2 \geq 0$, this implies $f_2(a, b) = 0$ for any a and b since $\mathbb{R}^2 = \{(a - \frac{y}{2\pi}, b^* - y) : a, y \in \mathbb{R}\}$. This contradicts with the fact that f_2 is a probability density. Hence, $f_0(a, b) > 0$ for any a and b.

Putting everything together, we have

$$\frac{P\Big(\sum_{|n|\geq N+1}\langle\widetilde{n}\rangle^{-2}|g_n|^2 \in A_\varepsilon(\widetilde{a}),\, \sum_{|n|\geq N+1}\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2 \in B_\varepsilon(\widetilde{b})\Big)}{P\Big(\sum_n\langle\widetilde{n}\rangle^{-2}|g_n|^2 \in A_\varepsilon(a),\, \sum_n\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2 \in B_\varepsilon(b)\Big)} \longrightarrow \frac{f_{N+1}(\widetilde{a}, \widetilde{b})}{f_0(a, b)}, \quad (2.10)$$

where the convergence is uniform in $\widetilde{a}$ and $\widetilde{b}$. Moreover, the right hand side of (2.10) is uniformly bounded for small $\varepsilon > 0$ (for *fixed* a and b), since $\|f_{N+1}\|_{L^\infty} \leq \|\hat{f}_{N+1}\|_{L^1} < \infty$. Hence, by (1.11), (2.1), and Lebesgue dominated convergence theorem, we have

$$P_0(E) = \lim_{\varepsilon\to 0} P_\varepsilon(E) = \int_F \frac{f_{N+1}(\widetilde{a}, \widetilde{b})}{f_0(a, b)} \frac{e^{-\frac{1}{2}\sum_{|n|\leq N}|g_n|^2}}{(2\pi)^{2N+1}} \prod_{|n|\leq N} dg_n.$$

This shows that P_0 is a well-defined probability measure. Lastly, note that it basically follows from the definition that P_ε converges weakly to P_0.

2.2. **Gibbs measure conditioned on mass and momentum.** In the previous subsection, we constructed the Wiener measure P_0 conditioned on mass and momentum as a limit of conditioned Wiener measures P_ε. In this subsection, we define the conditioned Gibbs measure $\mu_0 = \mu_{a,b}$ by (1.12). In the defocusing case, (1.12) defines a probability measure. In the focusing case, however, we need to show (1.13); the weight $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}$ is integrable with respect to P_0 for $p \leq 6$ (with sufficiently small mass when $p = 6$.)

Bourgain [B2] proved a similar integrability result of the weight $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}$ with respect to the (unconditioned) Wiener measure P in (1.7) via dyadic pigeonhole principle and a large deviation estimate. In the following, we also use dyadic pigeonhole principle and a large deviation estimate (for the conditioned Wiener measure P_0) to show that the conditioned Gibbs measure μ_0 is a well-defined probability measure. Indeed, Lemma 2.1 below establishes a uniform large deviation estimate for P_ε, $\varepsilon > 0$, and we prove uniform integrability of the weight $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}$ with respect to P_ε for sufficiently small $\varepsilon > 0$. See (2.17).

First, we present a *uniform* large deviation lemma for the conditioned Wiener measure P_ε, $\varepsilon > 0$.

Lemma 2.1. *Let $R \geq 5N^{\frac{1}{2}}$ and $M \sim N$. Then, we have*

$$P_\varepsilon\Bigg(\sum_{|n-M|\leq N} |g_n|^2 \geq R^2\Bigg) \leq Ce^{-\frac{1}{8}R^2} \quad (2.11)$$

uniformly for sufficiently small $\varepsilon \geq 0$.

