On the speed of biased random walk in translation invariant percolation

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Abstract. For biased random walk on the infinite cluster in supercritical i.i.d. percolation on $\mathbb{Z}^2$, where the bias of the walk is quantified by a parameter $\beta > 1$, it has been conjectured (and partly proved) that there exists a critical value $\beta_c > 1$ such that the walk has positive speed when $\beta < \beta_c$ and speed zero when $\beta > \beta_c$. In this paper, biased random walk on the infinite cluster of a certain translation invariant percolation process on $\mathbb{Z}^2$ is considered. The example is shown to exhibit the opposite behavior to what is expected for i.i.d. percolation, in the sense that it has a critical value β_c such that, for $\beta < \beta_c$, the random walk has speed zero, while, for $\beta > \beta_c$, the speed is positive. Hence the monotonicity in β that is part of the conjecture for i.i.d. percolation cannot be extended to general translation invariant percolation processes.

1. Introduction

This paper is concerned with biased random walk on infinite percolation clusters on the square lattice, whose vertex set is $\mathbb{Z}^2$ and whose edge set consists of pairs of vertices at Euclidean distance 1 from each other; with a slight abuse of notation we write $\mathbb{Z}^2$ for this lattice. Let there be two possible states for each edge in $e \in E$: open or closed. In general, a percolation model is a way of deciding which edges are to be open. In standard i.i.d. bond percolation with parameter $p \in [0, 1]$, each edge is independently open with probability p. The resulting configuration will almost surely contain an infinite open cluster if and only if $p > 1/2$; see Grimmett (1999) for this and other basics on percolation theory. When the origin belongs to the infinite cluster, we can define a random walk, starting at the origin, as follows: Let

$Z_t = (X_t, Y_t)$ denote the position of the random walk at time $t \in \mathbb{N}$ (we apologize to sensitive readers for using the letter t for a discrete time parameter; however, the integer indices $i, j, k, l, m, n, \ldots$ will be needed for other purposes later on). Write Λ_t for the set of neighbors of Z_t in the infinite cluster and define $l_t = |\Lambda_t|$. Also, fix $\beta > 1$. If $(X_t+1, Y_t) \in \Lambda_t$, then $Z_{t+1} = (X_t+1, Y_t)$ with probability $\beta(\beta+l_t-1)^{-1}$ and Z_{t+1} equals any other given vertex in Λ_t with probability $(\beta + l_t - 1)^{-1}$. If $(X_t + 1, Y_t) \notin \Lambda_t$, then Z_{t+1} is chosen uniformly from Λ_t, that is, Z_{t+1} equals any given vertex in Λ_t with probability l_t^{-1}. In case $\Lambda_t = \emptyset$, the walk stays put, i.e., $Z_{t+1} = Z_t$.

This model was introduced by Barma and Dhar (1983) and describes a random walk with drift towards the right, the strength of the drift being quantified by the parameter β. (Note that zero drift corresponds to $\beta = 1$.) The asymptotic speed, or simply the speed, is defined as $\lim_{t \to \infty} X_t/t$ (provided the limit exists). In Barma and Dhar (1983), it is conjectured that there is a critical drift $\beta_c = \beta_c(p) > 1$ such that the walk has positive speed for $\beta < \beta_c$ and speed zero for $\beta > \beta_c$. Intuitively, if the drift is large, the walk will tend to get stuck in "dead ends" of the percolation cluster, while, if the drift is weaker, it will be able to quickly backtrack and get out of the dead ends. The conjecture from Barma and Dhar (1983) was partly confirmed in two simultaneous and independent papers by Sznitman (2003) and Berger et al. (2003), respectively, where it is proved that there are β_l and β_u, with $1 < \beta_l \leq \beta_u$, such that the walk has positive speed for $\beta < \beta_l$ and speed zero for $\beta > \beta_u$. (Sznitman in fact obtained the same result in arbitrary dimension $d \geq 2$.) What remains here is to show that one can take $\beta_l = \beta_c$. Axelson-Fisk and Häggström (2009) demonstrated the same critical phenomenon with $\beta_l = \beta_c$ for a certain dependent percolation model on the lattice sometimes known as the infinite ladder. One might ask whether the monotonicity property suggested by the Barma–Dhar conjecture (namely that zero speed at a given $\beta > 1$ implies the same thing at all larger values of β) should be extended to a wider class of percolation processes, such as those that are translation invariant. Our main result, Theorem 1.1 below, shows that the answer is no.

More precisely, what we do in the present paper is as follows. We will construct a translation invariant percolation process on $\mathbb{Z}^2$ for which the above random walk dynamics give rise to a process which has speed zero when β is small and positive speed when β is large. For a translation invariant probability measure Ψ on $\{\text{open}, \text{closed}\}^E$ defining the percolation process, with the property that the existence of an infinite cluster has probability 1, write $\mathbb{P}_{\Psi, \beta}$ for the joint law of the percolation configuration and the random walk $\{Z_t\}_{t \geq 0}$. Furthermore, write $\{0 \leftrightarrow \infty\}$ for the event that the origin belongs to an infinite open cluster of the percolation configuration. We will prove the following:

Theorem 1.1. *For each $\gamma > 1$ there exists a translation invariant probability measure $\Psi = \Psi(\gamma)$ on $\{\text{open}, \text{closed}\}^E$ such that, for any $\beta > 1$,*

$$\lim_{t \to \infty} \frac{X_t}{t} = \begin{cases} 0 & \mathbb{P}_{\Psi, \beta}\text{-a.s. on the event } \{0 \leftrightarrow \infty\} & \text{if } \beta < \beta_c \\ \frac{\beta-1}{\beta+1} & \mathbb{P}_{\Psi, \beta}\text{-a.s. on the event } \{0 \leftrightarrow \infty\} & \text{if } \beta > \beta_c \end{cases}$$

with $\beta_c = \gamma$.

To work out the speed at criticality is probably doable with some more work, but might not be so important, since we suspect that you can get either answer (zero speed or full speed) by further fine-tuning of our model.

(a) Line with traps attached to it.
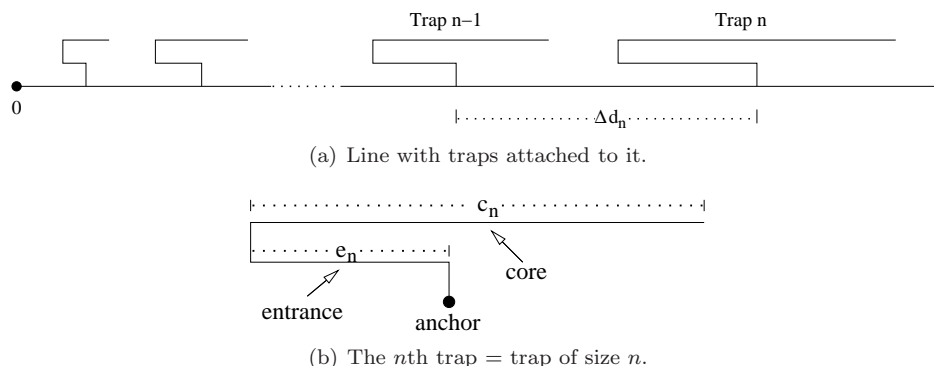

(b) The nth trap = trap of size n.

FIGURE 2.1. Schematic picture of configuration with line and traps.

The rest of the paper is organized as follows. A percolation process with the property described in Theorem 1.1 is constructed in detail in Section 3. First, however we illustrate one of the main ideas by describing in Section 2 a simpler (but not translation invariant) percolation process on which biased random walk behaves as in the theorem; the key concept here is the "trap" structure in Figure 1 below. These traps appear also in the main construction in Section 3. Section 4 concerns the main construction minus the traps, where the asymptotic speed is shown to equal $\frac{\beta-1}{\beta+1}$ for any $\beta > 1$. In Section 5 we show that including the traps as in the main construction makes no difference to the asymptotic speed as long as $\beta > \beta_c$, thus establishing the second half of Theorem 1.1. Finally, in Section 6, we consider the case $\beta < \beta_c$, and show that the traps slow down the speed to zero, thereby proving the first half of the theorem.

