Some Properties of an Infinite Family of Deformations of the Harmonic Oscillator

Christiane Quesne

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles,

Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

Abstract

In memory of Marcos Moshinsky, who promoted the algebraic study of the harmonic oscillator, some results recently obtained on an infinite family of deformations of such a system are reviewed. This set, which was introduced by Tremblay, Turbiner, and Winternitz, consists in some Hamiltonians H_k on the plane, depending on a positive real parameter k. Two algebraic extensions of H_k are described. The first one, based on the elements of the dihedral group D_{2k} and a Dunkl operator formalism, provides a convenient tool to prove the superintegrability of H_k for odd integer k. The second one, employing two pairs of fermionic operators, leads to a supersymmetric extension of H_k of the same kind as the familiar Freedman and Mende super-Calogero model. Some connection between both extensions is also outlined.

Keywords: quantum Hamiltonians; superintegrability; exchange operators; supersymmetry

PACS numbers: 03.65.Fd, 11.30.Pb

1 INTRODUCTION

During his long and prolific career, the study of the harmonic oscillator has been one of the prominent topics dealt with by Marcos Moshinsky [1, 2], with whom I have enjoyed the privilege of collaborating for many years including on this subject [3, 4, 5]. As a token of reminiscence and gratitude for all what I learnt from him, in this paper dedicated to his memory I will review some recent results on a related problem.

In [6], Tremblay, Turbiner and Winternitz (TTW) indeed introduced an infinite family of exactly solvable Hamiltonians

$$\begin{aligned} H_k &= -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{k^2}{r^2}[a(a-1)\sec^2 k\varphi + b(b-1)\csc^2 k\varphi], \\ 0 &\le r < \infty, \qquad 0 \le \varphi < \frac{\pi}{2k}, \end{aligned} \tag{1}$$

which may be considered as deformations of the harmonic oscillator on a plane and reduce for $k = 1, 2, 3$ to those of the familiar Smorodinsky-Winternitz [7, 8], BC_2 model [9] and Calogero-Marchioro-Wolfes [10, 11] systems. They showed that for any real k, H_k is integrable with $X_k = -\partial_\varphi^2 + k^2[a(a-1)\sec^2 k\varphi + b(b-1)\csc^2 k\varphi]$ an integral of motion. They also conjectured (and actually proved for $k = 1, 2, 3, 4$) that it is superintegrable for any integer k, the second integral of motion Y_{2k} being some $2k$th-order differential operator. Later on, this was established by myself for odd k [12], then by Kalnins, Kress and Miller for integer (or even rational) k [13].

In the following, two extensions of H_k will be considered. The first one [14], based on the elements of the dihedral group D_{2k}, was used in the superintegrability proof of Ref. [12] while the second one [15], employing two pairs of fermionic operators, led to a supersymmetric extension of the same kind as the familiar Freedman and Mende super-Calogero model [16]. Some connection between both extensions [17] will also be reviewed.

2 DIHEDRAL GROUP EXTENSION

The dihedral group D_{2k} has $4k$ elements $\mathcal{R}^i$ and $\mathcal{R}^i\mathcal{I}$, $i = 0, 1, \ldots, 2k-1$, satisfying the relations

$$\mathcal{R}^{2k} = \mathcal{I}^2 = 1, \qquad \mathcal{I}\mathcal{R} = \mathcal{R}^{2k-1}\mathcal{I}, \qquad \mathcal{R}^\dagger = \mathcal{R}^{2k-1}, \qquad \mathcal{I}^\dagger = \mathcal{I}. \tag{2}$$

They are realizable on the Euclidean plane as the rotation operator through angle π/k, $\mathcal{R} = \exp\left(\frac{1}{k}\pi\partial_\varphi\right)$, and the operator changing φ into $-\varphi$, $\mathcal{I} = \exp(\mathrm{i}\pi\varphi\partial_\varphi)$ [14].

