A uniqueness theorem for bounded analytic functions on the polydisc

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Abstract

For each $n, N \geq 1$ let $\mathcal{I}_N(\mathbb{D}^n)$ denote the set of rational inner functions on $\mathbb{D}^n$ of degree strictly less than N. We construct a set of points $\lambda_1, ..., \lambda_{N^n} \in \mathbb{D}^n$ with the following property: if $f \in \mathcal{I}_N(\mathbb{D}^n)$ and an analytic function g maps $\mathbb{D}^n$ to $\mathbb{D}$ and satisfies $g(\lambda_i) = f(\lambda_i)$ for each $i = 1, ..., N^n$, then $g = f$ on $\mathbb{D}^n$. In terms of the Pick problem on $\mathbb{D}^n$, our result implies that if $f \in \mathcal{I}_N(\mathbb{D}^n)$, then the Pick problem with data $\lambda_1, ..., \lambda_{N^n}$ and $f(\lambda_1), ..., f(\lambda_{N^n})$ has a unique solution.

1 Introduction

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$. Let $\mathbb{D}^n$ and $\mathbb{T}^n$ denote the cartesian products of n copies of $\mathbb{D}$ and $\mathbb{T}$ in $\mathbb{C}^n$, respectively. A rational function f on $\mathbb{D}^n$ is called **inner** if f is analytic and $|f(\tau)| = 1$ for almost every $\tau \in \mathbb{T}^n$. For a rational f, let $f = \frac{q}{r}$ for $q, r \in \mathbb{C}[z_1, ..., z_n]$ relatively prime and define the **degree** of f as the degree of q.

Throughout this paper N will denote a positive integer. Let $\mathcal{I}_N(\mathbb{D}^n)$ denote the set of rational inner functions on $\mathbb{D}^n$ of degree strictly less than N. Let $\mathcal{S}(\mathbb{D}^n)$ denote the **Schur class** of $\mathbb{D}^n$, the set of analytic functions mapping $\mathbb{D}^n$ to $\mathbb{D}$. Our main result is that for each N there exist points $\lambda_1, ..., \lambda_{N^n} \in \mathbb{D}^n$ such that each $f \in \mathcal{I}_N(\mathbb{D}^n)$ is uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on $\lambda_1, ..., \lambda_{N^n}$. In terms of the Pick problem on $\mathbb{D}^n$, our result implies that if $f \in \mathcal{I}_N(\mathbb{D}^n)$, then the Pick problem with data $\lambda_1, ..., \lambda_{N^n}$ and $f(\lambda_1), ..., f(\lambda_{N^n})$ has a unique solution.

The following is our main result.

Theorem 1.1. *For each N there exist points $\lambda_1, ..., \lambda_{N^n} \in \mathbb{D}^n$ with the following property: for each $f \in \mathcal{I}_N(\mathbb{D}^n)$, if $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for $i = 1, ..., N^n$, then $g = f$ on $\mathbb{D}^n$.*
Furthermore, the points $\lambda_1, ..., \lambda_{N^n}$ may be chosen as follows.
Let $M = N^{n-1}$, for $r = 2, ..., n$ let $\tau_1^r, ..., \tau_N^r \in \mathbb{T}$ be distinct and let $D_1, ..., D_M$ be distinct analytic discs given by

$$D_k : \mathbb{D} \to \mathbb{D}^n \text{ with } D_k(z) = (z, \tau^2_{i_{2,k}} z ..., \tau^n_{i_{n,k}} z).$$

If for each k, the points $\lambda_{k_1}, ..., \lambda_{k_N} \in D_k(\mathbb{D})$ are distinct, then the points $\{\lambda_{k_j}\}$ for $k = 1, ..., M$ and $j = 1, ..., N$ have the above property.

This work began as an investigation of the connections between the Pick problem, rational inner functions and algebraic varieties on $\mathbb{D}^2$, first discovered by Agler and McCarthy in [1]. I'm thankful to Jim Agler for his generous help in improving the exposition of this paper.