2. A warm-up construction

Consider first a configuration of open edges outlined in Figure 1(a): an infinite open path starting at the origin and going off straight along the positive x-axis, with so called *traps* attached to it. Each trap consists of an *entrance* and a *core*, the core being located one floor above the entrance; see Figure 1(b). The length (meaning the number of edges) of the entrance and the core of the n'th trap are denoted by e_n and c_n respectively. We furthermore write d_n for the x-coordinate where the entrance of the n'th trap begins, and define $\Delta d_n = d_n - d_{n-1}$. For each n, we need to have

$$e_n < \Delta d_n \tag{2.1}$$

and

$$e_n + c_{n-1} - e_{n-1} < \Delta d_n \tag{2.2}$$

in order for the traps not to overlap. The vertex $(d_n, 0)$ is called the *anchor* of the trap.

It is readily checked that a random walk with drift $\beta > 1$ along the infinite line without traps has positive speed equal to $\frac{\beta-1}{\beta+1}$. In particular, the walk is transient, and adding traps cannot change this, due to Rayleigh's monotonicity principle; see, e.g., Doyle and Snell (1984).

To obtain a configuration on which the walk has speed zero for small β and positive speed for large β we will choose the sequences $\{e_n\}$, $\{c_n\}$ and $\{d_n\}$ so that, if β is small, the walk will enter the core of infinitely many traps, which will cause a delay severe enough to bring the speed down to zero, while, if β is large, the walk will enter the core of only finitely many traps, causing a delay that is negligible in the limit. The details are as follows.

Fix $\alpha > 0$. It will turn out that choosing

$$d_n = n^3, \ \ e_n = \lceil \alpha \log n \rceil \ \text{ and } \ c_n = n \tag{2.3}$$

for all n sufficiently large (where $\lceil \cdot \rceil$ denotes rounding up to the nearest integer) yields an asymptotic speed which is zero for $\beta < \beta_c$ and strictly positive for $\beta > \beta_c$, with $\beta_c = e^{1/\alpha}$. The reason we say "for all n sufficiently large" is that the values of e_n and c_n may need to be lowered compared to (2.3) for small n in order to satisfy (2.1) and (2.2); by transience of the random walk, modifying a finite number of traps cannot change the asymptotic speed.

Proposition 2.1. *The infinite path on the positive x-axis, decorated with traps with parameters satisfying (2.3) for large n, yields the following almost sure behavior for random walk:*

$$\lim_{t\to\infty} \frac{X_t}{t} = \begin{cases} 0 & \text{if } \beta < \beta_c \\ \frac{\beta-1}{\beta+1} & \text{if } \beta > \beta_c \end{cases}$$

with $\beta_c = e^{1/\alpha}$.

Before proving this result, let us explain why it does not immediately imply our desired Theorem 1.1. The reason, of course, is that the construction in the present section is not translation invariant. Now, there is a standard way of turning a non-translation invariant construction into a translation invariant one, namely via random translation of the original construction. In this case, the natural thing to do would be the following.

(a) Make copies of the original configuration shifted k steps vertically, for $k = 3, 6, 9, \ldots$, and for $k = -3, -6, -9, \ldots$.
(b) Shift the configuration resulting from (a) K steps vertically, where K is chosen according to uniform distribution on $\{0, 1, 2\}$, thus making the model invariant under vertical translation.
(c) Shift the configuration resulting from (b) L steps horizontally to the left, where L is chosen according to uniform distribution on $\{1, \ldots, l\}$, and then consider the limit as $l \to \infty$, thus making the model invariant also under horizontal translation.

The problem with this approach is that since $\lim_{n\to\infty} \frac{d_n}{n} = \infty$ in (2.3), the density of trap entrances goes to zero, and they will disappear on us in step (c). This disappearance can be avoided by a more elaborate fractal-like construction of the percolation process described in Section 3.

For the proof of Proposition 2.1, we will need three lemmas concerning the behavior of random walk on traps; these lemmas will become useful also later on when we analyse random walk on our main construction, in Sections 5 and 6.

For $n, i \geq 1$, let $T_{n,i}$ denote the time spent by the random walk in trap number n during its i'th visit to the trap, and let

$$T_{n,i}^* = \begin{cases} 0 & \text{if the walk hits the trap's core during this visit} \\ T_{n,i} & \text{otherwise,} \end{cases}$$

with the convention that if the walk enters the trap exactly k times, then $T_{n,i} = T_{n,i}^* = 0$ for all $i > k$. The first lemma gives the probability, once a trap has been entered, of reaching its core.

Lemma 2.2. *Each time the random walk enters the n'th trap, it has probability $\frac{\beta-1}{\beta^{e_n+1}+\beta-2}$ of reaching the core before exiting the trap, so that*

$$\mathbf{P}\left[T_{n,i} > T_{n,i}^* \,|\, T_{n,i} > 0\right] = \frac{\beta - 1}{\beta^{e_n+1} + \beta - 2}$$

for any n and i.

Once the walk enters the core, it has a fair chance of spending a very long time (exponential in c_n) there. The second lemma quantifies this.

Lemma 2.3. *For each n and i, we have*

$$\mathbf{P}\left[T_{n,i} \geq \beta^{c_n} \,|\, T_{n,i} > 0\right] \geq \frac{(\beta - 1)^2}{2\beta(\beta^{e_n+1} + \beta - 2)}. \tag{2.4}$$

On the other hand, provided the walk does *not* hit the core of the trap, its expected time spent in the trap can be bounded uniformly in n. The third lemma makes this precise. Let $\preceq$ denote stochastic domination between random variables, i.e., $X \preceq X'$ means that $\mathbf{E}[f(X)] \leq \mathbf{E}[f(X')]$ for any bounded and increasing f.

Lemma 2.4. *We may define a positive random variable T^* such that*

: (a) $T_{n,i}^* \preceq T^*$ *for any n and i, and*
: (b) $\mathbf{E}[T^*] = \frac{2\beta-1}{\beta-1}$.

For the proofs of the lemmas (and also later on) it will be useful to consider an electrical analysis of the random walk à la Doyle and Snell (1984, Chapter 3). Each edge e of the network is assigned a resistance $R(e) = \beta^{-x(e)}$ where $x(e)$ is the largest x-coordinate amongst the two vertices incident to e. The rules of the random walk may then be reformulated as saying that a random walker standing at a vertex chooses among the incident edges with probabilities inversely proportional to their resistances. For two vertices v_1 and v_2, let $R_{eff}(v_1, v_2)$ denote the effective resistance between v_1 and v_2 in the electrical representation, see Doyle and Snell (1984, Section 3.4). Since the percolation network in this case is a tree, there is always a unique self-avoiding path between v_1 and v_2, and $R_{eff}(v_1, v_2)$ is simply the sum of the edge resistances along the path.

Proof of Lemma 2.2. When the walk enters the trap, it will find itself at the vertex $(d_n, 1)$. From there, it will eventually reach either $(d_n, 0)$ or $(d_n - e_n, 2)$. The probability that it hits the latter before the former equals

$$\frac{R_{eff}((d_n, 1), (d_n, 0))}{R_{eff}((d_n, 1), (d_n, 0)) + R_{eff}((d_n, 1), (d_n - e_n, 2))} = \frac{\beta^{-d_n}}{\beta^{-d_n} + \sum_{i=d_n-e_n}^{d_n} \beta^{-i}}$$

$$= \frac{1}{1 + \sum_{i=0}^{e_n} \beta^i}$$

$$= \frac{\beta - 1}{\beta^{e_n+1} + \beta - 2}. \tag{2.5}$$

$\square$

Proof of Lemma 2.3. Once the random walk hits the core, i.e., once it reaches the vertex $(d_n - e_n, 2)$, its conditional probability of hitting the second-to-last vertex $(d_n - e_n + c_n - 1, 2)$ of the core before going back to $(d_n - e_n, 1)$ is

$$\frac{R_{eff}((d_n - e_n, 2), (d_n - e_n, 1))}{R_{eff}((d_n - e_n, 2), (d_n - e_n, 1)) + R_{eff}((d_n - e_n, 2), (d_n - e_n + c_n - 1, 2))}$$

$$= \frac{1}{1 + \sum_{i=1}^{c_n - 1} \beta^{-i}}$$

$$\geq \frac{1}{1 + \sum_{i=1}^{\infty} \beta^{-i}} = \frac{\beta - 1}{\beta}. \tag{2.6}$$