Proof. By Chebyshev's inequality, we have

$$P_\varepsilon\bigg(\sum_{|n-M|\leq N}|g_n|^2\geq R^2\bigg)\leq e^{-tR^2}\mathbb{E}_{P_\varepsilon}\Big[e^{t\sum_{|n-M|\leq N}|g_n|^2}\Big]. \tag{2.12}$$

Set $t=\frac{1}{4}$. We estimate $\mathbb{E}_{P_\varepsilon}\Big[e^{\frac{1}{4}t\sum_{|n-M|\leq N}|g_n|^2}\Big]$ in the following. As in (2.1), we can write it as

$$\begin{aligned}
&\mathbb{E}_{P_\varepsilon}\Big[e^{\frac{1}{4}\sum_{|n-M|\leq N}|g_n|^2}\Big]\\
&\quad=\int_{\mathbb{C}^{2N+1}}\frac{P\Big(\sum_{|n-M|\geq N+1}\langle\widetilde{n}\rangle^{-2}|g_n|^2\in A_\varepsilon(\widetilde{a}),\,\sum_{|n-M|\geq N+1}\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\in B_\varepsilon(\widetilde{b})\Big)}{P\Big(\sum_n\langle\widetilde{n}\rangle^{-2}|g_n|^2\in A_\varepsilon(a),\,\sum_n\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\in B_\varepsilon(b)\Big)}\\
&\qquad\qquad\times\frac{e^{-\frac{1}{4}\sum_{|n-M|\leq N}|g_n|^2}}{(2\pi)^{2N+1}}\prod_{|n-M|\leq N}dg_n,
\end{aligned}\tag{2.13}$$

where $\widetilde{a}$ and $\widetilde{b}$ are given by

$$\widetilde{a}=a-\sum_{|n-M|\leq N}\langle\widetilde{n}\rangle^{-2}|g_n|^2,\quad\text{and}\quad\widetilde{b}=b-\sum_{|n-M|\leq N}\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2. \tag{2.14}$$

By repeating the argument in Subsection 2.1, we can show that the right hand side of (2.13) is uniformly bounded for small $\varepsilon>0$.

More precisely, define the density $\widetilde{f}_N(a,b)$ by

$$\widetilde{f}_N(a,b)\,dadb=P\bigg(\sum_{|n-M|\geq N}\langle\widetilde{n}\rangle^{-2}|g_n|^2\in da,\sum_{|n-M|\geq N}\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\in db\bigg).$$

Then, as in Subsection 2.1, one can prove

$$\frac{P\Big(\sum_{|n-M|\geq N+1}\langle\widetilde{n}\rangle^{-2}|g_n|^2\in A_\varepsilon(\widetilde{a}),\,\sum_{|n-M|\geq N+1}\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\in B_\varepsilon(\widetilde{b})\Big)}{P\Big(\sum_n\langle\widetilde{n}\rangle^{-2}|g_n|^2\in A_\varepsilon(a),\,\sum_n\langle\widetilde{n}\rangle^{-2}\widetilde{n}|g_n|^2\in B_\varepsilon(b)\Big)}\longrightarrow\frac{\widetilde{f}_{N+1}(\widetilde{a},\widetilde{b})}{f_0(a,b)}, \tag{2.15}$$

where the convergence is uniform in $\widetilde{a}$ and $\widetilde{b}$. Moreover, by showing $\|\widetilde{f}_N\|_{L^\infty}<\infty$ as before, we see that the right hand side of (2.15) is uniformly bounded for small $\varepsilon>0$. (Recall that a and b are fixed.) By (2.13), (2.15), and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{\varepsilon\to0}\mathbb{E}_{P_\varepsilon}\Big[e^{\frac{1}{4}\sum_{|n-M|\leq K}|g_n|^2}\Big]&=\int_{\mathbb{C}^{2N+1}}\frac{\widetilde{f}_{N+1}(\widetilde{a},\widetilde{b})}{f_0(a,b)}\frac{e^{-\frac{1}{4}\sum_{|n-M|\leq N}|g_n|^2}}{(2\pi)^{2N+1}}\prod_{|n-M|\leq N}dg_n\\
&\leq\frac{\|\widetilde{f}_{N+1}\|_{L^\infty}}{f_0(a,b)}\int_{\mathbb{C}^{2N+1}}\frac{e^{-\frac{1}{4}\sum_{|n-M|\leq N}|g_n|^2}}{(2\pi)^{2N+1}}\prod_{|n-M|\leq N}dg_n\\
&\leq\frac{\|\widetilde{f}_{N+1}\|_{L^\infty}}{f_0(a,b)}2^{2N+1},
\end{aligned}$$

where the last inequality follows from change of variables. Also, by examining the argument in Subsection 2.1, we see that $\|\widetilde{f}_{N+1}\|_{L^\infty}\leq\|(\widetilde{f}_{N+1})^\wedge\|_{L^1}$ is bounded at most by a power of