This paper is organized as follows. In section 2 we give background results on the Pick problem and prove the case $n = 1$ of Theorem 1.1. In section 3 we give background results about rational inner functions on $\mathbb{D}^n$. In section 4 we prove a result on $\mathbb{D}^2$ that we will use in the proof of Theorem 1.1. In section 5 we prove Theorem 1.1. In section 6 we prove a refinement of Theorem 1.1 that yields stronger results for singular inner functions.

2 The Pick problem on $\mathbb{D}^n$

The **Pick problem** on $\mathbb{D}^n$ is to determine, given N distinct nodes $\lambda_1, ..., \lambda_N \in \mathbb{D}^n$ and N target points $\omega_1, ..., \omega_N \in \mathbb{D}$, whether there is an analytic function $f \in \mathcal{S}(\mathbb{D}^n)$ that satisfies $f(\lambda_i) = \omega_i$ for each $i = 1, ..., N$. If such an f exists we call f a **solution**. For $n = 1$, Pick proved the following.

Theorem 2.1. *(Pick, 1916 [2]) Fix a Pick problem on $\mathbb{D}$ with data $\lambda_1, .., \lambda_M$ and $\omega_1, ..., \omega_N$. The following are equivalent.*
a. *The problem has a solution and the solution is unique.*
b. *The matrix $P = \left(\dfrac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}}\right)_{i,j=1}^N$ is positive semi-definite and singular.*
c. *The problem has a rational inner solution f and $\deg(f) < N$.*

Part **b** of Theorem 2.1 will not be used in the present work but is included to make the theorem look more recognizable to those familiar with the result.

The following lemma is equivalent to the implication **c** $\to$ **a** in Theorem 2.1. The lemma is case $n = 1$ of Theorem 1.1.

Lemma 2.2. *(Theorem 1.1 case $n = 1$)*
Fix $N > 1$, $\lambda_1, ...\lambda_N \in \mathbb{D}$ and $f \in \mathcal{I}_N(\mathbb{D})$.
If $g \in \mathcal{S}(\mathbb{D})$ satisfies $g(\lambda_i) = f(\lambda_i)$ for $i = 1, ..., N$, then $g = f$ on $\mathbb{D}$.

3 Rational Inner functions on $\mathbb{D}^n$

In [3], Rudin proved the following result about the structure of rational inner functions on $\mathbb{D}^n$.

Theorem 3.1. *(Rudin 1969) Every $f \in \mathcal{I}_N(\mathbb{D}^n)$ can be written*

$$\varphi(z_1, ..., z_n) = z_1^{d_1} \cdots z_n^{d_n} \frac{\overline{q(\frac{1}{\overline{z_1}}, ..., \frac{1}{\overline{z_n}})}}{q(z_1, ..., z_n)}$$

for some polynomial $q(z_1, ..., z_n)$ that does not vanish on $\mathbb{D}^n$.

We introduce two definitions before we employ Rudin's result.

Definition 3.2. For $n > m \geq 1$, we call an analytic function $E : \mathbb{D}^m \to \mathbb{D}^n$ an **analytic m-disc**. We use $E(\mathbb{D}^m)$ to denote the range of E.

Definition 3.3. Let $f \in \mathcal{S}(\mathbb{D}^n)$, $\tau \in \mathbb{T}$ and E an analytic (n-1)-disc

$$\text{given by } E : \mathbb{D}^{n-1} \to \mathbb{D}^n \text{ with } E(z_1, ..., z_{n-1}) = (z_1, ..., z_{n-1}, \tau z_1) \qquad (3.4)$$

We define f_E as follows

$$f_E(z_1, ..., z_{n-1}) = f(E(z_1, ..., z_{n-1})).$$

The function f_E is in $\mathcal{S}(\mathbb{D}^{n-1})$ and parametrizes the restriction of f to E.