Combining this with Lemma 2.2, we thus have that once the random walk enters the trap, it has probability at least

$$\frac{(\beta - 1)^2}{\beta(\beta^{e_n + 1} + \beta - 2)} \tag{2.7}$$

of reaching $(d_n - e_n + c_n - 1, 2)$ before exiting. A similar calculation as in (2.5) and (2.6) shows that once the walk has reached $(d_n - e_n + c_n - 1, 2)$, it has probability $\frac{\beta - 1}{\beta^{c_n + 1} - 1} < \beta^{-c_n}$ of hitting $(d_n - e_n, 1)$ before $(d_n - e_n + c_n, 2)$. Hence, upon reaching $(d_n - e_n + c_n - 1, 2)$, the number of visits to $(d_n - e_n + c_n, 2)$ before reaching $(d_n - e_n, 1)$ is geometric with mean at least β^{c_n}. Thus, the number of such visits exceeds $\frac{\beta^{c_n}}{2}$ with conditional probability at least $\frac{1}{2}$, and multiplying by (2.7) yields the corresponding unconditional probability (which is the desired right-hand side of (2.4)). Every visit to $(d_n - e_n + c_n, 2)$ is immediately followed by one to $(d_n - e_n + c_n - 1, 2)$, so by counting also the latter we can replace the count $\frac{\beta^{c_n}}{2}$ by simply β^{c_n}, and (2.4) follows. $\square$

Proof of Lemma 2.4. Write $(Z'_1, Z'_2, \ldots, Z'_{T_{n,i}})$ for the sequence of vertices visited during the i'th visit to trap n. Also, write $(Z''_1, Z''_2, \ldots, Z''_{T^*_{n,i}})$ for the thinned sequence obtained by deleting all visits to the core of the trap, and note that $(Z''_1, Z''_2, \ldots, Z''_{T^*_{n,i}})$ has the same distribution that the original sequence would have had if the trap had had no core.

Imagine now a trap whose entrance is infinite, i.e., consists of an infinite straight path going off to the left from the anchor and generate a sequence $(W'_1, W'_2, \ldots, W'_{T^*})$ describing the positions of the random walk during a single visit to this trap. Since the walk has a drift to the right ($\beta > 1$), we get that $T^* < \infty$ with $\mathbf{E}[T^*] < \infty$; a simple calculation shows that $\mathbf{E}[T^*] = \frac{2\beta - 1}{\beta - 1}$. Let $(W''_1, W''_2, \ldots, W''_{T^*_n})$ denote the thinned sequence obtained by deleting from $(W'_1, W'_2, \ldots, W'_{T^*})$ all visits to vertices more than e_n steps to the left of the anchor, and note, crucially, that $(W''_1, W''_2, \ldots, W''_{T^*_n})$ has the same distribution as $(Z''_1, Z''_2, \ldots, Z''_{T^*_{n,i}})$. Hence T^*_n and $T^*_{n,i}$ are identically distributed, and since $T^*_n \leq T^*$ the proof is complete. $\square$

Equipped with Lemmas 2.2, 2.3 and 2.4, we are now in a position to prove Proposition 2.1.

Proof of Proposition 2.1. For any vertex $(k, 0)$ on the positive x-axis, the resistance to infinity is given by

$$R_{eff}((k, 0), \infty) = \sum_{j=k+1}^{\infty} \beta^{-j} = \frac{\beta^{-k}}{\beta - 1}.$$

It is a standard fact from Doyle and Snell (1984) that the escape probability $p_{esc}((k,0))$ of the random walk from $(k,0)$ – that is, the probability that the walk leaves $(k,0)$ and never returns – is given by

$$p_{esc}((k,0)) = \left(R_{eff}((k,0),\infty) \sum_{\{e:\, e\sim(k,0)\}} R(e)^{-1} \right)^{-1} \qquad (2.8)$$

where $e \sim (k,0)$ means that the edge e is incident to the vertex $(k,0)$. If $k = d_n$ for some n (i.e., there is some trap connecting to the x-axis at $(k,0)$), we get

$$p_{esc}((k,0)) = \left(\frac{\beta^{-k}}{\beta-1}(\beta^{k+1} + \beta^k + \beta^k) \right)^{-1}$$
$$= \frac{\beta-1}{\beta+2}.$$

For such k, the probability that the walk immediately takes a step into the trap is $\frac{1}{\beta+2}$. Hence, the probability that it ever takes a step into the trap before escaping to ∞ is

$$\frac{\frac{1}{\beta+2}}{\frac{1}{\beta+2} + \frac{\beta-1}{\beta+2}} = \beta^{-1}.$$

It follows that

$$\text{the number of visits to the trap is geometrically}$$
$$\text{distributed with mean } (\beta-1)^{-1}. \qquad (2.9)$$

Define $T_{n,tot}$ as the total time $\sum_{i=1}^{\infty} T_{n,i}$ spent in the n'th trap, and analogously $T_{n,tot}^* = \sum_{i=1}^{\infty} T_{n,i}^*$. Combining (2.9) with Lemma 2.2 yields

$$\mathbf{P}[T_{n,i} > T_{n,i}^*] = \beta^{-i} \frac{\beta-1}{\beta^{e_n+1} + \beta - 2}.$$

Summing over i gives that the probability $\mathbf{P}[T_{n,tot} > T_{n,tot}^*]$ of ever hitting the core of the n'th trap satisfies

$$\mathbf{P}[T_{n,tot} > T_{n,tot}^*] \leq \sum_{i=1}^{\infty} \mathbf{P}[T_{n,i} > T_{n,i}^*]$$
$$\leq \frac{\beta-1}{\beta^{e_n+1} + \beta - 2} \sum_{i=1}^{\infty} \beta^{-i}$$
$$= \frac{1}{\beta^{e_n+1} + \beta - 2}. \qquad (2.10)$$

Consider first the case $\beta > \beta_c$, where we wish to show that the random walk has the same asymptotic speed $\frac{\beta-1}{\beta+1}$ that we would have seen on a naked x-axis without the traps. For $k \geq 0$ write $U(k)$ for the (random) time at which the random walk first arrives at the vertex $(k,0)$. Establishing asymptotic speed $\frac{\beta-1}{\beta+1}$ is clearly the same as showing that $\lim_{k\to\infty} \frac{U(k)}{k} = \frac{\beta+1}{\beta-1}$, and for this, it is enough to show that the time $\sum_{\{n:\, d_n < k\}} T_{n,tot}$ spent in traps to the left of $(k,0)$ satisfies

$$\lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} T_{n,tot} = 0 \quad \text{a.s.} \qquad (2.11)$$

By (2.10) and (2.3), we have for all n large enough that

$$
\begin{aligned}
\mathbf{P}[T_{n,tot} > T^*_{n,tot}] &\leq 2\beta^{-e_n} \\
&\leq 2\beta^{-\alpha \log n} \\
&= 2n^{-\alpha \log \beta} .
\end{aligned}
$$

Since $\beta > \beta_c = e^{1/\alpha}$ so that $\alpha \log \beta > 1$, we get

$$
\sum_{n=1}^{\infty} n^{-\alpha \log \beta} < \infty
$$

whence

$$
\sum_{n=1}^{\infty} \mathbf{P}[T_{n,tot} > T^*_{n,tot}] < \infty .
$$

By Borel–Cantelli, we get a.s. that $T_{n,tot} > T^*_{n,tot}$ for at most finitely many n, so that

$$
\sum_{n=1}^{\infty} (T_{n,tot} - T^*_{n,tot}) < \infty \quad \text{a.s.} \tag{2.12}
$$

The left-hand side in (2.11) decomposes as

$$
\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} T_{n,tot} &= \lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} (T_{n,tot} - T^*_{n,tot}) + T^*_{n,tot} \\
&= \lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} (T_{n,tot} - T^*_{n,tot}) \tag{2.13} \\
&\quad + \lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} T^*_{n,tot}
\end{aligned}
$$

where the limit in (2.13) is 0 a.s. due to (2.12). Hence, to settle the case $\beta > \beta_c$, it suffices to show that

$$
\lim_{k\to\infty} \frac{1}{k} \sum_{\{n:\, d_n < k\}} T^*_{n,tot} = 0
$$

or in other words that

$$
\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{n} T^*_{i,tot} = 0 . \tag{2.14}
$$

Lemma 2.4 in combination with (2.9) yields

$$
\mathbf{E}[T^*_{i,tot}] \leq \frac{2\beta - 1}{(\beta - 1)^2} ,
$$

so that

$$
\begin{aligned}
\mathbf{E}\left[\frac{1}{d_n} \sum_{i=1}^{n} T^*_{i,tot} \right] &= n^{-3} \sum_{i=1}^{n} \mathbf{E}[T^*_{i,tot}] \\
&= \frac{2\beta - 1}{n^2(\beta - 1)^2} .
\end{aligned}
$$

For any $\varepsilon > 0$, Markov's equality gives

$$
\mathbf{P}\left[\frac{1}{d_n} \sum_{i=1}^{n} T^*_{i,tot} > \varepsilon \right] \leq \frac{2\beta - 1}{\varepsilon n^2 (\beta - 1)^2} ,
$$

where we may note that the right-hand side is summable over n, so that, by another application of Borel–Cantelli, (2.14) follows, and the proof of the proposition for $\beta > \beta_c$ is complete.