N. Hence, we have

$$\mathbb{E}_{P_\varepsilon}\Big[e^{\frac{1}{4}\sum_{|n-M|\le K}|g_n|^2}\Big] \lesssim 2^{3N} \tag{2.16}$$

for all sufficiently small $\varepsilon > 0$. Therefore, (2.11) follows from (2.12) and (2.16) as long as $R^2 \geq (24\ln 2)N$. □

In the following, we show that the weight $e^{\int_{\mathbb{T}}|u|^p}$ is uniformly integrable with respect to P_ε, for sufficiently small $\varepsilon \geq 0$, for $p \leq 6$ (with sufficiently small mass when $p = 6$.) This, in particular, shows that μ_ε in (1.14) is a well-defined probability measure.

Note that it suffices to prove that

$$\begin{aligned}&\int_0^\infty e^\lambda\, P_\varepsilon\bigg(\int_{\mathbb{T}}|u|^p \geq p\lambda\bigg)d\lambda\\ &\quad= \int_0^\infty e^\lambda\, P\bigg(\int_{\mathbb{T}}|u|^p \geq p\lambda\,\bigg|\,\int_{\mathbb{T}}|u|^2 \in A_\varepsilon(a),\; i\int_{\mathbb{T}}u\overline{u}_x \in B_\varepsilon(b)\bigg)d\lambda \leq C_p < \infty\end{aligned} \tag{2.17}$$

for all sufficiently small $\varepsilon > 0$. The estimate (2.17) follows once we prove

$$P_\varepsilon\bigg(\int_{\mathbb{T}}|u|^p \geq p\lambda\bigg) \leq \begin{cases} Ce^{-c\lambda^{1+\delta}} & \text{when } p < 6.\\ Ce^{-(1+\delta)\lambda} & \text{when } p = 6.\end{cases} \tag{2.18}$$

for $\lambda > 1$ (with some $\delta > 0$), uniformly in small $\varepsilon > 0$.

Before proving (2.18), let us introduce some notations. Given $M_0 \in \mathbb{N}$, let $\mathbb{P}_{>M_0}$ denote the Dirichlet projection onto the frequencies $\{|n| > M_0\}$. i.e. $\mathbb{P}_{>M_0}u = \sum_{|n|>M_0}\hat{u}_n e^{2\pi inx}$. $\mathbb{P}_{\leq M_0}$ is defined in a similar manner. Given $j \in \mathbb{N}$, let $M_j = 2^j M_0$. We use the notation $|n| \sim M_j$ to denote the set of integers $|n| \in (M_{j-1}, M_j]$, and denote by $\mathbb{P}_{M_j}$ the Dirichlet projection onto the dyadic block $(M_{j-1}, M_j]$, i.e. $\mathbb{P}_{M_j}u = \sum_{|n|\sim M_j}\hat{u}_n e^{2\pi inx}$.

Without loss of generality, assume $\varepsilon \leq a$. Then, we have $\int |u|^2 \leq 2a =: K$. By Sobolev inequality,

$$\|\mathbb{P}_{\leq M_0}u\|_{L^p(\mathbb{T})} \leq cM_0^{\frac{1}{2}-\frac{1}{p}}\|\mathbb{P}_{\leq M_0}u\|_{L^2(\mathbb{T})}. \tag{2.19}$$

