STOCHASTIC POISSON EQUATIONS ASSOCIATED TO LIE ALGEBROIDS AND SOME REFINEMENTS OF A PRINCIPAL BUNDLE

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Dedicated to Professor doctor docent Dan I. Papuc at his 80th anniversary

Abstract. The aim of this paper is to present the stochastic Poisson equations associated to Lie algebroids. The obtained results are used for determination of stochastic Poisson equations associated to a refinement of a principal bundle having the affine group as structurgroup and defined by the linear group. [1]

1 Introduction

The stochastic Poisson equations has been introduced by J. -M. Bismut in [4] for Brownian motions. These have extended for semimartingales in [5]. In the paper [8] suggest to the study of stochastic Poisson equations on Lie algebroids, to have care in that the dual space of the algebroid is endowed with a Poisson structure.

In this paper we give an answer of the above question and one obtains in a canonical way the stochastic Poisson equations on Lie algebroids. These results are used for to write the stochastic Poisson equations associated to the principal bundles which compose a tissue defined by the principal bundle of affine tangent frames on a manifold and the sequence $GA(n,\mathbf{R}) \supset GL(n,\mathbf{R}) \supset \{e = (\delta^i_j)\}$, studied by Dan I. Papuc in [9](1972; MR 53 # 4058) and Dan I. Papuc and Ion P. Popescu in [10] (1973; MR 57 # 13739). For more details concerning the tissues and refinements of a differentiable principal bundles defined by closed subgroups of the structure group can be consult the paper [7] (Gh. Ivan and D. Opriş, 2002; MR 2005 b: 55032) and the references.

The paper is structured as follows. In Section 2, some basic facts on manifold valued semimartingale and stochastic Poisson equations are reviewed. In Section 3 are established the stochastic Poisson equations on a Lie algebroid. The stochastic Poisson equations associated to a refinement of principal bundles defined by the affine group and linear group are described in Section 4.

The study realized in this paper may be extended to other manifolds which are equipped with Poisson structures.

Throughout this paper all the geometrical objects like, manifolds, maps and functions always be assumed to be smooth.

[1] *AMS classification:* 60H10, 53D17, 55R05.
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2 Manifold valued semimartingale and stochastic Poisson equation

We recall the minimal necessary backgrounds on stochastic differential geometry (for notation, concepts and further details see [3], [8]).

Let M be a smooth manifold of dimension n. A continuous $M-$ valued stochastic process Γ defined on the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is called a *semimartingale* if, for any $f \in C^\infty(M)$, the process $f\circ\Gamma$ is a real valued semimartingale.

Let now V be a real vector space of dimension r. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, $X : \mathbf{R}_+ \times \Omega \to V$ a semimartingale that takes values on V with $X_0 = 0$ (X_0 is the initial value of X), and $h : M \to V^*$ is a smooth function (V^* denotes the dual of V).

Let $\{e^a | a = \overline{1,r}\}$ be a basis of V^*, and $h \in V^*$ such that $h = h_a e^a$.

The *Hamiltonian equation with stochastic component X, and Hamiltonian function h*, is the Stratonovich differential equation:

$$\delta\Gamma^h = H(x, \Gamma^h)\delta X, \tag{2.1}$$

defined by the Statonovich operator $H(v,z) : T_vV \to T_zM$ given by

$$H(v,z)u =< e^a, u > X_{h_a}(z). \tag{2.2}$$

We will refer to Γ^h as the *Hamiltonian semimartingale associated to h* with initial condition γ_0, ([8]).

Proposition 2.1. *([8]) Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, $X : \mathbf{R}_+ \times \Omega \to V$ a semimartingale and $h : M \to V^*$ a smooth function. Let Γ_0 be a $\mathcal{F}_0-$ measurable random variable and Γ^h the Hamiltonian semimartingale associated to h with initial condition Γ_0. Let ξ^h be the corresponding maximal stopping time. Then, for any stopping time $\tau < \xi^h$ the Hamiltonian semimartingale Γ^h satisfies*

$$f(\Gamma^h_{(\tau)}) - f(\Gamma^h_{(0)}) = \int_0^\tau \{f, h_a\}(\Gamma^h)dX^a + \frac{1}{2}\int_0^\tau \{\{f, h_a\}, h_b\}d[X^a, X^b], \tag{2.3}$$

for all $f \in C^\infty(M)$.

From (2.3) follows

$$x^i(\Gamma^h_{(\tau)}) - x^i(\Gamma^h_{(0)}) = \int_0^\tau \{x^i, h_a\}(\Gamma^h)dX^a + \frac{1}{2}\int_0^\tau \{\{x^i, h_a\}, h_b\}d[X^a, X^b], \tag{2.4}$$

for $i = \overline{1,n}, a, b = \overline{1,r}$.

