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A DUALITY OF QUANTALE-ENRICHED CATEGORIES

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Abstract. We describe a duality for quantale-enriched categories that extends the Lawson duality for continuous dcpos: for any saturated class J of modules that commute with certain weighted limits, and under an appropriate choice of morphisms, the category of J-cocomplete and J-continuous quantale-enriched categories is self-dual.

1. Introduction

In [12] we observed that the left adjoint to the Yoneda embedding in a quantale-enriched category X can be interpreted as a notion of approximation in X. Thus in directed-complete posets, approximation is the way-below relation [11].I.1.; in complete lattices the totally-below relation [22]; and in (generalised) metric spaces a distance $\Downarrow\colon X \times X \to [0,\infty]$ such that every $x \in X$ is a "metric supremum" of $\Downarrow(-,x)$ [12].

The purpose of this paper is to develop a duality theory for $\mathcal{Q}$-categories that extends the Lawson duality for continuous dcpos [20]. Recall that Lawson's theorem states that the category of continuous dcpos with Scott-open filter reflecting maps is self-dual. We show that under an appropriate choice of morphisms the category of J-cocomplete and J-continuous (= admitting approximation) $\mathcal{Q}$-categories is self-dual. Our duality theorem holds for any saturated class J of modules that preserve certain limits; therefore it works uniformly for continuous domains, completely distributive complete lattices, Yoneda-complete quasi-metric spaces, totally distributive $\mathcal{Q}$-categories, and perhaps many other familiar structures from the borderline of metric and order theory.

Our feet rest on shoulders of many. Hausdorff's point of view that a metric is a relation valued in non-negative real numbers, brought to light by [21], led to a development of an unified categorical/algebraic description of topology, uniformity, order and metric [5, 7, 6]. The idea of relative cocompleteness was developed in [14, 1, 17, 16, 15, 25]. Our primary examples of classes of modules have already been studied in [10, 25, 27]. We do hope that our results will be of interest to those who work with categories where the left adjoint to Yoneda embedding has a left adjoint; research in this direction include: [13, 18, 9, 24, 26].

2. Preliminaries

2.1. **Quantales.** A $\mathcal{Q} = (Q, \leqslant, \otimes, \mathbf{1})$ is a commutative unital quantale (in short: a quantale) such that the unit element $\mathbf{1}$ is greatest with respect to the order on $(Q, \leqslant)$. We also assume that $\bot \neq \mathbf{1}$. Examples of quantales include: the two element lattice $\mathbf{2} = (\{\bot, \mathbf{1}\}, \leqslant, \wedge, \mathbf{1})$; the unit interval $[0,1]$ in the natural order, with multiplication as tensor; the extended real half

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line $[0,\infty]$ in the order opposite to the natural one, with addition as tensor. In general, every Heyting algebra with infimum as tensor is a quantale.

2.2. **$\mathcal{Q}$-categories.** We recall that a *$\mathcal{Q}$-category* is a set X with a map $X\colon X\times X\to Q$, called *the structure of* X, with two properties: $\mathbf{1}\leqslant X(x,x)$ for all $x\in X$ (reflexivity), and $X(x,y)\otimes X(y,z)\leqslant X(x,z)$ for all $x,y,z\in X$ (transitivity). In our paper $\mathcal{Q}$-**Cat** denotes the category of $\mathcal{Q}$-categories, where morphisms, called $\mathcal{Q}$-functors, are maps $f\colon X\to Y$ such that $X(x,z)\leqslant Y(fx,fz)$ for all $x,z\in X$. For example **Met** $:=[0,\infty]$-**Cat** is Lawvere's category of generalised metric spaces [21], where reflexivity and transitivity correspond respectively to the assumption of self-distance being zero and to the triangle inequality. As another example we consider **2**-**Cat**, which is isomorphic to the category of preordered sets and monotone maps, and will henceforth be denoted by **Ord**.

A $\mathcal{Q}$-category is separated if $X(x,y)=X(y,x)=\mathbf{1}$ implies $x=y$, for all $x,y\in X$. For example a separated $[0,\infty]$-category is a quasi-metric space, where points can possibly be at infinite distance. Any $\mathcal{Q}$-category X is preordered by the relation $x\leqslant_X y$ iff $\mathbf{1}\leqslant X(x,y)$, which is antisymmetric iff X is separated. Clearly, $\mathcal{Q}$-functors are $\leqslant_X$-preserving.