Hence, we have

$$\int_{\mathbb{T}}|\mathbb{P}_{\leq M_0}u|^p \leq \tfrac{p}{2}\lambda \quad \text{on} \quad \int_{\mathbb{T}}u^2 \leq K, \tag{2.20}$$

by choosing

$$M_0 = c_0\lambda^{\frac{2}{p-2}}K^{-\frac{p}{p-2}} \sim c_0\lambda^{\frac{2}{p-2}}a^{-\frac{p}{p-2}} \tag{2.21}$$

for some $c_0 > 0$. Let $\sigma_j = C2^{-\delta j}$, $j = 1, 2, \dots$ for some small $\delta > 0$ where $C = C(\delta)$ is chosen such that $\sum_{j=1}^\infty \sigma_j = 1$. Then, we have

$$P_\varepsilon\bigg(\int_{\mathbb{T}}|\mathbb{P}_{>M_0}u|^p > \tfrac{p}{2}\lambda\bigg) \leq \sum_{j=0}^\infty P_\varepsilon\Big(\|\mathbb{P}_{M_j}u\|_{L^p(\mathbb{T})} > \sigma_j\big(\tfrac{p}{2}\lambda\big)^{\frac{1}{p}}\Big). \tag{2.22}$$

By Sobolev inequality as in (2.19), we have

$$\|\mathbb{P}_{M_j}u\|_{L^p(\mathbb{T})} \leq cM_j^{\frac{1}{2}-\frac{1}{p}}\|\mathbb{P}_{M_j}u\|_{L^2(\mathbb{T})}. \tag{2.23}$$

From (1.8), we have

$$\|\mathbb{P}_{M_j}u\|_{L^2(\mathbb{T})}^2 = \sum_{|n|\sim M_j}|\hat{u}_n|^2 = \sum_{|n|\sim M_j}\big(1 + (2\pi n)^2\big)^{-1}|g_n|^2. \tag{2.24}$$

From (2.23) and (2.24), the right hand side of (2.22) is bounded by

$$\sum_{j=0}^{\infty} P_\varepsilon\bigg(\sum_{|n|\sim M_j} |g_n|^2 \geq R_j^2\bigg), \quad \text{where } R_j := c'\sigma_j \lambda^{\frac{1}{p}} M_j^{\frac{1}{p}-\frac{1}{2}}(1+M_j^2)^{1/2}. \tag{2.25}$$

Note that $R_j \gtrsim M_j^{\frac{1}{2}+\frac{1}{p}} \gg M_j^{\frac{1}{2}}$. By applying Lemma 2.1 to (2.25), we obtain

$$\begin{aligned} P_\varepsilon\bigg(\int_{\mathbb{T}} |\mathbb{P}_{>M_0} u|^p > \tfrac{p}{2}\lambda\bigg) &\lesssim \sum_{j=0}^{\infty} e^{-\frac{1}{8}R_j^2} \lesssim \sum_{j=0}^{\infty} e^{-c''\sigma_j^2 \lambda^{\frac{2}{p}} M_j^{\frac{p+2}{p}}} \\ &\lesssim \sum_{j=0}^{\infty} e^{-\widetilde{c}(2^j)^{\frac{p+2}{p}-2\delta}\lambda^{\frac{2}{p}} M_0^{\frac{p+2}{p}}} \lesssim e^{-c\lambda^{\frac{2}{p}} M_0^{\frac{p+2}{p}}} \end{aligned} \tag{2.26}$$

Hence, from (2.26) and (2.21), we have

$$P_\varepsilon\bigg(\int_{\mathbb{T}} |u|^p > p\lambda\bigg) \leq C\exp\big\{ -c\,\lambda^{1+\frac{6-p}{p-2}} a^{-\frac{p+2}{p-2}}\big\} \tag{2.27}$$

and (2.18) follows. Note that when $p=6$, we need to take a sufficiently small such that the coefficient of λ in (2.27) is less than -1.

2.3. **Weak convergence.** Finally, we prove weak convergence of μ_ε defined in (1.14) to μ_0. Let f be a bounded continuous function on $H^{\frac{1}{2}-\gamma}(\mathbb{T})$ for some small $\gamma>0$.