The relations (2.4) can be written in the following form:

$$dx^i = \{x^i, h_a\}dX^a + \{\{x^i, h_a\}, h_b\}d[X^a, X^b], \quad i = \overline{1,n}, a, b = \overline{1,r}. \tag{2.5}$$

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and the smooth functions $h_a \in C^\infty(M), a = 0,1,2,...,r$. Let $h : M \to \mathbf{R}^{r+1}$ be the Hamiltonian function and consider the semimartingale $X : \mathbf{R}_+ \times \Omega \to \mathbf{R}^{r+1}$ given by $X(t,\omega) = (t, B_t^1(\omega), ..., B_t^r(\omega))$, where $B^a, a = \overline{1,r}$ are $r-$ independent Brownian motions. Lévy's characterization of Brownian motion shows ([4]) that $[B^a, B^b]_t = t\delta^{ab}$.

In this setup, the equation (2.3) reads

$$f(\Gamma^h_{(\tau)}) - f(\Gamma^h_{(0)}) = \int_0^\tau (\{f, h_a\}(\Gamma^h)dX^a + \delta^{ab}\{\{f, h_a\}, h_b\})dt + \int_0^\tau \{f, h_a\}dB^a, \tag{2.6}$$

$$dx^i = (\{x^i, h_0\} + \delta^{ab}\{\{x^i, h_a\}, h_b\})dt + \{x^i, h_a\}dB^a, \tag{2.7}$$

for $i = \overline{1,n},\ a, b = \overline{1,r}$.

These equations have been studied by Bismut in [4] in the particular case in which the Poisson manifold $(M, \{\cdot,\}$ is just the symplectic Euclidean space $\mathbf{R}^{2n}$ with the canonical symplectic form.

Proposition 2.2. *Let $(\mathbf{R}^n, \{\cdot,\cdot\})$ be a Poisson manifold with $\{x^i, x^j\} = \Lambda_k^{ij} x^k$, and $h_a = \alpha_{ai}x^i,\ a = \overline{1,r}$ with $\alpha_{ai} \in \mathbf{R}$. The equation (2.7) is given by*

$$dx^i = (\Lambda_\ell^{ij}\frac{\partial h_0}{\partial x^j} + \delta^{ab}\alpha_{aj}\alpha_{bk}\Lambda_p^{ij}\Lambda_{p\ell}^k)x^\ell dt + \alpha_{aj}\Lambda_\ell^{ij}x^\ell dB^a, \tag{2.8}$$

for $i, j, k, \ell, p = \overline{1,n},\ a, b = \overline{1,r}$.

The equations (2.8) are called the *stochastic Poisson equations* associated to Poisson manifold $(\mathbf{R}^n, \{\cdot,\cdot\})$.

Applying the relations (2.8) for the Poisson structures defined on $\mathbf{R}^3, \mathbf{R}^6, \mathbf{R}^9$ one obtains the stochastic Poisson equations for the rigid body on $SO(3), SO(2,1)$, heavy top etc. ([1]).

3 Stochastic Poisson equations associated to a Lie algebroid

The theory of Lie algebroids has recently proved to be extremely fruitful in tackling some problems in the context of geometric mechanics ([8]). Recall that the dual of a Lie algebroids admits a canonical Poisson structure and, therefore, one can naturally consider Hamiltonian systems on them. According to the results and the acceptance of this new formalism we shall investigate the consequences of having stochastic processes taking values on their duals for mechanical purposes.

A *Lie algebroid* A over a manifold M is a vector bundle $\pi : A \to M$ together with a Lie algebra structure $[\cdot,\cdot]$ on the space of sections $Sec(A)$ and a bundle map $b : A \to TM$ (called *anchor map*) such that:

(i) the induced map $b : Sec(A) \to Sec(TM) = \mathcal{X}(M)$ is a homomorphism of Lie algebras;

(ii) for any $a_1, a_2 \in Sec(A)$ and smooth function $f \in C^\infty(M)$, the *Leibniz identity* holds:

$$[a_1, fa_2] = f[a_1, a_2] + b(a_1)(f)a_2. \tag{3.1}$$

For a Lie algebroid $(E, \pi, M, [\cdot,\cdot], b)$, we consider the manifold M of dimension n and denote the rank of the vector bundle A with r. Recalling the construction of a canonical Poisson bracket on the dual A^* of the vector bundle A ([2]). If one fixes local coordinates $(x^i), i = \overline{1,n}$ over a trivializing neighborhood $U \subset M$ and choose a basis of local sections $\{e_\alpha | \alpha = \overline{1,r}\}$ of the vector bundle A, then the corresponding local coordinates on A are denoted by $(x^i, y^\alpha), i = \overline{1,n}, \alpha = \overline{1,r}$.