Corollary 3.5. *For $n \geq m > 1$ and $\tau \in \mathbb{T}$, if $f \in \mathcal{I}_N(\mathbb{D}^n)$ and $E(z_1, ..., z_{n-1}) = (z_1, ..., z_{n-1}, \tau z_1)$, then $f_E \in \mathcal{I}_N(\mathbb{D}^{n-1})$.*

PROOF: Since f is inner, the denominator of f has a non-zero constant term and Theorem 3.1 implies that f can be written as follows.

$$f(z_1, ..., z_n) = \tau \frac{z^{d_1} \cdots z^{d_n} + r_0(z_1, ..., z_n)}{1 + q_0(z_1, ..., z_n)} \text{ for some } \tau \in \mathbb{T} \qquad (3.6)$$

where the degree of f equals $d_1 + ... + d_n$ and each term of r_0 has degree less than or equal to d_i in each z_i and less than d_i in at least one z_i. The corollary follows from substituting $f(z_1, ..., z_{n-1}, \tau z_1)$ into equation 3.6. $\square$

4 A result on $\mathbb{D}^2$

We will use the case $n = 2$ of Lemma 4.2 in the proof of Theorem 1.1.

We will use the following technical lemma to prove the case $n = 2$ of Lemma 4.2. We use $B_\epsilon(z)$ to denote the ball of radius ϵ around z and we use $m_{t,a}(z)$ to denote the automorphism of $\mathbb{D}$ given by $t\frac{z-a}{1-\bar{a}z}$.

Lemma 4.1. *Let $\tau_1, ..., \tau_N \in \mathbb{T}$ be distinct and $E_1, ..., E_M$ be analytic discs*

$$\text{given by } E_i : \mathbb{D} \to \mathbb{D}^2 \text{ with } E_i(z) = (z, \tau_i z).$$

There exist $\tau \in \mathbb{T}$ and $\epsilon > 0$ such that for every $t \in B_\epsilon(\tau) \cap \mathbb{T}$ and $a \in B_\epsilon(0)$, the image of the analytic disc $C_{m_{t,a}}$ given by

$$C_{m_{t,a}} : \mathbb{D} \to \mathbb{D}^2 \text{ with } C_{m_{t,a}}(z) = (z, m_{t,a}(z))$$

intersects each $E_i(\mathbb{D})$ at a distinct point $(r_i, \tau_i r_i)$.
Furthermore, C, defined as the union of every $C_{m_{t,a}}(\mathbb{D})$ over $t \in B_\epsilon(\tau) \cap \mathbb{T}$ and $a \in B_\epsilon(0)$ is a set of uniqueness for analytic functions on $\mathbb{D}^2$.

PROOF: Fix $\tau \in \mathbb{T}$ such that $\tau \neq \tau_i$ for each i and let $\epsilon_1 > 0$ be small enough so that for each i, $\tau_i \notin B_{\epsilon_1}(\tau)$. Let $C_m = C_{m_{\tau,a}}$, with a to be specified later. The sets $C_m(\mathbb{D})$ and $E_i(\mathbb{D})$ intersect if and only if one of the roots of the equation $\tau_i z = m_{t,a}(z)$ lies in $\mathbb{D}$. Let r_i and s_i denote the roots. If $a = 0$ then $r_i = 0$ and $s_i = \infty$ for each i. For sufficiently small $\epsilon_1 > \epsilon > 0$, if a is perturbed away from zero and remains in $B_\epsilon(0)$, then each of the roots r_i becomes non-zero and stays in $\mathbb{D}$. That the roots $r_1, ..., r_M$ are distinct follows from that they are non-zero and that $\tau_i \neq \tau_j$.

To see that C is a set of uniqueness let f be analytic on $\mathbb{D}^2$ and suppose that $f|_C = 0$. Fix $x \in \mathbb{D}$, $a \in B_\epsilon(0)$ and let

$$A_x = \{(x, m_{t,a}(x)) \in \mathbb{D}^2 : t \in B_\epsilon(\tau) \cap \mathbb{T}\} \subset C.$$

Since $f(x, z)$ is an analytic function in the single variable z and vanishes on the arc A_x, $f = 0$. Since $f(x, \cdot) = 0$ for each $x \in \mathbb{D}$, $f = 0$ on $\mathbb{D}^2$. □

If Theorem 1.1 holds for n then the following lemma immediately follows for n. We prove the following lemma for $n = 2$.