With their use, one can extend the partial derivatives ∂_r and ∂_φ into some differential-difference operators,

$$
\begin{aligned}
D_r &= \partial_r - \frac{1}{r}(a\mathcal{R} + b)\left(\sum_{i=0}^{k-1}\mathcal{R}^{2i}\right)\mathcal{I}, \\
D_\varphi &= \partial_\varphi + a\sum_{i=0}^{k-1}\tan\left(\varphi + \frac{i\pi}{k}\right)\mathcal{R}^{k+2i}\mathcal{I} - b\sum_{i=0}^{k-1}\cot\left(\varphi + \frac{i\pi}{k}\right)\mathcal{R}^{2i}\mathcal{I},
\end{aligned}
\tag{3}
$$

similar to Dunkl operators [18]. They satisfy more complicated relations than ∂_r and ∂_φ, namely

$$
\begin{aligned}
& D_r^\dagger = -D_r - \frac{1}{r}\left[1 + 2(a\mathcal{R} + b)\left(\sum_{i=0}^{k-1}\mathcal{R}^{2i}\right)\mathcal{I}\right], \qquad D_\varphi^\dagger = -D_\varphi, \\
& \mathcal{R}D_r = D_r\mathcal{R}, \qquad \mathcal{I}D_r = D_r\mathcal{I}, \qquad \mathcal{R}D_\varphi = D_\varphi\mathcal{R}, \qquad \mathcal{I}D_\varphi = -D_\varphi\mathcal{I}, \\
& [D_r, D_\varphi] = -\frac{2}{r}(a\mathcal{R} + b)\left(\sum_{i=0}^{k-1}\mathcal{R}^{2i}\right)\mathcal{I}D_\varphi.
\end{aligned}
\tag{4}
$$

In (3), k may be any odd integer. For even k, D_r and D_φ assume a different form, which we will not consider here.

The TTW Hamiltonian H_k and its integral of motion X_k can be generalized by incorporating the elements of D_{2k}, as well as the new operators D_r and D_φ. The resulting D_{2k}-extended operators

$$
\begin{aligned}
\mathcal{H}_k &= -D_r^2 - \frac{1}{r}\left[1 + 2(a\mathcal{R} + b)\left(\sum_{i=0}^{k-1}\mathcal{R}^{2i}\right)\mathcal{I}\right]D_r - \frac{1}{r^2}D_\varphi^2 + \omega^2 r^2 \\
&= -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\left[D_\varphi^2 - k(a^2 + b^2 + 2ab\mathcal{R})\sum_{i=0}^{k-1}\mathcal{R}^{2i}\right] + \omega^2 r^2
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\mathcal{X}_k = -D_\varphi^2 &= -\partial_\varphi^2 + \sum_{i=0}^{k-1}\sec^2\left(\varphi + \frac{i\pi}{k}\right)a(a - \mathcal{R}^{k+2i}\mathcal{I}) \\
&\quad + \sum_{i=0}^{k-1}\csc^2\left(\varphi + \frac{i\pi}{k}\right)b(b - \mathcal{R}^{2i}\mathcal{I}) - k(a^2 + b^2 + 2ab\mathcal{R})\sum_{i=0}^{k-1}\mathcal{R}^{2i}
\end{aligned}
\tag{6}
$$

turn out to be invariant under D_{2k} and to give back H_k and X_k after projection in the D_{2k} identity representation, i.e., by letting $\mathcal{R}$ and $\mathcal{I}$ go to 1.

2.1 Application to the superintegrability problem of $\boldsymbol{H_k}$ for odd $\boldsymbol{k}$

The formalism considered above can still be enlarged [12] by introducing a set of k (dependent) pairs of modified boson creation and annihilation operators

$$\begin{aligned} A_i &= \frac{1}{\sqrt{2\omega}} \left[\cos\left(\varphi + \frac{i\pi}{k}\right)(\omega r + D_r) - \frac{1}{r}\sin\left(\varphi + \frac{i\pi}{k}\right) D_\varphi\right], \\ A_i^\dagger &= \frac{1}{\sqrt{2\omega}} \left[\cos\left(\varphi + \frac{i\pi}{k}\right)(\omega r - D_r) + \frac{1}{r}\sin\left(\varphi + \frac{i\pi}{k}\right) D_\varphi\right], \end{aligned} \tag{7}$$