The local expression of a section $a \in Sec(A)$ with to respect the basis $\{e_\alpha\}$ is $a = a^\alpha e_\alpha$, with $a^\alpha \in C^\infty(U),\ \alpha = \overline{1,r}$. Since $e_\alpha \in Sec(A)$, we have $b(e_\alpha) \in \mathcal{X}(U)$ and $[e_\alpha, e_\beta] \in Sec(A)$. Then there exists the functions $b^i_\alpha, C^\gamma_{\alpha\beta} \in C^\infty(U)$ such that:

$$\begin{cases} b(e_\alpha) & = & b^i_\alpha \dfrac{\partial}{\partial x^i}, & \text{for} \quad i = \overline{1,n},\ \alpha = \overline{1,r} \\ [e_\alpha, e_\beta] & = & C^\gamma_{\alpha\beta} e_\gamma, & \text{for} \quad \alpha, \beta, \gamma = \overline{1,r}. \end{cases} \tag{3.2}$$

The functions $b^i_\alpha, C^\gamma_{\alpha\beta} \in C^\infty(U)$ given by the relations (3.2) are called the *structure functions* of the Lie algebroid $(E, [\cdot,\cdot], b)$ with to respect the chosen local coordinates system.

The defining relations for a Lie algebroid translate into certain partially differential equations involving its structure functions.

One define a Poisson structure on A^* as follows. Let $\{\xi_\alpha\}$ the linear coordinates on the fibers of A^* associated with the basis of local sections $e_\alpha, \alpha = \overline{1,r}$. The Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(A^*)$ is defined by

$$\Lambda^{ij} = \{x^i, x^j\} = 0, \quad \Lambda^i_\alpha = \{x^i, \xi_\alpha\} = b^i_\alpha, \quad \Lambda_{\alpha,\beta} = \{\xi_\alpha, \xi_\beta\} = C^\gamma_{\alpha\beta}\xi_\gamma, \tag{3.3}$$

for $i, j = \overline{1,n},\ \alpha, \beta, \gamma = \overline{1,r}$.

One checks that this bracket is independent of the choice of local coordinates and basis.

Let $a \in Sec(A)$ be a section of the vector bundle A. Then it defines in a natural way a function $f_a : A^* \to \mathbf{R}$ which is linear in the fibers and is given by

$$f_a(x, \xi) = a^\alpha(x)\xi_\alpha, \quad \alpha = \overline{1,r}. \tag{3.4}$$

Proposition 3.1. *([2]) The assignement $a \mapsto f_a$ defines a Lie algebra homomorphism $(Sec(A), [\cdot,\cdot]) \to (C^\infty(A^*), \{\cdot,\cdot\})$. Moreover, the Hamiltonian vector field associated with f_a is given by*

$$X_{f_a} = b^i_\beta a^\beta \frac{\partial}{\partial x^i} + (a^\gamma C^\lambda_{\beta\gamma} - b^j_\beta \frac{\partial a^\lambda}{\partial x^j})\xi_\lambda \frac{\partial}{\partial \xi_\beta}, \quad i,j = \overline{1,n},\ \beta,\gamma,\lambda = \overline{1,r}. \tag{3.5}$$

Let be the functions $f_s : A^* \to \mathbf{R}$ for each $s = \overline{1,p}$, where

$$f_s(x,\xi) = a^\alpha_s(x)\xi_\alpha, \quad \alpha = \overline{1,r}. \tag{3.6}$$

Using the relations (3.3) and (3.6), from (2.7) we obtain the *stochastic Poisson equations* associated to $h : A^* \to \mathbf{R}$ and $f_s,\ s = \overline{1,p}$, given by

$$\begin{cases} dx^i &= (b^i_\alpha \dfrac{\partial h}{\partial \xi_\alpha} + \delta^{su} b^k_\lambda a^\lambda_s \dfrac{\partial}{\partial x^k}(b^i_\beta a^\beta_u))dt + b^i_\beta a^\beta_s dB^s, \\ d\xi_\alpha &= (b^i_\alpha \dfrac{\partial h}{\partial x^i} + C^\gamma_{\alpha\beta}\xi_\gamma \dfrac{\partial h}{\partial \xi_\beta} + \delta^{su} b^j_\gamma \dfrac{\partial}{\partial x^j}(b^i_\alpha \dfrac{\partial a^\gamma_u}{\partial x^i}) a^\varepsilon_s \xi_\varepsilon + \\ & \quad + \delta^{su} C^\varepsilon_{\theta\gamma} b^i_\alpha \dfrac{\partial a^\theta_u}{\partial x^i} a^\gamma_s \xi_\varepsilon)dt + (b^i_\alpha \dfrac{\partial a^\lambda_s}{\partial x^i}\xi_\lambda + C^\gamma_{\alpha\mu} a^\mu_s \xi_\gamma)dB^s. \end{cases} \tag{3.7}$$