It remains to handle the case $\beta < \beta_c$. Write A_n for the event that the first time the random walk reaches $(d_n, 0)$, it immediately enters the trap and spends at least time $\beta^{c_n} = \beta^n$ in there. Note that $A_1, A_2, \ldots$ are independent, and that, due to Lemma 2.3,

$$
\begin{aligned}
\mathbf{P}[A_n] &\geq \frac{(\beta-1)^2}{(\beta+2)2\beta(\beta^{e_n+1}+\beta-2)} \\
&\geq \frac{C}{\beta^{e_n}} \\
&\geq \frac{C}{2\beta^{\alpha \log n}} \\
&= \frac{C}{2n^{\alpha \log \beta}}
\end{aligned}
$$

for some $C > 0$ which may depend on β but not on n.

Next, for $i = 1, 2, \ldots$, define

$$
W_i = \sum_{n=2^{i-1}+1}^{2^i} I_{A_n} \tag{2.15}
$$

as the number of events happening amongst $A_{2^{i-1}+1}, A_{2^{i-1}+2}, \ldots, A_{2^i}$. We get

$$
\begin{aligned}
\mathbf{E}[W_i] &= \sum_{n=2^{i-1}+1}^{2^i} \mathbf{P}[A_n] \\
&\geq 2^{i-1}\frac{C}{2 \cdot 2^{i\alpha \log \beta}} \\
&= \frac{C}{4}2^{i(1-\alpha \log \beta)}
\end{aligned}
$$

where we may note that $1 - \alpha \log \beta > 0$ (this is where we use $\beta < \beta_c$). Note that for large i, W_i is approximately Poisson, because it counts independent events with small probabilities. Hence, for i large enough,

$$
\mathbf{P}[W_i = 0] \leq 2\exp\left(-\frac{C}{4}2^{i(1-\alpha \log \beta)}\right) \tag{2.16}
$$

which decays to 0 (faster than) exponentially, so that by Borel–Cantelli we get a.s. that $W_i > 0$ for all but at most finitely many i. On the event that $W_i > 0$, the time $U(2^{3i})$ of the first arrival of the random walk at the vertex $(d_{2^i}, 0) = (2^{3i}, 0)$ satisfies $U(2^{3i}) \geq \beta^{2^{i-1}}$. Hence we get a.s. that

$$
\lim_{k \to \infty} \frac{U(k)}{k} = \infty \tag{2.17}
$$

along the subsequence $k = 2^3, 2^6, 2^9, \ldots$. Since $U(k)$ is increasing, $\frac{U(k)}{k}$ can drop by at most a factor $\frac{7}{8}$ as k increases in the interval $[2^{3i}, 2^{3(i+1)})$, so convergence to ∞ along the full sequence in (2.17) follows. This implies zero asymptotic speed. □

3. The main construction

In this section we specify the percolation process to be used as a witness for proving Theorem 1.1. We proceed in three steps. First, in Section 3.1, we specify a (deterministic) fractal-like percolation configuration that will play roughly the same role as the path along the x-axis did in Section 2. Then, in Section 3.2, we add traps to the construction. Finally, in Section 3.3, we make the percolation process translation invariant by means of a more successful application of random translation than in Section 2.

3.1. *Fractal structure.* For $k = 1, 2, \ldots$, a *branch* of order k (see Figure 2) consists of a horizontal path, called the *main part*, of length $b_k - 1$, linked at its rightmost vertex to a vertical path, called the *abutment*, of length $3 \cdot 2^{k-1}$, going either upwards or downwards; here $(b_1, b_2, \ldots)$ is a fairly rapidly growing sequence to be specified more precisely in what follows. The point where the main part and the abutment meet is called the *corner* of the branch, the other endpoint of the main part is called the *tip*, and the other endpoint of the abutment is called the *root*. The root of a branch of order k will always be situated somewhere on the main part of a branch of order $k + 1$; in this way, the random walk will be able to escape to infinity via branches of higher and higher order.
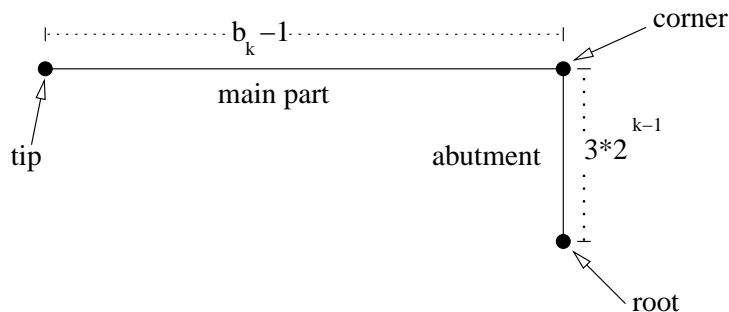

FIGURE 3.2. Branch of order k with abutment pointing downwards.

Define a sequence $(q_1, q_2, \ldots)$ prescribing how many branches of order k should attach to each branch of order $k + 1$. Each branch of order $k + 1$ will be attached to by $2q_k$ branches of order k, q_k of them attaching from above and the other q_k from below. In order for there to be room for all these branches of order k, we will need the lengths to satisfy $b_{k+1} \geq q_k b_k$ for each k. To give room for inserting traps in Section 3.2, we will need some extra margin, and will take

$$b_{k+1} = (q_k + 1)b_k. \tag{3.1}$$

A useful choice of $(q_1, q_2, \ldots)$ turns out to be

$$q_k = 3^k - 1$$

so that bootstrapping (3.1) gives

$$
\begin{aligned}
b_k &= b_1 \prod_{i=1}^{k-1} (q_i + 1) \\
&= b_1 \prod_{i=1}^{k-1} 3^i \\
&= b_1 3^{k(k-1)/2} .
\end{aligned}
$$

Somewhat arbitrarily we set $b_1 = 4$, so that

$$
b_k = 4 \cdot 3^{k(k-1)/2} . \tag{3.2}
$$

Here is how we arrange the branches. First, y-coordinates satisfying $y = 3l$ with integer l will be reserved for (the main parts of) branches. (Other y-coordinates will be used for traps later on.) More specifically,

$$
\begin{cases}
\quad y\text{-coordinates with } y = 3l \text{ with } l \text{ odd are reserved for branches of order } 1 \\
\quad y\text{-coordinates with } y = 6l \text{ with } l \text{ odd are reserved for branches of order } 2 \\
\qquad\qquad\qquad\qquad \vdots \\
\quad y\text{-coordinates with } y = 2^{k-1}3l \text{ with } l \text{ odd are reserved for branches of order } k \\
\qquad\qquad\qquad\qquad \vdots
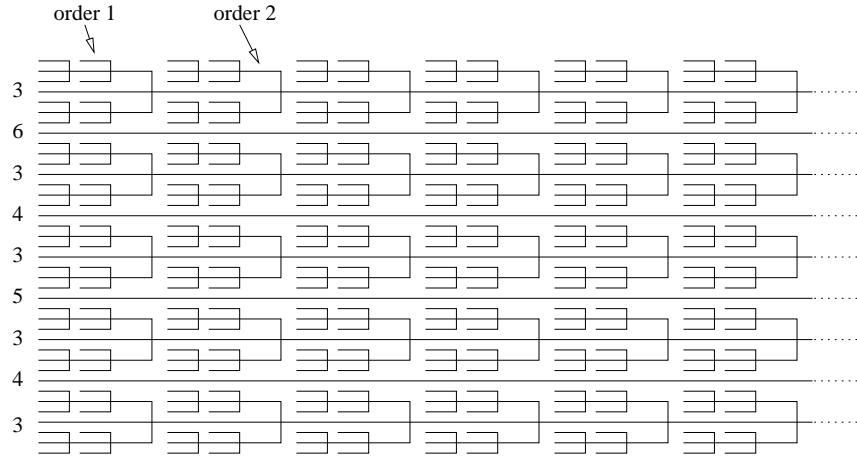\end{cases}
$$

FIGURE 3.3. Self-similar structure of branches. The orders of the larger branches are indicated to the left.