The internal hom of $\mathcal{Q}$-**Cat** is the set Y^X of all $\mathcal{Q}$-functors of type $X\to Y$ considered with the structure $Y^X(f,g):=\bigwedge_{x\in X}Y(fx,gx)$. The induced order on Y^X is pointwise. The quantale $\mathcal{Q}$ is made into a separated $\mathcal{Q}$-category by its internal hom. The induced order $\leqslant_{\mathcal{Q}}$ coincides with the original order on Q. By X^{op} we mean the $\mathcal{Q}$-category dual to X. $\widehat{X}$ is defined as $\mathcal{Q}^{X^{\mathrm{op}}}$, that is $\widehat{X}(f,g)=\bigwedge_{x\in X}\mathcal{Q}(fx,gx)$. For any X, we have the $\mathcal{Q}$-functor $\mathsf{y}_X\colon X\to\widehat{X}$, $\mathsf{y}_X x=X(-,x)$, called the *Yoneda embedding.* The Yoneda embedding is fully faithful. Furthermore, for all $x\in X$ and $f\in\widehat{X}$, we have $\widehat{X}(\mathsf{y}_X x,f)=fx$, and this equality is the statement of the Yoneda Lemma for $\mathcal{Q}$-categories.

Lastly, $\mathcal{Q}$-**Cat** admits a tensor product $X\otimes Y((x,y),(z,w))=X(x,z)\otimes Y(y,w)$. Since tensor is left adjoint to internal hom, every $\mathcal{Q}$-functor $g\colon X\otimes Y\to Z$ has its *exponential mate* $\ulcorner g\urcorner\colon Y\to Z^X$. It is worth noting that the structure of X is always a $\mathcal{Q}$-functor of type $X^{\mathrm{op}}\otimes X\to Q$, and its exponential mate is the Yoneda embedding $\mathsf{y}_X : X\to\widehat{X}$.

2.3. **$\mathcal{Q}$-modules.** A $\mathcal{Q}$-functor of type $X^{\mathrm{op}}\otimes Y\to Q$ is called a *$\mathcal{Q}$-module* (or plainly: a *module*). For example, the structure of any $\mathcal{Q}$-category X is a module. Moreover, any two modules $\phi\colon X^{\mathrm{op}}\otimes Y\to Q$ and $\psi\colon Y^{\mathrm{op}}\otimes Z\to Q$ can be composed to give a module of type $X^{\mathrm{op}}\otimes Z\to Q$:

$$(\psi\cdot\phi)(x,z):=\bigvee_{y\in Y}(\phi(x,y)\otimes\psi(y,z)).$$

Therefore we think of $\phi\colon X^{\mathrm{op}}\otimes Y\to Q$ as an arrow $\phi\colon X{\multimap}Y$, which, by the above, can be composed with $\psi\colon Y{\multimap}Z$ to give $\psi\cdot\phi\colon X{\multimap}Z$. Note also that $Y\cdot\phi=\phi=\phi\cdot X$.

Any function $f\colon X\to Y$ gives rise to two modules, namely $f_*\colon X{\multimap}Y$, $f_*(x,y)=Y(fx,y)$ and $f^*\colon Y{\multimap}X$, $f^*(y,x)=Y(y,fx)$. We further observe that for any element $x\colon 1\to X$ (1 is the one-element $\mathcal{Q}$-category that should not be confused with the unit of the quantale), the module $x^*\colon X{\multimap}1$ is in fact the same as the $\mathcal{Q}$-functor $\mathsf{y}_X x:=X(-,x)\in\widehat{X}$. Dually, the module $x_*\colon 1{\multimap}X$ corresponds to the $\mathcal{Q}$-functor $\lambda_X x:=X(x,-)$.