Let the tangent bundle $TM \to M$ and cotangent bundle $T^*M \to M$. The total space of the vector bundle $T^*M \oplus A^*$ has the Poisson structure $\{\cdot,\cdot\}$, defined by

$$\begin{cases} \Lambda^{ij} = \{x^i, x^j\} = 0, & \Lambda^i_j = \{x^i, p_j\} = \delta^i_j, & \Lambda^i_\alpha = \{x^i, \xi_\alpha\}, \\ \Lambda_{ij} = \{p_i, p_j\}, & \Lambda_{\alpha\beta} = \{\xi_\alpha, \xi_\beta\} = C^\gamma_{\alpha\beta}\xi_\gamma, & \Lambda_{i\alpha} = \{p_i, \xi_\alpha\}. \end{cases} \tag{3.8}$$

Proposition 3.2. *The stochastic Poisson equations defined by $h : T^*M \oplus A^* \to \mathbf{R}$ and functions $g_s : T^*M \oplus A^* \to \mathbf{R},\ s = \overline{1,p}$, given by*

$$\begin{cases} g_s(x,p,\xi) &= a^\alpha_s(x)\xi_\alpha + d^i_s p_i,\ s = \overline{1,p}, \\ h(x,p,\xi) &= \dfrac{1}{2}k^{ij}(x)p_i p_j + k^{i\alpha}(x)p_i\xi_\alpha + \dfrac{1}{2}k^{\alpha\beta}(x)\xi_\alpha\xi_\beta, \end{cases} \tag{3.9}$$

are

$$\begin{cases} dx^i &= (k^{ij} + b^i_\alpha k^{j\alpha})p_j + (k^{i\beta} + b^i_\alpha k^{\alpha\beta})p_\beta + \delta^{su}(d^j_u + b^j_\alpha a^\alpha_u)\cdot \\ & \cdot \dfrac{\partial}{\partial x^j}(a^i_s + b^i_\alpha a^\alpha_s) + (d^i_s + b^i_\alpha a^\alpha_s)dB^s(t), \\ dp_j &= (-(\dfrac{1}{2}\dfrac{\partial k^{h\ell}}{\partial x^j}p_h p_\ell + \delta^{us}(b^m_\alpha a^\alpha_u + d^m_u)(\dfrac{\partial^2 a^\gamma_s}{\partial x^m \partial x^j}\xi_\gamma + \dfrac{\partial^2 d^i_s}{\partial x^m \partial x^j}p_i) - \\ & -\delta^{su}\dfrac{\partial d^\alpha_s}{\partial x^j}(\dfrac{\partial a^\alpha_u}{\partial x^\ell}\xi_\alpha + \dfrac{\partial d^i_u}{\partial x^\ell}p_i))dt + (\dfrac{\partial a^\alpha_s}{\partial x^j}\xi_\alpha + \dfrac{\partial d^i_s}{\partial x^j}p_i)dB^s(t), \\ d\xi_\alpha &= (-b^i_\alpha(\dfrac{1}{2}\dfrac{\partial k^{h\ell}}{\partial x^i}p_h p_\ell + \dfrac{\partial k^{\alpha\beta}}{\partial x^i}\xi\alpha\xi_\beta + \dfrac{1}{2}\dfrac{\partial k^{j\beta}}{\partial x^i}p_j\xi_\beta) - \\ & -\delta^{us}(b^\ell_\beta a^\beta_u + d^\ell_u)\cdot\dfrac{\partial}{\partial x^\ell}(b^i_\alpha\dfrac{\partial a^\beta_s}{\partial x^i}\xi_\beta + b^i_\alpha\dfrac{\partial d^j_s}{\partial x^i}p_j) + \delta^{su}b^i_\alpha(\dfrac{\partial d^\ell_s}{\partial x^i} + b^\ell_\gamma\dfrac{\partial a^\gamma_s}{\partial x^i})\cdot \\ & \cdot(\dfrac{\partial a^\mu_u}{\partial x^\ell}\xi_\mu + \dfrac{\partial d^j_s}{\partial x^i}p_j))dt - b^i_\alpha(\dfrac{\partial a^\beta_s}{\partial x^i}\xi_\beta + \dfrac{\partial d^j_s}{\partial x^i}p_j)dB^s(t). \end{cases} \tag{3.10}$$

4 Stochastic Poisson equations associated to refinement of a principal bundle having the affine group as structure group

We start with some definitions and results of [3] that we will use later.