Now, for each k and each y with $y = 2^{k-1}3l$ for l odd, we set up a branch of order k with its tip at $(0, y)$, its corner at $(b_k - 1, y)$, and its root at either $(b_k - 1, y + 3 \cdot 2^{k-1})$ (i.e., the abutment pointing upwards)) or $(b_k - 1, y - 3 \cdot 2^{k-1})$ (the abutment pointing downwards) chosen as follows. Since $y = 2^{k-1}3l$ with l odd, we have that exactly one of the numbers $y + 3 \cdot 2^{k-1}$ and $y - 3 \cdot 2^{k-1}$ equals $2^k 3j$ for some j odd, and we choose to direct the abutment so that the root ends up at such a y-coordinate, thus ensuring that it sits on (the main part of) a branch of order $k + 1$.

In this way, for any $k \geq 2$, any branch of order k having its tip at, say, (x, y) will have received exactly two branches of order $k - 1$ attaching to it, one from above and one from below, and both of them with the root at $(x + b_{k-1} - 1, y)$. For each such branch of order k, we attach another $2(q_{k-1} - 1)$ branches of order $k - 1$ to it, two (one from above and one from below) at each of the points $(x + 2b_{k-1} - 1, y), (x + 3b_{k-1} - 1, y), \ldots, (x + q_{k-1}b_{k-1} - 1, y)$.

This creates, for any $k \geq 2$, further branches of order k to which presently there are no branches of order $k-1$ attached. To such a branch of order k with tip at (x, y) we attach $2q_{k-1}$ branches of order $k - 1$, two (one from above and one from below) at each of the points $(x + b_{k-1} - 1, y), (x + 2b_{k-1} - 1, y), \ldots, (x + q_{k-1}b_{k-1} - 1, y)$.

Repeating this procedure *ad infinitum* produces a percolation configuration $\xi \in \{\text{open}, \text{closed}\}^E$, where as before E is the edge set of the square lattice. This is the fractal structure that forms the foundation of our construction.

3.2. *Adding traps.* Now we will add traps to the configuration ξ of Section 3.1. Each branch of order $k \geq 2$ will be equipped with exactly one trap. The trap will be situated in the final region of the main part of the branch. This part contains a stretch of length $(b_k - 1) - (q_{k-1}b_{k-1} - 1)$ from the last attachment of a lower-order branch to its corner point. By (3.1) and (3.2), this length equals

$$(b_k - 1) - (q_{k-1}b_{k-1} - 1) \quad = \quad b_{k-1}$$
$$= \quad 4 \cdot 3^{(k-1)(k-2)/2} \, .$$

We choose to place the anchor of the trap exactly at the midpoint of this stretch, i.e., exactly $2 \cdot 3^{(k-1)(k-2)/2}$ steps to the left of the corner point of the order k-branch. The lengths e_k and c_k of the trap's entrance and core, respectively, are preliminarily chosen as

$$e_k \quad = \quad \left\lceil \frac{\log k}{\log \gamma} \right\rceil \tag{3.3}$$

and

$$c_k \quad = \quad 3^{(k-1)(k-2)/2}$$

respectively. For values of γ very close to 1, it may turn out that the chosen value of e_k in (3.3) exceeds $2 \cdot 3^{(k-1)(k-2)/2}$ so that the trap bumps into the abutment of the last branch of order $k - 1$ attaching from above to the branch of order k to which the trap is attached. This cannot be allowed to happen, and we therefore replace (3.3) by

$$e_k \quad = \quad \min \left\{ \left\lceil \frac{\log k}{\log \gamma} \right\rceil, 3^{(k-1)(k-2)/2} \right\} \, , \tag{3.4}$$

and note that this coincides with (3.3) for all k large enough.

Write $\eta \in \{\text{open}, \text{closed}\}^E$ for the percolation configuration obtained by adding traps in this manner to the configuration ξ.

3.3. *Stationarizing.* Let Ψ'' be the probability measure on $\{\text{open}, \text{closed}\}^E$ corresponding to picking the configuration $\eta \in \{\text{open}, \text{closed}\}^E$ from Section 3.2 deterministically. Ψ'' is turned into a vertically translation invariant measure Ψ' on $\{\text{open}, \text{closed}\}^E$ by shifting the configuration η vertically by an amount S_y chosen uniformly from $\{-m, -m + 1, \ldots, m\}$, and taking weak limits as $m \to \infty$, if

necessary after passing to a subsequence. It is readily checked that, for any $k \geq 1$,

$$\Psi'(\text{the origin is the tip of a branch of order } k) = \frac{1}{3 \cdot 2^k}. \qquad (3.5)$$

To achieve translation invariance also in the horizontal direction, we first choose a configuration $\eta' \in \{\text{open}, \text{closed}\}^E$ according to Ψ', and then we pick a shift $S_x \in \{0, \ldots, b_n - 1\}$ independently of η', resulting in a probability measure Ψ'_n on $\{\text{open}, \text{closed}\}^E$. We finally take the probability measure Ψ on $\{\text{open}, \text{closed}\}^E$ as a weak limit of the Ψ'_n measures as $n \to \infty$, if necessary after passing to a subsequence.[1]

It is clear that Ψ is both vertically and horizontally translation invariant. We now need to check that local structures (branches of order k for given k, and traps) do not disappear upon us in the limit as in the failed attempt at stationarizing in Section 2. For this, it suffices to show that for any $k \geq 1$,

$$\lim_{n \to \infty} \Psi'_n(\text{the origin sits on the main part of a branch of order } k) > 0. \qquad (3.6)$$

From (3.5), we get immediately that

$$\Psi'_n(\text{the origin is at a } y\text{-coordinate devoted to branches of order } k) = \frac{1}{3 \cdot 2^k}. \qquad (3.7)$$

Writing A_k for the event in (3.7), we get for $n > k$ using a direct count of the b_n different horizontal translations available to Ψ'_n that

$$\Psi'_n(\text{the origin is on the main part of a branch of order } k \,|\, A_k)$$
$$= \frac{b_k \prod_{i=k}^{n-1} q_i}{b_n} = \frac{b_k \prod_{i=k}^{n-1} q_i}{b_k \prod_{i=k}^{n-1}(q_i + 1)}$$
$$= \prod_{i=k}^{n-1} \frac{q_i}{q_i + 1} = \prod_{i=k}^{n-1}(1 - 3^{-i}),$$

where the second equality derives from the recursive definition (3.1). Multiplying by (3.7) and sending $n \to \infty$ gives

$$\Psi(\text{the origin is on the main part of a branch of order } k)$$
$$= \frac{1}{3 \cdot 2^k} \prod_{i=k}^{\infty}(1 - 3^{-i}) > 0,$$

so that the local structures do not disappear upon us in the limit.

4. Positive speed without the traps

In this section, we study what happens to the random walk in the modified percolation process obtained by removing all the traps. To this end, write Ψ^* for the probability measure on $\{\text{open}, \text{closed}\}^E$ corresponding to picking a percolation configuration according to Ψ and then deleting all the traps. Similarly as in Theorem 1.1, we write $\mathbb{P}_{\Psi^*, \beta}$ for the joint law of the percolation configuration chosen according to Ψ^* and the random walk $\{Z_t\}_{t \geq 0}$ with drift parameter β starting at

---

[1] In fact, it turns out that the limits exist without passing to a subsequence, both here and in going from Ψ'' to Ψ', but we do not need this.

the origin. Recall that $\{0 \leftrightarrow \infty\}$ denotes the event that the origin is in the infinite cluster of the percolation configuration.