Lemma 4.2. *Fix N, let $f \in \mathcal{I}_N(\mathbb{D}^n)$, let $\tau_1, ..., \tau_N \in \mathbb{T}$ be distinct and let $E_1, ..., E_N$ be analytic (n-1)-discs given by*

$$E_k : \mathbb{D}^{n-1} \to \mathbb{D}^n \text{ with } E_k(z_1, ..., z_{n-1}) = (z_1, ..., z_{n-1}, \tau_k z_1)$$

If $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g = f$ on each $E_k(\mathbb{D}^{n-1})$, then $g = f$ on $\mathbb{D}^n$.

Proof of lemma 4.2(case n=2): By Lemma 4.1 there exists an analytic disc $C_m(\mathbb{D})$ that intersects each of $E_1(\mathbb{D}), ..., E_N(\mathbb{D})$ at a distinct point $R_i = (r_i, \tau_i r_i)$. Fix $f \in \mathcal{I}_N(\mathbb{D}^2)$ and assume that $g \in \mathcal{S}(\mathbb{D}^2)$ satisfies $g = f$ on each $E_k(\mathbb{D}^{n-1})$. Let $f_m = f_{C_m}$ and $g_m = g_{C_m}$. Notice that $g_m \in \mathcal{S}(\mathbb{D})$ and by Lemma 3.5, $f_m \in \mathcal{I}_N(\mathbb{D})$. It follows that for $i = 1, ..., N$,

$$g_m(r_i) = g(D_i(r_i)) = g(r_i, \tau_i r_i) = f(r_i, \tau_i r_i) = f(D_i(r_i)) = f_m(r_i)$$

Since $g_m(r_i) = f_m(r_i)$ for $i = 1, ..., N$, lemma 2.2 implies that $g_m = f_m$ on $\mathbb{D}$ and thus, $g = f$ on each $C_m(\mathbb{D})$. By Lemma 4.1, the discs $C_m(\mathbb{D})$ sweep out a set of uniqueness and thus, $g = f$ on $\mathbb{D}^2$. $\square$

5 Proof of Theorem 1.1

In this section we use induction to prove Theorem 1.1. The case $n = 1$ is Lemma 2.2. Fix $n \geq 2$ and suppose that Theorem 1.1 holds for each $m < n$. We show that Theorem 1.1 holds holds for n in 3 steps.

In the first step we fix N, fix a set of analytic (n-1)-discs $E_1, ..., E_N$, and fix a set of N^{n-1} points $\{\lambda_{j_s}\} \subset \mathbb{D}^{n-1}$ to which we will imply the induction hypothesis. We lift the set $\{\lambda_{j_s}\}$ to the set of N^n points $\{\lambda_{k_{j_s}}\}$ in $\mathbb{D}^n$ by letting $\lambda_{k_{j_s}} = E_k(\lambda_{j_s})$.

In the second step we apply the induction hypothesis to show that for each $f \in \mathcal{I}_N(\mathbb{D}^n)$, if $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_{k_{j_s}}) = f(\lambda_{k_{j_s}})$ for k, j, s, then $g = f$ on $E_1, ..., E_N$.

In the third step we use Lemma 4.2 (which holds for $n-1$ by the induction hypothesis) to show that since g equals f on $E_1, ..., E_N$, $g = f$ on $\mathbb{D}^n$.

STEP 1: Fix N and let $\tau_1, ..., \tau_N \in \mathbb{T}$ be distinct and $E_1, ..., E_N$ be analytic (n-1)-discs given by

$$E_k : \mathbb{D}^{n-1} \to \mathbb{D}^n \text{ with } E_k = (z_1, ..., z_m, \tau_i z_1).$$

Let $M = N^{n-2}$. For each $r = 2, ..., n-1$ let $\tau_1^r, ..., \tau_N^r \in \mathbb{T}$ be distinct. Let $D_1, ..., D_{N^{n-2}}$ be the N^{n-2} analytic discs given by

$$D_j : \mathbb{D} \to \mathbb{D}^{n-1} \text{ with } D_j(z) = (z, \tau^2_{i_{2,j}} z ..., \tau^{n-1}_{i_{n-1,j}} z).$$

For each j, let $\lambda_{j_1}, ..., \lambda_{j_N} \in D_j(\mathbb{D}) \subset \mathbb{D}^{n-1}$ be distinct and lift each point λ_{j_s} to $\mathbb{D}^n$, N times, by letting $\lambda_{k_{j_s}} = E_k(\lambda_{j_s})$.