Let $\pi_G : P \to M$ be a left principal bundle with the Lie group G as structure group, where $M = P/G$. Let $\mathcal{G}$ the Lie algebra of the Lie group G. The associated bundle with standard fibre $\mathcal{G}$, where the action of G on $\mathcal{G}$ is the adjoint action is called the *adjoint bundle* and it is denoted by $\widetilde{\mathcal{G}}^G = Ad_G(P)$. We let $\widetilde{\pi}_G : \widetilde{\mathcal{G}}^G \to M = P/G$ denote the projection given by $\widetilde{\pi}_G([q,\xi]_G = [q]_G$.

Consider now the bundle $TM \otimes \widetilde{\mathcal{G}}^G \to M$ and we assume that is given a (principal) connection A^G on the principal bundle $\pi_G : P \to M$, determined by the local functions $\{A^a_i(x)\}$ on M. Given the basis $\{\varepsilon_a | a = \overline{1,p}\}$ for the Lie algebra $\mathcal{G}$ having $\{C^a_{bc}\}$ as structure constants, one obtains the local basis $\{\dfrac{\partial}{\partial x^i}, \varepsilon_a\}$ for $Sec(TM \otimes \widetilde{\mathcal{G}}^G)$ such that $[\varepsilon_a, \varepsilon_b] = C^c_{ab}\varepsilon_c$.

The corresponding covariant derivative $\widetilde{\nabla}^{A^G}\xi$ of a section $\xi = \xi^a\varepsilon_a$ and $X \in Sec(TM)$ reads

$$\widetilde{\nabla}^{A^G}_X \xi \ = X^i(\frac{\partial \xi^a}{\partial x^i} + C^a_{bc}A^{Gb}_i \xi^c)\varepsilon_a. \tag{4.1}$$

The curvature $\widetilde{B}^{A^G}$ of the connection A is given by

$$\widetilde{B}^{A^G} = \frac{1}{2}\widetilde{B}^{Ga}_{ij} dx^i \wedge dx^j \varepsilon_a, \quad \text{where} \tag{4.2}$$

$$\widetilde{B}^{Ga}_{ij} = \frac{\partial A^{Ga}_j}{\partial x^i} - \frac{\partial A^{Ga}_i}{\partial x^j} + C^a_{bc} A^{Gb}_i A^{Gc}_j. \tag{4.3}$$

Let $X_i \oplus \overline{\xi}_i \in Sec(TM \oplus \widetilde{\mathcal{G}}^G)$, $i = 1,2$ be given two sections. Then

$$[X_1 \oplus \overline{\xi}_1, X_2 \oplus \overline{\xi}_2] = [X_1, X_2] \oplus \widetilde{\nabla}^{A^G}_{X_1}\xi_2 - \widetilde{\nabla}^{A^G}_{X_2}\xi_1 - \widetilde{B}^{A^G}(X_1, X_2) + [\overline{\xi}_1, \overline{\xi}_2]. \tag{4.4}$$

For $\{ \frac{\partial}{\partial x^i} \oplus \varepsilon_a,\ i = \overline{1,n},\ a = \overline{1,p}$ we have

$$[\frac{\partial}{\partial x^i} \oplus \varepsilon_a, \frac{\partial}{\partial x^j} \oplus \varepsilon_b] = (C^d_{cb} A^{Gc}_i - C^d_{ca} A^{Gc}_j - \widetilde{B}^{A^G d}_{ij} + C^d_{ab})\varepsilon_d. \tag{4.5}$$

Let $(x^i, \dot{x}^i, \xi^a)$ the local coordinates of $TM \oplus \widetilde{\mathcal{G}}^G$ and (x^i, p_i, μ_a) the local coordinates of $T^*M \oplus \widetilde{\mathcal{G}}^{G^*}$. The structure Poisson on $T^*M \oplus \widetilde{\mathcal{G}}^{G^*}$ is given by

$$\begin{cases} \{x^i, x^j\} = 0, & \{x^i, p_j\} = \delta^i_j, & \{p_i, p_j\} = -B^c_{ij}\mu_c, \\ \{p_i, \mu_a\} = -C^d_{ca} A^c_i \mu_d, & \{\mu_a, \mu_b\} = C^c_{ab}\mu_c, & \{x^i, \mu_a\} = 0. \end{cases} \tag{4.6}$$

Using the method for determination of Poisson equations in the case of Lie algebroids one obtains the following proposition.