We first consider the defocusing case. If a sequence of functions u_n converges to u in $H^{\frac{1}{2}-\gamma}(\mathbb{T})$ with $\gamma < p^{-1}$, then we have $u_n \to u$ in $L^p(\mathbb{T})$ by Sobolev inequality. Thus, $e^{-\int_{\mathbb{T}}|u|^p}$ is bounded and continuous on $H^{\frac{1}{2}-\gamma}(\mathbb{T})$. Then, by weak convergence of P_ε to P_0, we have

$$Z_\varepsilon = \int e^{-\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_\varepsilon \longrightarrow \int e^{-\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_0 = Z_0 \quad \text{as } \varepsilon\to 0.$$

Since $f(u)e^{-\int_{\mathbb{T}}|u|^p}$ is also bounded and continuous on $H^{\frac{1}{2}-\gamma}(\mathbb{T})$, we have

$$\int f d\mu_\varepsilon = Z_\varepsilon^{-1}\int f(u) e^{-\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_\varepsilon \longrightarrow Z_0^{-1}\int f(u) e^{-\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_0 = \int f d\mu_0 \quad \text{as } \varepsilon\to 0.$$

This shows that μ_ε converges weakly to μ_0 in the defocusing case.

Next, we consider the focusing case. First, we prove

$$Z_\varepsilon = \int e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_\varepsilon \longrightarrow \int e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p} dP_0 = Z_0 \quad \text{as } \varepsilon\to 0. \tag{2.28}$$

For small $\varepsilon\geq 0$, define $Z_{\varepsilon,N}$ by

$$Z_{\varepsilon,N} = \int e^{\frac{1}{p}\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p} dP_\varepsilon.$$

By Sobolev inequality (see (2.19)), we have $\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p \leq cN^{\frac{p}{2}-1}a^{\frac{p}{2}}$ on $\int_{\mathbb{T}}|u| \leq a+\varepsilon\leq 2a$. In particular, $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}$ is bounded and continuous on $H^{\frac{1}{2}-\gamma}(\mathbb{T})$. Thus, by weak convergence of P_ε to P_0, we have

$$Z_{\varepsilon,N} \longrightarrow Z_{0,N} \quad \text{as } \varepsilon\to 0. \tag{2.29}$$

The following lemma on uniform convergence of $Z_{\varepsilon,N}$ to Z_ε is a consequence of the uniform tail estimate (2.26) and (2.27).

Lemma 2.2. *Let $Z_{\varepsilon,N}$ and Z_ε be as above. Then, $Z_{\varepsilon,N}$ converges to Z_ε as $N\to\infty$, uniformly in small $\varepsilon\geq 0$.*

Assume Lemma 2.2 for the moment. Fix $\delta>0$. By Lemma 2.2, choose large N^* such that $|Z_{\varepsilon,N^*}-Z_\varepsilon|<\frac{\delta}{3}$ for all small $\varepsilon\geq 0$. By (2.29), there exists small $\varepsilon^*>0$ such that such that $|Z_{\varepsilon,N^*}-Z_{0,N^*}|<\frac{\delta}{3}$ for all $\varepsilon\in[0,\varepsilon^*]$. Then, we have

$$|Z_\varepsilon-Z_0|\leq|Z_{\varepsilon,N}-Z_\varepsilon|+|Z_{\varepsilon,N}-Z_{0,N}|+|Z_{0,N}-Z_0|<\delta$$

for all $\varepsilon\in[0,\varepsilon^*]$. Hence, (2.28) follows.

Proof of Lemma 2.2. By Mean Value Theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned}|Z_\varepsilon-Z_{\varepsilon,N}|&=\bigg|\int\Big(e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}-e^{\frac{1}{p}\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p}\Big)dP_\varepsilon\bigg|\\&\lesssim\int\max\Big(e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p},e^{\frac{1}{p}\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p}\Big)\bigg|\int_{\mathbb{T}}|u|^p-\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p\bigg|dP_\varepsilon\\&\leq\bigg(\int e^{\frac{2}{p}\int_{\mathbb{T}}|u|^p+\frac{2}{p}\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p}dP_\varepsilon\bigg)^{\frac12}\bigg(\int\bigg|\int_{\mathbb{T}}|u|^p-\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p\bigg|^2dP_\varepsilon\bigg)^{\frac12}.\end{aligned}\tag{2.30}$$