Proposition 4.1. *For any $\beta > 1$ we have*

$$\lim_{t \to \infty} \frac{X_t}{t} = \frac{\beta - 1}{\beta + 1} \ \mathbb{P}_{\Psi^*, \beta}\text{-}a.s. \text{ on the event } \{0 \leftrightarrow \infty\}.$$

A first simplification for the proof of Proposition 4.1 is the following reduction, where we write B_k for the event that the origin is on the main part of a branch of order k.

Lemma 4.2. *Suppose for given $\beta > 1$ and $\theta \geq 0$ that*

$$\lim_{t \to \infty} \frac{X_t}{t} = \theta \ \mathbb{P}_{\Psi^*, \beta}\text{-}a.s. \text{ on the event } B_1. \tag{4.1}$$

Then, in fact,

$$\lim_{t \to \infty} \frac{X_t}{t} = \theta \ \mathbb{P}_{\Psi^*, \beta}\text{-}a.s. \text{ on the event } \{0 \leftrightarrow \infty\}. \tag{4.2}$$

The same result holds with $\mathbb{P}_{\Psi, \beta}$ in place of $\mathbb{P}_{\Psi^*, \beta}$.

Proof. Fix a probability distribution $\mathbf{Q}$ on $\mathbb{Z}^2$ with full support, and pick $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbb{Z}^2$ according to $\mathbf{Q}$. Given the percolation configuration chosen according to Ψ^*, run *two* random walks $\{Z_t\}_{t \geq 0} = \{(X_t, Y_y)\}_{t \geq 0}$ and $\{\tilde{Z}_t\}_{t \geq 0} = \{(\tilde{X}_t, \tilde{Y}_t)\}_{t \geq 0}$ starting at the origin and at $\tilde{z}$, respectively, coupled as follows. Let $\{\tilde{Z}_t\}_{t \geq 0}$ run independently of $\{Z_t\}_{t \geq 0}$ except that from the first time T that $\tilde{Z}_t$ hits the origin (if ever), $\tilde{Z}_t$ plagiarizes the trajectory of Z_t from then on, meaning that $\{\tilde{Z}_T, \tilde{Z}_{T+1}, \ldots\} = \{Z_0, Z_1, \ldots\}$. By translation invariance of Ψ^*,

$$\{\tilde{Z}_t - \tilde{z}\}_{t \geq 0} \text{ has the same distribution as } \{Z_t\}_{t \geq 0}. \tag{4.3}$$

Define $D = D_1 \cap D_2 \cap D_3$ where

$$D_1 = \{0 \leftrightarrow \infty\} \cap \left\{ \lim_{t \to \infty} \frac{X_t}{t} \neq \theta \right\},$$

$$D_2 = \{\tilde{z} \text{ is on the main part of an order-1 branch}\},$$

and $D_3 = \{T < \infty\}$. Assume for contradiction that (4.1) holds and that (4.2) fails. Then D_1 has positive probability, and it is easy too see that in that case, $D = D_1 \cap D_2 \cap D_3$ has positive probability too. On the event D we get that

$$\lim_{t \to \infty} \frac{\tilde{X}_t - \tilde{x}}{t} = \lim_{t \to \infty} \frac{X_{t-T} - \tilde{x}}{t} = \lim_{t \to \infty} \frac{X_t}{t} \neq \theta$$

which, in view of (4.3), contradicts (4.1). This concludes the argument for $\mathbb{P}_{\Psi^*, \beta}$, and the same argument goes through with $\mathbb{P}_{\Psi, \beta}$ in place of $\mathbb{P}_{\Psi^*, \beta}$. $\qquad \square$

For the purpose of proving Proposition 4.1, we may (due to Lemma 4.2) assume that the origin sits on the main part of a branch of order 1, and go on to analyze random walk from there. In this case, there is a unique self-avoiding path P in the percolation configuration from the origin to infinity. This path goes through (parts of) the main parts and the abutments of branches of increasing order $1, 2, 3, \ldots$. Write a_k for the x-coordinate of the abutment of the branch of order k in this path. For $k \geq 2$, a crude lower bound for a_k, which follows directly from the construction, is

$$a_k \geq b_{k-1} = 4 \cdot 3^{(k-1)(k-2)/2}. \tag{4.4}$$

A couple of further lemmas will be convenient to isolate for the proof of Proposition 4.1.

Lemma 4.3. *For any k and any vertex z which sits on the main part of a branch of order k and which sits at least $3 \cdot 2^{k-1}$ steps to the left of the corner point of that branch, we have that the escape probability for the random walk starting from z is at least*

$$\frac{\beta - 1}{2(3 + \beta)}.$$

Proof. We proceed electrically as in Section 2, attaching a resistance $R(e) = \beta^{-x(e)}$ to each edge e of the percolation configuration, where $x(e)$ is the largest x-coordinate amongst the two endpoints of e. We recall from (2.8) that the escape probability from a vertex $z = (x, y)$ is given by

$$p_{esc}(z) = \left(R_{eff}(z, \infty) \sum_{\{e:\, e \sim z\}} R(e)^{-1} \right)^{-1} \tag{4.5}$$

where the sum is over all edges e that are incident to vertex z. The sum in (4.5) is bounded above by the sum

$$3\beta^x + \beta^{x+1}, \tag{4.6}$$

corresponding to the case where all four possible edges incident to z are present in the percolation configuration. The effective resistance from z to ∞ is simply the sum of the resistances along the unique self-avoiding path from z to ∞. Counting only the horizontal edges of this path would give simply the sum $\sum_{i=x+1}^{\infty} \beta^{-i}$. The point of the choice of the bound $3 \cdot 2^{k-1}$ in the lemma is that the set of vertical edges on the path can be paired with a subset of the set of horizontal edges on the path, in such a way that a vertical edge is always paired with a horizontal edge with smaller x-coordinate and therefore larger resistance. Hence the set of vertical edges can contribute at most as much as the set of horizontal edges to $R_{eff}(z, \infty)$, so

$$R_{eff}(z, \infty) \leq 2 \sum_{i=x+1}^{\infty} \beta^{-i} = \frac{2\beta^{-x}}{\beta - 1}.$$

Plugging this bound and (4.6) into (4.5) gives

$$
\begin{aligned}
p_{esc}(z) &\geq \left((3\beta^x + \beta^{(x+1)}) \frac{2\beta^{-x}}{\beta - 1} \right)^{-1} \\
&= \frac{\beta - 1}{2(3 + \beta)}
\end{aligned}
$$

as desired. $\qquad \square$

Lemma 4.4. *For any $\beta > 1$, there exists a constant $C' = C'_\beta$ independent of k, such that*

    : *(a) a random walk taking a step to the left from a corner point of a branch of order k has an expected time until return to the corner point which is at most C'_β, and*

    : *(b) a random walk taking a step into a branch of order k from its root has an expected return time to the root which is at most*

$$3 \cdot 2^k + C'_\beta.$$

Proof. Imagine random walk with bias β on a finite connected subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ of the square lattice. This can be described as a finite-state Markov chain with a unique stationary distribution, where it is easily checked that each vertex $v \in \tilde{V}$ receives a probability $\pi(v)$ proportional to the sum

$$\sum_{e \sim v} R(e)^{-1} \tag{4.7}$$

of inverse edge resistances (defined in the same way as in the proof of Lemma 4.3) among edges incident to v. Define $R^{-1}(v)$ as the sum in (4.7). Standing at a given $v \in \tilde{V}$, the expected return time to v is

$$\pi(v)^{-1} = \frac{1}{R^{-1}(v)} \sum_{w \in \tilde{V}} R^{-1}(w). \tag{4.8}$$

For a vertex $z = (x, y)$, we have that

$$R^{-1}(z) \leq 3\beta^x + \beta^{x+1}. \tag{4.9}$$

For our percolation process, formula (4.8) applies when the random walk leaves a vertex v to enter a finite region of the percolation configuration cut of from v from the rest of the configuration. Applying this when $z = (x, y)$ is a corner point of a branch of order k, we have that the entire finite structure $\tilde{G} = (\tilde{V}, \tilde{E})$ cut off by z is contained in the cone

$$\{z' = (x', y') \in \mathbb{Z}^2 : x' \leq x, y' \in [y - (x - x'), y + (x - x')]\}. \tag{4.10}$$

Summing (4.9) over this cone gives

$$\sum_{w \in \tilde{V}} R^{-1}(w) \leq \sum_{i=0}^{\infty} (2i + 1)(3\beta^{x-i} + \beta^{x-i+1}).$$

Furthermore, the R^{-1} value of the corner point itself is $R^{-1}(z) = 2\beta^x$. Plugging these observations into (4.8) yields that the expected return time to z is bounded by

$$\frac{1}{2\beta^x} \sum_{i=0}^{\infty} (2i + 1)(3\beta^{x-i} + \beta^{x-i+1}) = \frac{3 + \beta}{2} \sum_{i=0}^{\infty} (2i + 1)\beta^{-i} < \infty, \tag{4.11}$$

so part (a) of the lemma is established with C'_β equal to the right-hand side in (4.11).