Proposition 4.1. *The stochastic Poisson equations defined by the functions* $h : T^*M \oplus \widetilde{\mathcal{G}}^{G^*} \to \mathbf{R}$ *and* $f : T^*M \oplus \widetilde{\mathcal{G}}^{G^*} \to \mathbf{R}$ *with* $f(x,p,\mu) = a^j(x)p_j + d^a(x)\mu_a$ *are*

$$\begin{cases} dx^i &= (\frac{\partial h}{\partial x^i} + \frac{\partial a^i}{\partial x^\ell} a^\ell)dt + a^i dB(t), \\ dp_i &= (-\frac{\partial h}{\partial x^i} - B^c_{ij}\mu_c \frac{\partial h}{\partial p_j} - C^d_{ca}\mu_a A^c_i \frac{\partial h}{\partial \mu_a} + \{\{p_i, f\}, f\})dt - \\ & -(B^c_{ij}\mu_c a^j + C^d_{ca}\mu_d A^c_i d^a)dB(t), \\ d\mu_a &= (C^d_{ca}\mu_d A^c_j \frac{\partial h}{\partial p_j} + C^c_{ab}\mu_c \frac{\partial h}{\partial \mu_b} + \{\{\mu_a, f\}, f\})dt + \\ & +(C^d_{ca}\mu_d A^c_j a^j + C^c_{ab}\mu_c d^b)dB(t). \end{cases} \tag{4.7}$$

Let $\pi_G : P \to M = P/G$ the principal bundle with the structure group G. We assume that is given a sequence $\mathcal{N}_2 = (G \supset K \supset \{e\})$ of closed subgroups of G. If we denote $\eta = (P, \pi_G, M = P/G, G)$, then the pair $(\eta, \mathcal{N}_2)$ determines a refinement $(\eta; \eta_{01}, \eta_{12})$ of η defined by K, where $\eta_{01} = (P/K, \pi_{GK}, M, G/K, G/N)$

and $\eta_{12} = (P, \pi_K, P/K, K)$, and N is the largest normal subgroup of G included in K (see Papuc [9], Ivan and Opriş [7]).

Let A^G and A^K two connections on P given by the forms $A^G : TP \to \mathcal{G}$, $A^K : TP \to \mathcal{K}$, where $\mathcal{G}$ resp., $\mathcal{K}$ is the Lie algebra of G resp., K.

Let the adjoint bundles $\widetilde{\mathcal{G}}^G = Ad_G(P)$ and $\widetilde{\mathcal{K}}^K = Ad_K(P)$. The vector bundles $TM \oplus \widetilde{\mathcal{G}}^G \to M$ and $T(P/K) \oplus \widetilde{\mathcal{K}}^K \to P/K$ are called the *reduced bundles associated to refinement defined by the pair* $(\eta, \mathcal{N}_2)$.

Let us we apply the above considerations in the case when the group $G = GA(n, \mathbf{R})$ is the affine group and $K = GL(n, \mathbf{R})$ is the linear group. We obtain thus the sequence $\mathcal{N}_2 = (G = GA(n, \mathbf{R}) \supset K = GL(n, \mathbf{R}) \supset \{e\})$. The Lie algebra $\mathcal{G}$ of G has the base $\{e^i_j, e_j\}$ and we have $[e^i_j, e^\ell_k] = \delta^i_k e^\ell_j - \delta^\ell_j e^i_k$, $[e^i_j, e_k] = \delta^i_k e_j$, $[e_i, e_j] = 0$. The Lie algebra $\mathcal{K}$ of K has the base $\{e^i_j\}$ and we have $[e^i_j, e^\ell_k] = \delta^i_k e^\ell_j - \delta^\ell_j e^i_k$.

Let $\pi_G : P \to M$ the principal bundle having the affine group G as structure group and the local coordinates (x^i, y^i_j, y^i) on P. The base of sections of the vector bundle $\widetilde{\mathcal{G}}^G \to M$ is $\varepsilon^i_j = y^h_j \dfrac{\partial}{\partial y^h_i}$, $\varepsilon_j = y^h_j \dfrac{\partial}{\partial y^h}$.