In view of (2.27), the first factor on the right hand side is bounded uniformly in small $\varepsilon\geq 0$. (When $p=6$, the L^2-cutoff needs to be sufficiently small so that $e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p}\in L^2(dP_\varepsilon)$, uniformly in small $\varepsilon>0$.) Hence, it remains to show that the second factor tends to zero uniformly in small $\varepsilon\geq 0$ as $N\to\infty$.

In the following, we use the following elementary inequality. For $p\geq 1$ and small $\eta>0$, we have

$$|a+b|^p-|a|^p\leq\eta|a|^p+C\eta^{1-p}|b|^p.\tag{2.31}$$

First, we use (2.31) to show that the second factor in (2.30) tends to zero uniformly in small $\varepsilon\geq 0$ as $N\to\infty$. Then, we prove (2.31).

Fix small $\eta>0$ (to be chosen later.) By (2.31), we have

$$\begin{aligned}\bigg(\int\bigg|\int_{\mathbb{T}}|u|^p-\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p\bigg|^2dP_\varepsilon\bigg)^{\frac12}&\leq\bigg(\int\bigg|\eta\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^p+C\eta^{1-p}\int_{\mathbb{T}}|\mathbb{P}_{>N}u|^p\bigg|^2dP_\varepsilon\bigg)^{\frac12}\\&\leq C_1\eta\bigg(\int\int_{\mathbb{T}}|\mathbb{P}_{\leq N}u|^{2p}\,dxdP_\varepsilon\bigg)^{\frac12}+C_2\eta^{1-p}\bigg(\int\int_{\mathbb{T}}|\mathbb{P}_{>N}u|^{2p}\,dxdP_\varepsilon\bigg)^{\frac12}=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

Fix $\delta>0$. By (2.27) (applied to $\mathbb{P}_{\leq N}u$ with $2p$ instead of p), we have

$$\mathrm{I}=C_1\eta\bigg(\int_0^\infty e^{-c\lambda^{\frac{2}{p-1}}a^{-\frac{p+1}{p-1}}}d\lambda\bigg)^{\frac12}=C(p)\eta<\frac{\delta}{2}$$

by choosing $\eta\sim\delta$. Next, we estimate II. The contribution of II from $\int_{\mathbb{T}}|\mathbb{P}_{>N}u|^{2p}\leq c_2\delta^{2p}$ (with $\eta\sim\delta$) can be easily estimated by $\frac{\delta}{4}$. Thus, we assume $\int_{\mathbb{T}}|\mathbb{P}_{>N}u|^{2p}>c_2\delta^{2p}$ in the following. Then, by (2.26) with $N=M_0$ and $2p$ instead of p, we can estimate the contribution of II in this case by

$$\begin{aligned}&C_2'\delta^{1-p}\bigg(\int_0^{c_2\delta^{2p}}1\,d\lambda\bigg)^{\frac12}+C_2'\delta^{1-p}\bigg(\int_{c_2\delta^{2p}}^\infty P_\varepsilon\Big(\int_{\mathbb{T}}|\mathbb{P}_{>N}u|^{2p}>\lambda\Big)d\lambda\bigg)^{\frac12}\\&\qquad<\frac{\delta}{8}+C_2''\delta^{1-p}\bigg(\int_{c_2\delta^{2p}}^\infty e^{-c\lambda^{\frac1p}N^{\frac{p+1}{p}}}d\lambda\bigg)^{\frac12}<\frac{\delta}{4}\end{aligned}$$

by making $N = N(p, \delta)$ sufficiently large. This shows that $Z_{\varepsilon,N}$ converges to Z_ε uniformly in small $\varepsilon > 0$, since the estimates (2.26) and (2.27) are uniform in small $\varepsilon \geq 0$.