STEP 2: Fix $f \in \mathcal{I}_N(\mathbb{D}^n)$ and suppose $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_{k_{j_s}}) = f(\lambda_{k_{j_s}})$ for each k, j, s. For each k, let $f_k = f_{E_k}$ and $g_k = g_{E_K}$. Notice that $g_k \in \mathcal{S}(\mathbb{D}^{n-1})$ and by Lemma 3.5, $f_k \in \mathcal{I}_N(\mathbb{D}^{n-1})$. It follows that for $k = 1, ...N$, $j = 1, ..., N^{n-2}$ and $s = 1, ..., N$,

$$g_k(\lambda_{j,s}) = g(E_k(\lambda_{j,s})) = g(\lambda_{k,j,s}) = f(\lambda_{k,j,s}) = f(E_k(\lambda_{l,s})) = f_k(\lambda_{j,s}).$$

Since for each k, $g_k(\lambda_{j_s}) = f_k(\lambda_{j_s})$ for each j and s, the induction hypothesis implies that $g_k = f_k$ on $\mathbb{D}^{n-1}$. Thus, $g = f$ on each E_k.

STEP 3: If $n = 2$, then case $n = 2$ of Lemma 4.2 implies that $g = f$ on $\mathbb{D}^2$. Suppose $n \geq 3$.

For $\rho \in \mathbb{T}$ let C_ρ be the analytic (n-1)-disc given by

$$C_\rho : \mathbb{D}^{n-1} \to \mathbb{D}^n \text{ with } C_\rho(z_1, ..., z_{n-1}) = (z_1, ..., z_{n-2}, z_{n-1}, \bar{\rho} z_{n-1}).$$

For each ρ, let $f_\rho = f_{C_\rho}$, $g_\rho = g_{C_\rho}$. Let $I_{\rho,k} : \mathbb{D}^{n-2} \to \mathbb{D}^n$ and $H_{\rho,k} : \mathbb{D}^{n-2} \to \mathbb{D}^{n-1}$ be analytic (n-2)-discs such that

$$I_{\rho,k}(\mathbb{D}^{n-2}) = C_\rho(\mathbb{D}^{n-1}) \cap E_k(\mathbb{D}^{n-1}) \text{ and } H_{\rho,k}(\mathbb{D}^{n-2}) = C_\rho^{-1}(I_{\rho,k}(\mathbb{D}^{n-2})).$$

Since $g = f$ on $I_{\rho,1}(\mathbb{D}^{n-2}), ..., I_{\rho,N}(\mathbb{D}^{n-2})$ it follows that $g_\rho = f_\rho$ on $H_{\rho,1}(\mathbb{D}^{n-2}), ..., H_{\rho,N}(\mathbb{D}^{n-2})$ and Lemma 4.2 (which holds for $n-1$ by the induction hypothesis) implies that $g_\rho = f_\rho$. Thus, $g = f$ on C_ρ and since $\mathbb{D}^n = \bigcup_{\rho \in \mathbb{T}} C_\rho$, it follows that $g = f$ on $\mathbb{D}^n$. □

6 A refinement and a question

We call a rational inner function f on $\mathbb{D}^n$ **singular**, if f has a singular point on $\mathbb{T}^n$. In this section we show how Theorem 1.1 may be refined to yield stronger results for singular inner functions.

For an analytic disc D of the form in Theorem 1.1, let $\deg_D(f)$ equal the number of zeros of f on D. Plugging $f(D(z))$ into formula 3.6 implies that $\deg_D(f)$ is less than or equal to the degree of f. If f has a singular point on D, then $\deg_D(f)$ is strictly less than the degree of f. Our proof of Theorem 1.1 actually established the following refined theorem.

Theorem 6.1. *If in Theorem 1.1 the condition that there lie N points on each analytic disc D_k is replaced with the condition that there lie $N_k = \deg_{D_k}(f) + 1$ points on each D_k, then the conclusion still holds.*

References

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