Let A^G a connection on the principal bundle $\pi_G : P \to M$ given by the functions (A^h_{kr}, A^h_k) on M. From (4.1) follows

$$\begin{cases} \widetilde{\nabla}^{A^G}_{\frac{\partial}{\partial x^i}} \varepsilon^\ell_k & = & (A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\varepsilon^q_p - A^\ell_i \varepsilon_k, \\ \widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial x^r}} \varepsilon_k & = & A^i_{kr}\varepsilon_i, \\ \widetilde{B}^{A^G} & = & \dfrac{1}{2}(B^\ell_{kij} dx^i \wedge dx^j \otimes \varepsilon^k_\ell + B^\ell_{ij} dx^i \wedge dx^j \otimes \varepsilon_\ell). \end{cases} \tag{4.8}$$

Let $(x^i, p_i, \mu^\ell_k, \mu_\ell)$ the local coordinates on $T^*M \oplus \widetilde{\mathcal{G}}^{G^*}$. The structure Poisson is given by the following relations

$$\begin{cases} \{x^i, x_j\} = 0, \quad \{x^i, \mu^\ell_k\} = 0, \quad \{x^i, \mu_\ell\} = 0, \quad \{\mu_i, \mu_j\} = 0, \\ \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = -B^\ell_{kij}\mu^k_\ell - B^\ell_{ij}\mu_\ell, \\ \{p_i, \mu^k_\ell\} = (A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\mu^q_p - A^\ell_i \mu_k, \quad \{p_i, \mu_k\} = A^p_{ki}\mu_p, \\ \{\mu^i_j, \mu^\ell_k\} = \delta^i_k \mu^\ell_j - \delta^\ell_j \mu^i_k, \quad \{\mu^i_k, \mu_j\} = \delta^i_k \mu_j. \end{cases} \tag{4.9}$$

Using the method for determination of Poisson equations in the case of Lie algebroids one obtains the following proposition.

Proposition 4.2. *The stochastic Poisson equations defined by the functions* $h : T^*M \oplus \widetilde{\mathcal{G}}^{G^*} \to \mathbf{R}$ *and* $f : T^*M \oplus \widetilde{\mathcal{G}}^{G^*} \to \mathbf{R}$ *with*

$$f(x^i, p_j, \mu^\ell_k, \mu_\ell) = a^j(x)p_j + d^k_\ell(x)\mu^\ell_k + g^\ell(x)\mu_\ell$$

are

$$\begin{cases} dx^i &= (\frac{\partial h}{\partial p_i} + \frac{\partial a^i}{\partial x^k}a^k)dt + a^i dB(t), \\ dp_i &= (\frac{\partial h}{\partial x^i} - (B^\ell_{kij}\mu^k_\ell + B^\ell_{ij}\mu_\ell)\frac{\partial h}{\partial p_j} + ((A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\mu^q_p - \\ & -A^\ell_i\mu_k)\frac{\partial h}{\partial \mu^\ell_k} + A^p_{ki}\mu_p\frac{\partial h}{\partial \mu_k}) + \{\{p_i, f\}, f\})dt + \{p_i, f\}dB(t), \\ d\mu^\ell_k &= (((A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\mu^q_p - A^\ell_i\mu_k)d^k_\ell + A^p_{ki}\mu^\ell_p g^i + \\ & +\{\{\mu^\ell_k, f\}, f\})dt + \{\mu^\ell_k, f\}dB(t), \\ d\mu_i &= (-A^p_{ik}\mu_p a^k - \mu_i\delta^\ell_k d^k_\ell + \{\{\mu_i, f\}, f\})dt + \{\mu_i, f\}dB(t). \end{cases} \tag{4.10}$$

Let $\pi_K : P \to P/K$ the principal bundle having the affine group $K = GL(n, \mathbf{R})$ as structure group and the local coordinates (x^i, q^i) on P/K. The base of sections of the vector bundle $\widetilde{\mathcal{K}}^K \to P/K$ is $\varepsilon^i_j = y^h_j \frac{\partial}{\partial x^h_i}$.