Part (b) follows similarly, the only difference being that in the sum in (4.8) we have to take into account the additional $3 \cdot 2^{k-1}$ vertices in the abutment, each of which contributes an amount $2\beta^x$ to the sum. $\qquad \square$

Proof of Proposition 4.1. To establish that $\lim_{t \to \infty} \frac{X_t}{t} = \frac{\beta - 1}{\beta + 1}$, it suffices to show that

$$\lim_{x \to \infty} \frac{U(x)}{x} = \frac{\beta + 1}{\beta - 1} \tag{4.12}$$

where, similarly as in the proof of Proposition 2.1, we define $U(x)$ as the time of first arrival at x-coordinate x:

$$U(x) = \min\{t : X_t = x\}.$$

As a means towards estimating $U(x)$ well enough to establish (4.12), we decompose it as

$$U(x) = U'(x) + U''(x)$$

where $U'(x)$ is the time spent on the path P to infinity before hitting x-coordinate x, and $U''(x)$ is the time spent outside the path P before first hitting x-coordinate x. Our plan is to establish

$$\lim_{x \to \infty} \frac{U'(x)}{x} = \frac{\beta + 1}{\beta - 1} \quad \text{a.s.} \tag{4.13}$$

and

$$\lim_{x \to \infty} \frac{U''(x)}{x} = 0 \quad \text{a.s.} \tag{4.14}$$

Together, (4.13) and (4.14) will of course imply (4.12).

We begin with (4.13). Note that for the purpose of studying $U'(x)$ we may simply pretend that paths other than P do not exist. Now, if P had no abutments and instead consisted of a single straight path along the x-axis, (4.13) would follow immediately from the strong law of large numbers. Furthermore, it is easy to see that $\{U'(x)\}_{x \geq 1}$ stochastically dominates the corresponding process in such an ideal scenario. Hence

$$\lim_{x \to \infty} \frac{U'(x)}{x} \geq \frac{\beta + 1}{\beta - 1}. \tag{4.15}$$

To strengthen this to an equality, we need to show that the delay caused by abutments is small. More precisely, define $S(k)$ as the time spent on the abutment of order k in P, *plus* the time spent on P to the left of this abutment after first having visited its corner point. The inequality (4.15) is strengthened to the equality (4.13) if we can show that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\{k : a_k \leq x\}} S(k) = 0 \quad \text{a.s.}$$

which is equivalent to

$$\lim_{k \to \infty} \frac{1}{a_k} \sum_{j=1}^{k} S(j) = 0 \quad \text{a.s.} \tag{4.16}$$

We go on to estimate $\mathbf{E}[S(k)]$. On the abutment itself, the random walk behaves like simple random walk, and it is a standard fact that the expected time on it until first hitting the root point equals its length squared, i.e., $(3 \cdot 2^{k-1})^2$. During this walk, the expected number of returns to the corner point is linear in the length $3 \cdot 2^{k-1}$, and to each such return corresponds a geometric number with mean 1 of excursions to the left of the corner point, so that the expected total number of such excursions is again linear in the abutment's length. Lemma 4.4 (a) ensures that the expected duration of such an excursion is bounded uniformly in k. Furthermore, Lemma 4.3 ensures that once the walk has reached the root point, the number of times it goes back into the abutment again is dominated by a geometric variable with mean $(\frac{\beta-1}{2(3+\beta)})^{-1}$, and Lemma 4.4 (b) ensures that each such excursion has expected duration at most $3 \cdot 2^k + C'_\beta$. Summing up the contributions to $S(k)$, we get that there exists a constant $C = C_\beta$ independent of k such that

$$\mathbf{E}[S(k)] \leq C_\beta 2^{2k}.$$

Taking expectation in the left-hand side of (4.16) and plugging in (4.4) gives

$$
\mathbf{E}\left[\frac{1}{a_k}\sum_{j=1}^{k}S(j)\right] \leq \frac{1}{4\cdot 3^{(k-1)(k-2)/2}}\sum_{j=1}^{k}\mathbf{E}[S(j)]
$$

$$
\leq \frac{C_\beta}{4\cdot 3^{(k-1)(k-2)/2}}\sum_{j=1}^{k}2^{2j}
$$

$$
\leq \frac{C_\beta k 2^{2k}}{4\cdot 3^{(k-1)(k-2)/2}}\,.
$$

Markov's inequality gives, for any $\varepsilon > 0$, that

$$
\mathbf{P}\left[\frac{1}{a_k}\sum_{j=1}^{k}S(j)\geq \varepsilon\right] \leq \frac{C_\beta k 2^{2k}}{4\cdot 3^{(k-1)(k-2)/2}\varepsilon}
$$

which decays to 0 exponentially fast as $k \to \infty$, so Borel–Cantelli gives (4.16). Hence, (4.13) is established, and it only remains to prove (4.14).

For $k \geq 2$, define W_k as the total time spent in parts of the percolation configuration away from P that attach to P in the part of P that belongs to a branch of order k. Regions contributing to W_k are of two kinds, namely,

: (i) branches of order $k-1$ (together with their respective subbranches), and
: (ii) the section not contained in P of the branch of order k itself.

There are at most $2q_k = 2(3^k - 1)$ branches of order $k-1$ contributing to W_k. By Lemma 4.3, the expected number of times that each such branch is visited is at most $(\frac{\beta-1}{2(3+\beta)})^{-1}$, and by Lemma 4.4 (b) the expected duration of each such visit is at most $3\cdot 2^k + C'_\beta$. Hence, the contribution from (i) to $\mathbf{E}[W_k]$ is at most

$$
2(3^k - 1)\left(\frac{\beta-1}{2(3+\beta)}\right)^{-1}(3\cdot 2^k + C'_\beta)\,.
$$

The contribution from (ii) is obtained by multiplying the expected number of visits to the section in question, by the expected duration of each visit; the former is bounded by $(\frac{\beta-1}{2(3+\beta)})^{-1}$ due to Lemma 4.3, and the latter is bounded by C'_β by arguing as in the proof of Lemma 4.4 (a). Summing the contributions from (i) and (ii) gives

$$
\mathbf{E}[W_k] \leq \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1}\left(2(3^k - 1)(3\cdot 2^k + C'_\beta) + C'_\beta\right)\,.
$$

For $k \geq 3$ and with a_k as in (4.4), we get

$$
\mathbf{E}\left[\frac{W_2 + W_3 + \cdots + W_k}{a_{k-1}}\right]
$$

$$
\leq \frac{1}{4\cdot 3^{(k-2)(k-3)/2}}\sum_{j=2}^{k}\left(\frac{\beta-1}{2(3+\beta)}\right)^{-1}\left(2(3^j - 1)(3\cdot 2^j + C'_\beta) + C'_\beta\right)
$$

$$
\leq \frac{k-1}{4\cdot 3^{(k-2)(k-3)/2}}\left(\frac{\beta-1}{2(3+\beta)}\right)^{-1}\left(2(3^k - 1)(3\cdot 2^k + C'_\beta) + C'_\beta\right)
$$

which tends to 0 exponentially fast in k. Hence, by Markov's inequality and Borel–Cantelli, a.s. $(W_2 + \cdots + W_k)/a_{k-1}$ will exceed any given $\varepsilon > 0$ at most finitely

many times. In other words, we have a.s. that

$$\lim_{k\to\infty} \frac{W_2 + W_3 + \cdots + W_k}{a_{k-1}} = 0. \tag{4.17}$$

Now, it is easy to see that $\frac{U''(x)}{x} \leq \frac{W_2+W_3+\cdots+W_{k'}}{a_{k'-1}}$ where k' is the smallest k such that $a_k \geq x$. Hence (4.17) implies the desired (4.14), so the proof is complete. $\square$

5. Main construction: positive speed regime

We are almost ready to switch from considering the modified percolation configuration gotten from Ψ^*, to the full percolation configuration, including traps, obtained from Ψ. But before taking the full step we make an intermediate stop at the probability measure Ψ^{**} on $\{\text{open}, \text{closed}\}^E$ corresponding to picking a configuration according to Ψ and then deleting all traps situated directly on the path P from 0 to ∞, but leaving all other traps undeleted. We have the following variation of Proposition 4.1.