where $i = 0, 1, \ldots, k-1$. One can show that $A_i^\dagger$ is the Hermitian conjugate of A_i and that the $2k$ operators A_i and $A_i^\dagger$ satisfy the modified commutations relations

$$\begin{aligned} &[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0, \\ &[A_i, A_j^\dagger] = [A_j, A_i^\dagger] = \cos\frac{(j-i)\pi}{k} + 2a\sum_l \cos\frac{(l-i)\pi}{k}\cos\frac{(l-j)\pi}{k}\mathcal{R}^{k+2l}\mathcal{I} \\ &\qquad + 2b\sum_l \sin\frac{(l-i)\pi}{k}\sin\frac{(l-j)\pi}{k}\mathcal{R}^{2l}\mathcal{I}, \end{aligned} \tag{8}$$

for $i, j = 0, 1, \ldots, k-1$. Here all summations over l run from 0 to $k-1$. The A_i also fulfil the exchange relations

$$\begin{aligned} &\mathcal{R}A_i\mathcal{R}^{-1} = A_{i+1}, \qquad i = 0, 1, \ldots, k-2, \qquad \mathcal{R}A_{k-1}\mathcal{R}^{-1} = -A_0, \\ &\mathcal{I}A_0\mathcal{I}^{-1} = A_0, \qquad \mathcal{I}A_i\mathcal{I}^{-1} = -A_{k-i}, \qquad i = 1, 2, \ldots, k-1. \end{aligned} \tag{9}$$

These modified boson operators can be used to define k modified oscillator Hamiltonians

$$H_i = \tfrac{1}{2}\{A_i^\dagger, A_i\}, \qquad i = 0, 1, \ldots, k-1, \tag{10}$$

which transform among themselves under D_{2k}:

$$\begin{aligned} &\mathcal{R}H_i\mathcal{R}^{-1} = H_{i+1}, \qquad i = 0, 1, \ldots, k-2, \qquad \mathcal{R}H_{k-1}\mathcal{R}^{-1} = H_0, \\ &\mathcal{I}H_0\mathcal{I}^{-1} = H_0, \qquad \mathcal{I}H_i\mathcal{I}^{-1} = H_{k-i}, \qquad i = 1, 2, \ldots, k-1. \end{aligned} \tag{11}$$

From the explicit expressions of the H_i's, namely

$$\begin{aligned}
2\omega H_i = & -\cos^2\left(\varphi+\frac{i\pi}{k}\right)D_r^2+\frac{1}{r}\sin\left(\varphi+\frac{i\pi}{k}\right)\cos\left(\varphi+\frac{i\pi}{k}\right)(D_rD_\varphi+D_\varphi D_r) \\
& -\frac{1}{r^2}\sin^2\left(\varphi+\frac{i\pi}{k}\right)D_\varphi^2 \\
& -\frac{1}{r}\left[\sin^2\left(\varphi+\frac{i\pi}{k}\right)+2a\sum_l\cos^2\frac{(l-i)\pi}{k}\mathcal{R}^{k+2l}\mathcal{I}+2b\sum_l\sin^2\frac{(l-i)\pi}{k}\mathcal{R}^{2l}\mathcal{I}\right]D_r \\
& +\frac{1}{r^2}\left[-2\sin\left(\varphi+\frac{i\pi}{k}\right)\cos\left(\varphi+\frac{i\pi}{k}\right)-2a\sum_l\sin\frac{(l-i)\pi}{k}\cos\frac{(l-i)\pi}{k}\mathcal{R}^{k+2l}\mathcal{I}\right. \\
& \left.+2b\sum_l\sin\frac{(l-i)\pi}{k}\cos\frac{(l-i)\pi}{k}\mathcal{R}^{2l}\mathcal{I}\right]D_\varphi+\omega^2r^2\cos^2\left(\varphi+\frac{i\pi}{k}\right),
\end{aligned} \tag{12}$$

it can be shown that such operators are connected with the D_{2k}-extended Hamiltonian through the relation

$$2\omega\sum_{i=0}^{k-1}H_i=\frac{k}{2}\mathcal{H}_k. \tag{13}$$

Hence $\mathcal{H}_k$ may be considered as a modified boson oscillator Hamiltonian.