Let A^K a connection on the principal bundle $\pi_K : P \to P/K$ given by the functions (A^k_{ij}, B^k_{ij}) on P/K. From the relations (4.1) follows:

$$\begin{cases} \widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial x^i}} \varepsilon^\ell_k &= (A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\varepsilon^q_p \\ \widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial q^i}} \varepsilon^\ell_k &= (B^p_{ki}\delta^\ell_q - B^\ell_{qi}\delta^p_k)\varepsilon^q_p \\ \widetilde{B}^{A^K} &= \frac{1}{2}(B^\ell_{kij}dx^i \wedge dx^j + B^\ell_{kij}dq^i \wedge dq^j + B^\ell_{kij}dx^i \wedge dq^j) \otimes \varepsilon^k_\ell. \end{cases} \tag{4.11}$$

Let $(x^i, q^i, \dot{x}^i, \dot{q}^i, \xi^\ell_k)$ the local coordinates on $T(P/K) \oplus \widetilde{\mathcal{K}}^K$ and $(x^i, q^i, p_i, \lambda_i, \mu^\ell_k)$ the local coordinates on $T^*(P/K) \oplus \widetilde{\mathcal{K}}^{K^*}$. The structure Poisson is given by the following relations:

$$\begin{cases} \{x^i, x^j\} = 0, \quad \{x^i, q^k\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{x^i, \lambda_j\} = \delta^i_j, \\ \{x^i, \mu^\ell_k\} = 0, \quad \{q^i, q^j\} = 0, \quad \{q^i, p_j\} = 0, \quad \{q^i, \lambda_j\} = 0, \\ \{q^i, \mu^\ell_k\} = 0, \quad \{p_i, p_j\} = -\frac{1}{2}B^\ell_{kij}\mu^k_\ell, \quad \{p_i, \lambda_j\} = -\frac{1}{2}B^\ell_{kij}\mu^k_\ell, \\ \{p_i, \mu^\ell_k\} = (A^p_{ki}\delta^\ell_q - A^\ell_{qi}\delta^p_k)\mu^q_p, \quad \{\lambda_i, \lambda_j\} = -\frac{1}{2}B^\ell_{kij}\mu^k_\ell, \\ \{\lambda_i, \mu^\ell_k\} = (B^p_{ki}\delta^\ell_q - B^\ell_{qi}\delta^p_k)\mu^q_p, \quad \{\mu^i_j, \mu^\ell_k\} = \delta^i_k\mu^\ell_j - \delta^\ell_j\mu^i_k. \end{cases} \tag{4.12}$$

Using the method for determination of Poisson equations in the case of Lie algebroids one obtains the following proposition.

Proposition 4.3. *The stochastic Poisson equations defined by the functions* $h : T^*(P/K) \oplus \widetilde{\mathcal{K}}^{K^*} \to \mathbf{R}$ *and* $f : T^*(P/K) \oplus \widetilde{\mathcal{K}}^{K^*} \to \mathbf{R}$ *with*

$$f(x^i, q^i, p_j, \lambda_j, \mu_k^\ell) = a^j(x,q)p_j + d^j(x,q)\lambda_j + g_k^j(x,q)\mu_j^k$$

are

$$\left\{\begin{array}{lcl} dx^i & = & (\dfrac{\partial h}{\partial p_i} + \{\{x^i, f\}, f\})dt + \{x^i, f\}dB(t), \\ dp_i & = & (-\dfrac{\partial h}{\partial x^i} - \dfrac{1}{2}B_{kij}^\ell \mu_\ell^k + (A_{ki}^p \delta_q^\ell - A_{qi}^\ell \delta_k^p)\mu_p^q \dfrac{\partial h}{\partial \mu_k^\ell} + \\ & & +\{\{p_i, f\}, f\})dt + \{p_i, f\}dB(t), \\ dq^i & = & (\dfrac{\partial h}{\partial \lambda_i} + \{\{q^i, f\}, f\})dt + \{q^i, f\}dB(t), \\ d\lambda_i & = & (-\dfrac{\partial h}{\partial q^i} - \dfrac{1}{2}B_{kij}^\ell \mu_\ell^k + (B_{ki}^p \delta_q^\ell - B_{qi}^\ell \delta_k^p)\mu_p^q \dfrac{\partial h}{\partial \mu_k^\ell} + \\ & & +\{\{\lambda_i, f\}, f\})dt + \{\lambda_i, f\}dB(t), \\ d\mu_k^\ell & = & (-(A_{kj}^p \delta_q^\ell + A_{qj}^\ell \delta_k^p)\mu_p^q \dfrac{\partial h}{\partial p_j} - (B_{kj}^p \delta_q^\ell - B_{kj}^\ell \delta_k^p)\mu_p^q \dfrac{\partial h}{\partial \lambda_j} + \\ & & +(\delta_j^\ell \mu_k^i - \delta_k^i \mu_j^\ell)\dfrac{\partial h}{\partial \mu_j^i} + \{\{\mu_k^\ell, f\}, f\})dt + \{\mu_k^\ell, f\}dB(t). \end{array}\right. \tag{4.13}$$

The study of equations (3.10), (4.7), (4.10) and (4.13) enable by choosing of the functions h and f_a.

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