It remains to prove (2.31). It suffices to prove

$$|a + b|^p \leq (1 + \eta)|a|^p + C\eta^{1-p}|b|^p \tag{2.32}$$

for $a, b > 0$. By Mean Value Theorem and Young's inequality, we have

$$|a + b|^p \leq |a|^p + p|a + b|^{p-1}|b| \leq |a|^p + (p - 1)\theta^{\frac{p}{p-1}}|a + b|^p + \theta^{-p}|b|^p$$

Given $\eta > 0$, choose θ such that $1 + \eta = (1 - (p - 1)\theta^{\frac{p}{p-1}})^{-1}$. This choice of θ gives $\theta^{-p} \sim \eta^{1-p}$, and hence (2.32) follows. $\square$

Let f be a bounded continuous function f on $H^{\frac{1}{2}-\gamma}(\mathbb{T})$. Then, by writing

$$\begin{aligned}\int f d\mu_\varepsilon - \int f d\mu_0 &= Z_\varepsilon^{-1} \int f(u)e^{\frac{1}{p}\int_\mathbb{T} |u|^p} dP_\varepsilon - Z_0^{-1} \int f(u)e^{\frac{1}{p}\int_\mathbb{T} |u|^p} dP_0 \\ &= Z_0^{-1}\bigg(\int f(u)e^{\frac{1}{p}\int_\mathbb{T} |u|^p} dP_\varepsilon - \int f(u)e^{\frac{1}{p}\int_\mathbb{T} |u|^p} dP_0\bigg) \\ &\quad + (Z_\varepsilon^{-1} - Z_0^{-1}) \int f(u)e^{\frac{1}{p}\int_\mathbb{T} |u|^p} dP_\varepsilon,\end{aligned}$$

it follows from (2.28) that the second term on the right hand side goes to zero. By a slight modification of the proof of (2.28), we can easily show that the first term goes to zero. Hence, μ_ε converges weakly to μ_0. This completes the proof of Theorem 1.

3. Proof of Theorem 2: Invariance of the conditioned Gibbs measures

In this section, we show that the conditioned Gibbs measure μ_0 is invariant under the flow of NLS (1.1). In fact, one can directly establish the invariance of the conditioned Gibbs measure μ_0 by following the argument developed by Bourgain [B2, B3]. This argument is based on approximating the PDE flow by finite dimensional Hamiltonian systems with invariant finite dimensional Gibbs measures. For such an argument, one needs the the following large deviation estimate (with $\varepsilon = 0$.)

Lemma 3.1. *Let $s < \frac{1}{2}$. Then, we have*

$$P_\varepsilon\Big(\|u\|_{H^s} > \Lambda\Big) \leq C_s e^{-c\Lambda^2}, \tag{3.1}$$

uniformly in small $\varepsilon \geq 0$.

Proof. This basically follows from the proof of (2.27) in Subsection 2.2. Given $s < \frac{1}{2}$, choose $p > 2$ such that $s = \frac{1}{2} - \frac{1}{p}$. Then, we have

$$\|\mathbb{P}_{\leq M_0} u\|_{H^s(\mathbb{T})} \leq cM_0^{\frac{1}{2}-\frac{1}{p}} \|\mathbb{P}_{\leq M_0} u\|_{L^2(\mathbb{T})}. \tag{3.2}$$

(Compare this with (2.19).) By repeating the computation in Subsection 2.2 (with $\Lambda = \lambda^{\frac{1}{p}}$), we obtain

$$P_\varepsilon\Big(\|u\|_{H^s} > \Lambda\Big) \leq C_s \exp\big\{ - c\,\Lambda^{p(1+\frac{6-p}{p-2})} a^{-\frac{p+2}{p-2}} \big\}. \tag{3.3}$$

Then, (3.1) follows since $p(1 + \frac{6-p}{p-2}) > 2$ for $p > 2$. $\square$

Bourgain's argument [B2, B3] requires a combination of PDE and probabilistic techniques. In the following, however, we simply show how the invariance of the conditioned Gibbs measure μ_0 follows, as a corollary, from a priori invariance of Gibbs measures μ_ε, $\varepsilon > 0$.