The set of all modules of type $X \rightarrow\!\!\!\!\circ\, Y$ becomes a complete lattice via the pointwise order where the supremum ϕ of a family $\phi_i\colon X \rightarrow\!\!\!\!\circ\, Y$ $(i \in I)$ of modules can be calculated as $\phi(x,y) = \bigvee_{i\in I} \phi_i(x,y)$. Furthermore, composition of modules preserves this suprema on both sides, and therefore the maps $-\cdot\phi$ and $\phi\cdot-$ have right adjoints $- \bullet\!\!- \phi$ and $\phi -\!\!\bullet -$ respectively. Explicitly, given $\phi\colon X \rightarrow\!\!\!\!\circ\, Y$,

$$(\psi \bullet\!\!- \phi)(y,z) = \bigwedge_{x\in X} \mathcal{Q}(\phi(x,y), \psi(x,z))$$

for any $\psi\colon X \rightarrow\!\!\!\!\circ\, Z$, and

$$(\phi -\!\!\bullet \psi)(z,x) = \bigwedge_{y\in Y} Q(\phi(x,y), \psi(z,y))$$

for any $\psi\colon Z \rightarrow\!\!\!\!\circ\, Y$. We call $\psi \bullet\!\!- \phi$ the *extension* of ψ along ϕ, and $\phi -\!\!\bullet \psi$ the *lifting* of ψ along ϕ. This construction will be used to define the so called *way-below module* in Section 2.5.

In **Ord**, modules of type $X \rightarrow\!\!\!\!\circ\, 1$ are precisely (characteristic maps of) lower sets, and modules of type $1 \rightarrow\!\!\!\!\circ\, X$ are upper sets of the poset X. Furthermore, the up-set of all upper bounds of $\psi\colon\ : X \rightarrow\!\!\!\!\circ\, 1$ is given by $\phi = (\leqslant\ \bullet\!\!- \psi)$, and $x \in X$ is a smallest upper bound of ψ if and only if $x_* = (\leqslant\ \bullet\!\!- \psi)$. On the other hand, in **Met**, any Cauchy sequence $(x_n)_{n\in\omega}$ induces a module $\phi\colon 1 \rightarrow\!\!\!\!\circ\, X$ via $\phi(x) = \lim_{n\to\infty} X(x_n, x)$, and a module $\psi\colon X \rightarrow\!\!\!\!\circ\, 1$ via $\psi(x) = \lim_{n\to\infty} X(x, x_n)$. Observe that $\psi\cdot\phi \leqslant 0$ and $\phi\cdot\psi \geqslant X$ in the pointwise order. Conversely, any pair of modules that satisfies the above equations comes from some Cauchy sequence on X. More generally, we will say that modules $\phi\colon Z \rightarrow\!\!\!\!\circ\, X$, $\psi\colon X \rightarrow\!\!\!\!\circ\, Z$ are adjoint iff $\phi\cdot\psi \leqslant X$ and $\psi\cdot\phi \geqslant Z$. In this case we say that ϕ is a left adjoint to ψ and ψ is a right adjoint to ϕ.

2.4. **J-cocomplete $\mathcal{Q}$-categories.** We recall here briefly the notions of weighted limit and weighted colimit, for further details we refer to [14, 16]. For a module $\phi\colon 1 \rightarrow\!\!\!\!\circ\, I$, a ϕ-weighted limit of a $\mathcal{Q}$-functor $h\colon I \to X$ is an element $x \in X$ with $x^* = \phi -\!\!\bullet h^*$. Dually, for a module $\psi\colon I \rightarrow\!\!\!\!\circ\, 1$, a ψ-weighted colimit of a $\mathcal{Q}$-functor $h\colon I \to X$ is an element $x \in X$ with $x_* = h_* \bullet\!\!- \psi$. A $\mathcal{Q}$-category X is called *complete* if X admits all weighted limits, and *cocomplete* if X admits all weighted colimits. For instance, $\mathcal{Q}$ is both complete and cocomplete where the limit of h and ϕ is given by $\bigwedge_{i\in I} \mathcal{Q}(\phi(i), h(i))$ and the colimit of h and ψ by $\bigvee_{i\in I} \psi(i)\otimes h(i)$. This argument extends pointwise to $\widehat{X}$, and we also note that a $\mathcal{Q}$-category X is complete if and only if X is cocomplete.