Next, it can be proved that all the H_i's commute with $\mathcal{H}_k$ and are therefore integrals of motion for the latter. From them, one can form two D_{2k} invariants, namely their sum proportional to $\mathcal{H}_k$ and their symmetrized product

$$\mathcal{Y}_{2k}=(2\omega)^k\sum_p H_{p(0)}H_{p(1)}\cdots H_{p(k-1)}, \tag{14}$$

where the summation runs over all $k!$ permutations of 0, 1, ..., $k-1$.

Projection in the D_{2k} identity representation then leads to H_k, on one hand, and to an integral of motion Y_{2k} of the latter, on the other hand. From its definition, it is clear that Y_{2k} is a differential operator of order $2k$. It is also functionally independent of X_k because it can be established that one of the highest-order terms in $[\mathcal{X}_k,\mathcal{Y}_{2k}]$ does not vanish and remains nonvanishing after projection in the D_{2k} identity representation, thereby proving that $[X_k,Y_{2k}]\neq 0$.

We conclude that for any odd k, the operators X_k and Y_{2k} provide us with a set of two functionally independent integrals of motion of H_k, which is therefore superintegrable as claimed in the TTW conjecture.

3 $\mathcal{N} = 2$ supersymmetric extension

By using two independent pairs of fermionic creation and annihilation operators $(b_x^\dagger, b_x)$ and $(b_y^\dagger, b_y)$, the TTW Hamiltonian H_k can be extended into a supersymmetric Hamiltonian $\mathcal{H}^s$ [15]. This can be carried out in the framework of an $osp(2/2, \mathbb{R})$ superalgebra with even generators K_0, $K_\pm$, Y (closing the $sp(2, \mathbb{R}) \times so(2)$ Lie algebra) and odd generators $V_\pm$, $W_\pm$ (which are two $sp(2, \mathbb{R})$ spinors). The corresponding (nonvanishing) commutation or anticommutation relations and Hermiticity properties are given by

$$\begin{aligned} &[K_0, K_\pm] = \pm K_\pm, && [K_+, K_-] = -2K_0, \\ &[K_0, V_\pm] = \pm\tfrac{1}{2} V_\pm, && [K_0, W_\pm] = \pm\tfrac{1}{2} W_\pm, \\ &[K_\pm, V_\mp] = \mp V_\pm, && [K_\pm, W_\mp] = \mp W_\pm, \\ &[Y, V_\pm] = \tfrac{1}{2} V_\pm, && [Y, W_\pm] = -\tfrac{1}{2} W_\pm, \\ &\{V_\pm, W_\pm\} = K_\pm, && \{V_\pm, W_\mp\} = K_0 \mp Y \end{aligned} \tag{15}$$

and

$$K_0^\dagger = K_0, \qquad K_\pm^\dagger = K_\mp, \qquad Y^\dagger = Y, \qquad V_\pm^\dagger = W_\mp, \tag{16}$$

respectively.

Standard supersymmetric quantum mechanics [19], with supersymmetric Hamiltonian $\mathcal{H}^s$ and supercharges Q, $Q^\dagger$ such that

$$[\mathcal{H}^s, Q] = [\mathcal{H}^s, Q^\dagger] = 0, \qquad \{Q, Q^\dagger\} = \mathcal{H}^s, \tag{17}$$

is realized by the three operators

$$\mathcal{H}^s = 4\omega(K_0 + Y), \qquad Q = 2\sqrt{\omega}\, W_+, \qquad Q^\dagger = 2\sqrt{\omega}\, V_- \tag{18}$$

generating a $sl(1/1)$ subsuperalgebra of $osp(2/2, \mathbb{R})$.

To get simpler expressions of the generators, it is useful to introduce new 'rotated' fermionic operators defined by

$$\bar{b}_x^\dagger = b_x^\dagger \cos\varphi + b_y^\dagger \sin\varphi, \qquad \bar{b}_y^\dagger = -b_x^\dagger \sin\varphi + b_y^\dagger \cos\varphi \tag{19}$$

and similarly for $\bar{b}_x$, $\bar{b}_y$. Such a transformation, however, breaks the commutativity of bosonic and fermionic degrees of freedom (e.g., $[\partial_\varphi, \bar{b}_x] = \bar{b}_y$).