• **Case 1:** $p \leq 6$. In this case, the flow of (1.1) is globally defined in $H^{\frac{1}{2}-\delta}(\mathbb{T})$ for small $\delta = \delta(p) > 0$, thanks to [B1, B5]. Let $\mathcal{S}_t$ be the flow map of (1.1): $u_0 \mapsto u(t) = \mathcal{S}_t u_0$. Then, $\mathcal{S}_t$ is well-defined and continuous on $H^{\frac{1}{2}-\delta}(\mathbb{T})$

Given a bounded continuous function ϕ on $H^{\frac{1}{2}-\delta}(\mathbb{T})$, $\phi \circ \mathcal{S}_t$ is bounded and continuous on $H^{\frac{1}{2}-\delta}(\mathbb{T})$. By weak convergence of μ_ε to μ_0 and invariance of μ_ε under the flow of (1.1), we have

$$\int \phi \, d\mu_0 = \lim_{\varepsilon \to 0} \int \phi \, d\mu_\varepsilon = \lim_{\varepsilon \to 0} \int \phi \circ \mathcal{S}_t \, d\mu_\varepsilon = \int \phi \circ \mathcal{S}_t \, d\mu_0.$$

This proves invariance of μ_0 for $p \geq 6$.

• **Case 2:** $p > 6$. (This is relevant only in the defocusing case.)

In this case, there is no a priori global-in-time flow of (1.1) on $H^{\frac{1}{2}-\delta}(\mathbb{T})$. However, by Bourgain's argument [B2, B3], μ_ε is invariant under the flow of NLS (1.1) for each $\varepsilon > 0$, and we show invariance of μ_0 as a corollary to the invariance of μ_ε, $\varepsilon > 0$.

Let K be a compact set in $H^s(\mathbb{T})$ with $s = \frac{1}{2}-$. Then, there exists $\Lambda = \Lambda(K) > 0$ such that $\|u\|_{H^s} \leq \Lambda$ for $u \in K$. By the (deterministic) local well-posedness [B2], there exists $t_0 > 0$ such that NLS (1.1) is well-posed on $[0, t_0]$ for initial data u_0 with $\|u_0\|_{H^s} \leq \Lambda + 1$. Moreover, for each small $\theta > 0$, there exists $\delta > 0$ such that

$$\mathcal{S}_{t_0}(K + B_\delta) \subset \mathcal{S}_{t_0} K + B_\theta. \tag{3.4}$$

Then, by weak convergence of μ_ε to μ_0, we have

$$\mu_0(K) \leq \mu_0(K + B_\delta) \leq \liminf_{\varepsilon \to 0} \mu_\varepsilon(K + B_\delta)$$

By invariance of μ_ε and (3.4),

$$\begin{aligned}
&= \liminf_{\varepsilon \to 0} \mu_\varepsilon\big(\mathcal{S}_{t_0}(K + B_\delta)\big) \leq \liminf_{\varepsilon \to 0} \mu_\varepsilon(\mathcal{S}_{t_0} K + B_\theta\,) \\
&\leq \limsup_{\varepsilon \to 0} \mu_\varepsilon(\mathcal{S}_{t_0} K + B_\theta\,) \leq \limsup_{\varepsilon \to 0} \mu_\varepsilon(\mathcal{S}_{t_0} K + \overline{B_\theta}\,) \\
&\leq \mu_0(\mathcal{S}_{t_0} K + \overline{B_\theta}\,),
\end{aligned}$$

where the last inequality follows once again from the weak convergence of μ_ε to μ_0. By letting $\theta \to 0$, we have $\mu_0(K) \leq \mu_0(\mathcal{S}_{t_0} K)$. Given arbitrary $t > 0$, we can iterate the above argument and obtain $\mu_0(K) \leq \mu_0(\mathcal{S}_t K)$. By the time-reversibility of the NLS flow, we obtain

$$\mu_0(K) = \mu_0(\mathcal{S}_t K).$$

This proves invariance of μ_0 for $p > 6$.

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