One says that a $\mathcal{Q}$-functor $f : X \to Y$ preserves the ϕ-weighted limit x of $h\colon I \to X$ if $f(x)$ is a ϕ-weighted limit of $fh : I \to Y$, likewise, $f : X \to Y$ preserves the ψ-weighted colimit x of $h\colon I \to X$ if $f(x)$ is a ψ-weighted colimit of $fh : I \to Y$. Then $f : X \to Y$ is called *continuous* if f preserves all existing weighted limits in X, and f is called *cocontinuous* if f preserves all existing weighted colimits in X.

In the sequel we will be interested in special kinds of colimits, hence we suppose that there is given a collection J of modules of type $X \rightarrow\!\!\!\!\circ\, 1$, called thereafter *$J$-ideals*. The set of those modules in J with domain X we denote as JX. Then we define X to be *J-cocomplete* if X admits all ψ-weighted colimits with ψ in J, and a $\mathcal{Q}$-functor $f : X \to Y$ is called *J-cocontinuous* if f preserves all existing J-weighted colimits in X. We will also assume that our class J of modules is *saturated*, which amounts to saying that JX contains all modules $x^* : X \rightarrow\!\!\!\!\circ\, 1$ and is closed in $\widehat{X}$ under J-weighted colimits. In this case, X is J-cocomplete if and only if X admits all ψ-weighted colimits with $\psi : X \rightarrow\!\!\!\!\circ\, 1$ in J-**Mod**, which in turn is equivalent to $\mathsf{y}_X\colon X \to JX$ having a left adjoint in $\mathcal{Q}$-**Cat**. That is, there must exist a $\mathcal{Q}$-functor $\mathsf{S}_X\colon JX \to X$ such that for all $\phi \in JX$ and all $x \in X$:

$$X(\mathsf{S}_X\phi, x) = \widehat{X}(\phi, \mathsf{y}_X x). \tag{2.1}$$

The element $\mathsf{S}_X\phi \in X$ is called the *supremum* of ϕ. If $JX = \widehat{X}$ and $\Psi : \widehat{X} \relbar\joinrel\mapsto 1$, then $\mathsf{S}_X(\Psi)(x) = \bigvee_{\psi \in \widehat{X}} \Psi(\psi) \otimes \psi(x) = \bigvee_{\psi \in \widehat{X}} \Psi(\psi) \otimes [\mathsf{y}(x), \psi]$, hence $\mathsf{S}_X(\Psi) = \Psi \cdot \mathsf{y}_*(x)$. Since JX is closed in $\widehat{X}$ under J-colimits, the same formula describes J-suprema in JX. For example, if $\mathcal{Q} = \mathbf{2}$, then $\widehat{X}$ is a poset of lower subsets of the poset X ordered by inclusion, ψ is a lower set of lower sets of X, and the supremum of ψ is nothing else but $\bigcup \psi$.

A $\mathcal{Q}$-functor $f\colon X \to Y$ between J-cocomplete $\mathcal{Q}$-categories is J-cocontinuous if and only if $f(\mathsf{S}\phi) = \mathsf{S}(Jf(\phi))$, for all $\phi \in JX$. Here we make use of the fact that J defines a functor $J : \mathcal{Q}\text{-}\mathbf{Cat} \to J\text{-}\mathbf{Cocts}$ which sends a $\mathcal{Q}$-category X to JX, and a $\mathcal{Q}$-functor $f : X \to Y$ to $Jf : JX \to JY, \psi \mapsto \psi \cdot f^*$. We use the occasion to remark that $J : \mathcal{Q}\text{-}\mathbf{Cat} \to J\text{-}\mathbf{Cocts}$ is left adjoint to the inclusion functor $J\text{-}\mathbf{Cocts} \to \mathcal{Q}\text{-}\mathbf{Cat}$. Even better, $J\text{-}\mathbf{Cocts} \to \mathcal{Q}\text{-}\mathbf{Cat}$ is monadic which we need here only to conclude that $J\text{-}\mathbf{Cocts}$ is complete and limits in $J\text{-}\mathbf{Cocts}$ are calculated as in $\mathcal{Q}\text{-}\mathbf{Cat}$. For details we refer to [15].

There is a well-known general procedure to specify a saturated class J of modules which we describe now.