The resulting even and odd $osp(2/2,\mathbb{R})$ generators can be expressed as

$$
\begin{aligned}
& K_0 = K_{0,\mathrm{B}} + \Gamma, \qquad K_\pm = K_{\pm,\mathrm{B}} - \Gamma, \\
& K_{0,\mathrm{B}} = \frac{1}{4\omega} H_k, \qquad K_{\pm,\mathrm{B}} = \frac{1}{4\omega}[-H_k + 2\omega^2 r^2 \mp 2\omega(r\partial_r + 1)] \\
& \Gamma = \frac{k}{2\omega r^2}\big\{a\big[\bar{b}_x^\dagger \bar{b}_x - \tan k\varphi\big(\bar{b}_x^\dagger \bar{b}_y + \bar{b}_y^\dagger \bar{b}_x\big) + (k\sec^2 k\varphi - 1)\bar{b}_y^\dagger \bar{b}_y\big] \\
& \quad + b\big[\bar{b}_x^\dagger \bar{b}_x + \cot k\varphi\big(\bar{b}_x^\dagger \bar{b}_y + \bar{b}_y^\dagger \bar{b}_x\big) + (k\csc^2 k\varphi - 1)\bar{b}_y^\dagger \bar{b}_y\big]\big\}, \\
& Y = \frac{1}{2}\big[\bar{b}_x^\dagger \bar{b}_x + \bar{b}_y^\dagger \bar{b}_y - k(a+b) - 1\big]
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
V_\pm &= \frac{1}{2\sqrt{\omega}}\left[\bar{b}_x^\dagger\left(\mp\partial_r + \omega r \pm \frac{k(a+b)}{r}\right) \mp \bar{b}_y^\dagger \frac{1}{r}(\partial_\varphi + ka\tan k\varphi - kb\cot k\varphi)\right], \\
W_\pm &= \frac{1}{2\sqrt{\omega}}\left[\bar{b}_x\left(\mp\partial_r + \omega r \mp \frac{k(a+b)}{r}\right) \mp \bar{b}_y \frac{1}{r}(\partial_\varphi - ka\tan k\varphi + kb\cot k\varphi)\right],
\end{aligned}
\tag{21}
$$

respectively.

On starting from the wavefunctions of the TTW Hamiltonian H_k

$$
\begin{aligned}
& \Psi_{N,n}(r,\varphi) = \mathcal{N}_{N,n} Z_N^{(2n+a+b)}(z)\Phi_n^{(a,b)}(\varphi), \\
& Z_N^{(2n+a+b)}(z) = \left(\frac{z}{\omega}\right)^{\left(n+\frac{a+b}{2}\right)k} L_N^{((2n+a+b)k)}(z) e^{-\frac{1}{2}z}, \qquad z = \omega r^2, \\
& \Phi_n^{(a,b)}(\varphi) = \cos^a k\varphi \sin^b k\varphi P_n^{\left(a-\frac{1}{2}, b-\frac{1}{2}\right)}(\xi), \qquad \xi = -\cos 2k\varphi, \\
& N, n = 0, 1, 2, \ldots,
\end{aligned}
\tag{22}
$$

defined in terms of Laguerre and Jacobi polynomials and such that

$$
\begin{aligned}
& H_k \Psi_{N,n}(r,\varphi) = E_{N,n}\Psi_{N,n}(r,\varphi), \qquad E_{N,n} = 2\omega[2N + (2n+a+b)k + 1], \\
& \int_0^\infty dr\, r \int_0^{\pi/(2k)} d\varphi\, |\Psi_{N,n}(r,\varphi)|^2 = 1,
\end{aligned}
\tag{23}
$$

one gets eigenstates of the supersymmetrized TTW Hamiltonian $\mathcal{H}^s$ after multiplication by the fermionic vacuum state $|0\rangle$. The corresponding eigenvalues are $\mathcal{E}_{N,n} = E_{N,n} - E_{0,0} = 4\omega(N + nk)$. Such extended states are also eigenstates of the $osp(2/2,\mathbb{R})$ weight generators K_0 and Y, corresponding to the eigenvalues

$$
\tau = \left(n + \frac{a+b}{2}\right)k + \frac{1}{2}, \qquad q = -\frac{1}{2}[(a+b)k + 1], \tag{24}
$$

respectively.