Proposition 5.1. *For any $\beta > 1$ we have*

$$\lim_{t\to\infty} \frac{X_t}{t} = \frac{\beta - 1}{\beta + 1} \quad \mathbb{P}_{\Psi^{**},\beta}\text{-a.s. on the event } \{0 \leftrightarrow \infty\}.$$

Proof. The proof of Proposition 4.1 translates verbatim to this case. The crucial point to note is that the estimates in Lemma 4.4 for the time spent in branches outside of P are still valid when traps are added, because any trap added to such a branch (or any of its subbranches) will be contained in the cone (4.10). $\square$

In this section we consider the large drift regime $\beta > \beta_c$. In view of Proposition 5.1, all we need to keep track of is the time spent in the traps directly attached to the path P. The trap attached to the order-k branch part of P will henceforth be called *trap number k*. We go on to consider random walk on the full percolation configuration obtained from Ψ. Define $U'''(x)$ as the time spent in traps directly attached to the path P before first hitting x-coordinate x. By reasoning similarly as in the decomposition of $U(x)$ at the beginning of the proof of Proposition 4.1, what we need to show is that a.s.

$$\lim_{x\to\infty} \frac{U'''(x)}{x} = 0. \tag{5.1}$$

Analogously to the notation in Section 2, we write (for $k \geq 2$ and $i \geq 1$) $T_{k,i}$ for the time spent in the trap attached to the order-k branch (trap number k, for short) in P during the i'th visit to this trap; if i exceeds the number of visits to the trap, we set $T_{k,i} = 0$. We also define the total time spent in the trap

$$T_{k,tot} = \sum_{i=1}^{\infty} T_{k,i}.$$

Still following Section 2, we define

$$T_{k,i}^* = \begin{cases} 0 & \text{if the walk hits the trap's core during this visit} \\ T_{k,i} & \text{otherwise,} \end{cases}$$

and $T_{k,tot}^* = \sum_{i=1}^{\infty} T_{k,i}^*$.

Proof of Theorem 1.1, case $\beta > \beta_c$. By Lemma 4.2, we may assume that the origin sits on a branch of order 1. We begin by noting that

$$\frac{U'''(x)}{x} \leq \frac{1}{a_{k'-1}} \sum_{j=2}^{k'} T_{j,tot}$$

where k' is the smallest k such that $a_k \geq x$. Hence, to establish the desired (5.1), it suffices to show that a.s.

$$\lim_{k \to \infty} \frac{1}{a_{k-1}} \sum_{j=2}^{k} T_{j,tot} = 0. \tag{5.2}$$

Next, we note that the expected number of times that trap number k is visited is at most $(\frac{\beta-1}{2(3+\beta)})^{-1}$ due to Lemma 4.3. In combination with Lemma 2.4, this gives

$$\mathbf{E}[T_{j,tot}^*] \leq \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{2\beta-1}{\beta-1}\right)$$

and, using (4.4),

$$\mathbf{E}\left[\frac{1}{a_{k-1}} \sum_{j=2}^{k} T_{j,tot}^*\right] \leq \frac{k \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{2\beta-1}{\beta-1}\right)}{4 \cdot 3^{(k-1)(k-2)/2}}$$

which decays to 0 (faster than) exponentially as $k \to \infty$. This allows us to exploit the familiar combination of Markov's inequality and Borel–Cantelli: for any $\varepsilon > 0$ the probability that $\frac{1}{a_{k-1}} \sum_{j=2}^{k} T_{j,tot}^*$ exceeds ε is summable over k, so that a.s.

$$\lim_{k \to \infty} \frac{1}{a_{k-1}} \sum_{j=2}^{k} T_{j,tot}^* = 0. \tag{5.3}$$

This will imply the desired (5.2) as soon as we can establish that $T_{j,tot} > T_{j,tot}^*$ for at most finitely many j. For this we proceed as in the proof of Proposition 2.1: Lemma 2.2 tells us that each time the walk enters trap number j, it has probability $\frac{\beta-1}{\beta^{e_j+1}+\beta-2}$ of hitting the core. Using again that the expected number of visits to the trap is at most $(\frac{\beta-1}{2(3+\beta)})^{-1}$, we get that

$$\mathbf{P}[T_{j,tot} > T_{j,tot}^*] \leq \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{\beta-1}{\beta^{e_j+1}+\beta-2}\right). \tag{5.4}$$

The choice (3.4) of e_j gives $e_j \geq \frac{\log j}{\log \beta_c}$, so that the estimate (5.4) may be further bounded as

$$\begin{aligned}
\mathbf{P}[T_{j,tot} > T_{j,tot}^*] &\leq \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{\beta-1}{\beta^{e_j+1}+\beta-2}\right) \\
&\leq \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{\beta-1}{\beta^{\frac{\log j}{\log \beta_c}+1}+\beta-2}\right) \\
&= \left(\frac{\beta-1}{2(3+\beta)}\right)^{-1} \left(\frac{\beta-1}{j^{\frac{\log \beta}{\log \beta_c}+1}+\beta-2}\right).
\end{aligned}$$

The assumption $\beta > \beta_c$ makes the last expression summable over j. Hence

$$\sum_{j=2}^{\infty} \mathbf{P}[T_{j,tot} > T_{j,tot}^*] < \infty$$

so that by using Borel–Cantelli yet again we get a.s. that $T_{j,tot} > T_{j,tot}^*$ for at most finitely many j. This takes us from the already-established (5.3) to the desired (5.2), and we are done. $\square$

6. Main construction: zero speed regime

Having established, in the previous section, the $\beta > \beta_c$ part of Theorem 1.1, it only remains to prove the $\beta < \beta_c$ part.

Proof of Theorem 1.1, case $\beta < \beta_c$. As usual, we assume (without loss of generality due to Lemma 4.2) that the origin sits on a branch of order 1. We need to show that for $\beta < \beta_c$ we have a.s. $\lim_{t\to\infty} \frac{X_t}{t} = 0$. For this it is enough to show that a.s.

$$\lim_{x\to\infty} \frac{U(x)}{x} = \infty \tag{6.1}$$

where, as before, $U(x)$ is the time of first arrival to x-coordinate x. To this end, we proceed as in the last part of the proof of Proposition 2.1, writing A_k for the event that the first time the random walk reaches the anchor of trap number k, it enters the trap and spends at least time $\beta^{c_k} = \beta^{3^{(k-1)(k-2)/2}}$ there. Furthermore, defining W_i as the number of events happening amongst $A_{2^{i-1}+1}, A_{2^{i-1}+2}, \ldots, A_{2^i}$ (recall (2.15)), we get using the same estimates as those leading up to (2.16) that $\mathbf{P}(W_i = 0)$ decays exponentially in i. Hence, Borel–Cantelli tells us that a.s.

$$W_i > 0 \text{ for all but finitely many } i. \tag{6.2}$$

Write x_i for the (random) x-coordinate at which trap number 2^i attaches to the path P, and note that x_i does not exceed $b_{2^i} = 4 \cdot 3^{2^i(2^i-1)/2}$. We have on the event $\{W_i > 0\}$ that

$$U(x) > \beta^{3^{2^{i-1}(2^{i-1}-1)/2}} \text{ for all } x > x_i$$

and consequently that

$$\frac{U(x)}{x} \geq \frac{1}{x_{i+1}} \beta^{3^{2^{i-1}(2^{i-1}-1)/2}}$$

$$\geq \frac{\beta^{3^{2^{i-1}(2^{i-1}-1)/2}}}{4 \cdot 3^{2^{i+1}(2^{i+1}-1)/2}} \tag{6.3}$$

for all $x \in [x_i, x_{i+1}]$. This bound tends to ∞ as $i \to \infty$. Using (6.2), we thus get (6.1), so the proof is complete. $\square$

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