For each value of $n \in \{0, 1, 2, \ldots\}$ (specifying the angular wavefunctions of H_k as well as the eigenvalues of the first integral of motion X_k), it is possible to construct an $osp(2/2, \mathbb{R})$ irreducible representation (irrep) characterized by (τ, q). Its nature, however, depends on the value assumed by n.

For $n = 0$, one obtains a lowest-weight state (LWS) irrep based on the extended ground state $\Psi_{0,0}(r, \varphi)|0\rangle$. This state is indeed annihilated by all the lowering generators K_-, V_-, W_- of $osp(2/2, \mathbb{R})$. The irrep is a so-called atypical one with $\tau = -q$ (which means that the vanishing Casimir operators C_2 and C_3 cannot specify the irrep). It contains only two $sp(2, \mathbb{R}) \times so(2)$ irreps: $(\tau)(q)$ and $(\tau + \frac{1}{2})(q + \frac{1}{2})$ (spanned by zero- and one-fermion states, respectively).

For any $n \neq 0$, one gets an $osp(2/2, \mathbb{R})$ irrep containing four $sp(2, \mathbb{R}) \times so(2)$ irreps: $(\tau)(q)$, $(\tau - \frac{1}{2})(q + \frac{1}{2})$, $(\tau + \frac{1}{2})(q + \frac{1}{2})$, and $(\tau)(q + 1)$. The first one is spanned by zero-fermion states, the next two by a mixture of one-fermion states and the last one by two-fermion states. No state is annihilated by all the $osp(2/2, \mathbb{R})$ lowering generators.

The eigenvalues of the two Casimir operators of $osp(2/2, \mathbb{R})$ are given by

$$C_2 \to n(n + a + b)k^2, \qquad C_3 \to -\tfrac{1}{2}(a + b)n(n + a + b)k^2, \tag{25}$$

which proves the above-mentioned result for $n = 0$.

The supersymmetric extension presented in this section is valid for any real value of k. For the special cases of $k = 1$, 2, 3, it gives back some known results related to the super-Calogero model [16] and to the supersymmetrization of other Calogero-like systems [20].

4 CONNECTION BETWEEN BOTH EXTENSIONS

To start with, it is possible to realize the elements $\mathcal{R}^i$ and $\mathcal{R}^i\mathcal{I}$, $i = 0, 1, \ldots, 2k - 1$, of the dihedral group D_{2k} in terms of two independent pairs of fermionic operators $(b_x^\dagger, b_x)$ and

$(b_y^\dagger, b_y)$ [17]. On starting from the definitions

$$\begin{aligned}
\mathcal{R} &\equiv 1 + \left(\cos\frac{\pi}{k} - 1\right)(b_x^\dagger b_x + b_y^\dagger b_y) + \sin\frac{\pi}{k}(b_x^\dagger b_y - b_y^\dagger b_x) \\
&\quad + 2\left(1 - \cos\frac{\pi}{k}\right) b_x^\dagger b_x b_y^\dagger b_y, \\
\mathcal{I} &\equiv 1 - 2b_y^\dagger b_y = -[b_y^\dagger, b_y],
\end{aligned} \tag{26}$$

one can indeed show that for any $i = 0, 1, \ldots, 2k-1$

$$\begin{aligned}
\mathcal{R}^i &= 1 + \left(\cos\frac{i\pi}{k} - 1\right)(b_x^\dagger b_x + b_y^\dagger b_y) + \sin\frac{i\pi}{k}(b_x^\dagger b_y - b_y^\dagger b_x) \\
&\quad + 2\left(1 - \cos\frac{i\pi}{k}\right) b_x^\dagger b_x b_y^\dagger b_y, \\
\mathcal{R}^i\mathcal{I} &= 1 + \left(\cos\frac{i\pi}{k} - 1\right) b_x^\dagger b_x - \left(\cos\frac{i\pi}{k} + 1\right) b_y^\dagger b_y - \sin\frac{i\pi}{k}(b_x^\dagger b_y + b_y^\dagger b_x)
\end{aligned} \tag{27}$$

and that such operators satisfy all defining relations of D_{2k}.

The next step consists in making the substitution (27) in the D_{2k}-extended TTW Hamiltonian, given in (5). As a result, the latter is mapped onto the difference between the supersymmetric TTW Hamiltonian $\mathcal{H}^s$ and its purely fermionic term $4\omega Y$ provided the trigonometric identities

$$\begin{aligned}
\sum_{i=0}^{k-1} \tan\left(\varphi + \frac{i\pi}{k}\right)\cos\frac{2i\pi}{k} &= -k\frac{\sin[(k-2)\varphi]}{\cos k\varphi}, \\
\sum_{i=0}^{k-1} \tan\left(\varphi + \frac{i\pi}{k}\right)\sin\frac{2i\pi}{k} &= k\frac{\cos[(k-2)\varphi]}{\cos k\varphi} - \delta_{k,1}
\end{aligned} \tag{28}$$

are satisfied. A simple proof of these relations has been found, thereby establishing a connection between the D_{2k} and the supersymmetric extensions of the TTW Hamiltonian.

References

[1] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets*, Gordon and Breach, New York, 1967.

[2] M. Moshinsky and Yu. F. Smirnov, *The Harmonic Oscillator in Modern Physics*, Harwood, Amsterdam, 1996.

[3] M. Moshinsky and C. Quesne, "Oscillator Systems," in *Symmetry Properties of Nuclei, Proceedings 15th Solvay Conference on Physics, Brussels, 1970*, Gordon and Breach, London, 1974, pp. 233–300.

[4] C. Quesne and M. Moshinsky, *J. Phys. A* **23**, 2263–2272 (1990).

[5] M. Moshinsky, C. Quesne, and Yu. F. Smirnov, *J. Phys. A* **28**, 6447–6457 (1995).

[6] F. Tremblay, A. V. Turbiner and P. Winternitz, *J. Phys. A* **42**, 242001, 10 pp. (2009).

[7] J. Friš, V. Mandrosov, Ya. A. Smorodinsky, M. Uhlir, and P. Winternitz, *Phys. Lett.* **16**, 354–356 (1965).

[8] P. Winternitz, Ya. A. Smorodinsky, M. Uhlir, and J. Friš, *Sov. J. Nucl. Phys.* **4**, 444–450 (1967).

[9] M. A. Olshanetsky and A. M. Perelomov, *Phys. Rep.* **94**, 313–404 (1983).

[10] J. Wolfes, *J. Math. Phys.* **15**, 1420–1424 (1974).

[11] F. Calogero and C. Marchioro, *J. Math. Phys.* **15**, 1425–1430 (1974).

[12] C. Quesne, *J. Phys. A* **43**, 082001, 10 pp. (2010).

[13] E. G. Kalnins, J. D. Kress and W. Miller Jr., *J. Phys. A* **43**, 265205, 21 pp. (2010).

[14] C. Quesne, *Mod. Phys. Lett. A* **25**, 15–24 (2010).

[15] C. Quesne, *J. Phys. A* **43**, 305202, 12 pp. (2010).

[16] D. Z. Freedman and P. F. Mende, *Nucl. Phys. B* **344**, 317–343 (1990).

[17] C. Quesne, *Mod. Phys. Lett. A* **25**, 2373–2380 (2010).

[18] C. F. Dunkl, *Trans. Am. Math. Soc.* **311**, 167–183 (1989).

[19] F. Cooper, A. Khare, and U. Sukhatme, *Phys. Rep.* **251**, 267–385 (1995).

[20] L. Brink, A. Turbiner, and N. Wyllard, *J. Math. Phys.* **39**, 1285–1315 (1998).