Example 2.1. Fix a collection Φ of modules $\phi : 1 \relbar\joinrel\mapsto I$, and define J as the class of all those modules $\psi : X \relbar\joinrel\mapsto 1$ where the $\mathcal{Q}$-functors

$$\psi \cdot - : \mathcal{Q}^X \to \mathcal{Q},\ \alpha \mapsto \psi \cdot \alpha = \bigvee_{x \in X} \alpha(x) \otimes \psi(x).$$

preserve Φ-weighted limits. Here we identify a $\mathcal{Q}$-functor $\alpha : X \to \mathcal{Q}$ with a module $\alpha : 1 \relbar\joinrel\mapsto X$. Explicitly, we require that, for any $\phi : 1 \relbar\joinrel\mapsto I$ in Φ and any $\mathcal{Q}$-functor $\alpha_- : I \to \mathcal{Q}^X$,

$$\bigwedge_{i \in I} \mathcal{Q}(\phi(i), \bigvee_{x \in X} \alpha_i(x) \otimes \psi(x)) = \bigvee_{x \in X} \left(\bigwedge_{i \in I} \mathcal{Q}(\phi(i), \alpha_i(x)) \right) \otimes \psi(x).$$

Note that $\mathcal{Q}$-functoriality of $\psi \cdot -$ implies already that the left hand side is larger or equal to the right hand side.

Cocompleteness relative to J allows for a unified presentation of seemingly unrelated notions of order- and metric completeness:

Example 2.2. For any $\mathcal{Q}$, there is a largest and a smallest choice of J: let either J consist of all modules of type $X \relbar\joinrel\mapsto 1$, or only of representable modules $x^* : X \relbar\joinrel\mapsto 1$ where $x \in X$. In the first case a $\mathcal{Q}$-category X is J-cocomplete if and only if it is cocomplete, and in the second case every $\mathcal{Q}$-category is J-cocomplete.

Example 2.3. For $\mathcal{Q} = \mathbf{2}$, we consider all modules of type $X \relbar\joinrel\mapsto 1$ corresponding to order-ideals in X (i.e. directed and lower subsets of X), and write $J = \mathbf{Idl}$. Then X is $\mathbf{Idl}$-cocomplete iff X is a directed-complete.

Example 2.4. For $\mathcal{Q} = [0, \infty]$ we consider all modules of type $X \relbar\joinrel\mapsto 1$ corresponding to ideals in X in the sense of [4], and write $J = \mathbf{FC}$. These ideals in turn correspond to equivalence classes of forward Cauchy sequences on X. Hence, X is $\mathbf{FC}$-cocomplete if and only if each forward Cauchy sequence on X converges if and only if X is sequentially Yoneda complete.

Example 2.5. For any $\mathcal{Q}$ we can choose J to consist of all right adjoint modules (i.e. modules that have left adjoints). Recall from [21] that, for $\mathcal{Q} = [0, \infty]$, a right adjoint module $X \looparrowright 1$ corresponds to an equivalence class of Cauchy sequences on X. A generalised metric space X is J-cocomplete if and only if each Cauchy sequence on X converges.

Example 2.6. For a completely distributive quantale $\mathcal{Q}$ with totally below relation $\prec$ and any $\mathcal{Q}$-category X, a module $\psi\colon X \looparrowright 1$ is a **FSW**-ideal if: (a) $\bigvee_{z\in X} \psi z = \mathbf{1}$, and (b) for all $e_1, e_2, d \prec \mathbf{1}$, for all $x_1, x_2 \in X$, whenever $e_1 \prec \psi x_1$ and $e_2 \prec \psi x_2$, then there exists $z \in X$ such that $d \prec \psi z$, $e_1 \prec X(x_1, z)$ and $e_2 \prec X(x_2, z)$. Now for $\mathcal{Q} = [0, \infty]$ **FSW**-ideals on X are in a bijective correspondence with equivalence classes of forward Cauchy nets on X [10]; for $\mathcal{Q} = \mathbf{2}$, **FSW**-ideals are characteristic maps of order-ideals on X. Therefore this example unifies Examples 2.3, 2.4.

Example 2.7. For any quantale $\mathcal{Q}$, a module $\psi\colon X \looparrowright 1$ is called *flat* if the map $(\psi \cdot -)$ taking modules of type $1 \looparrowright X$ to $\mathcal{Q}$ preserves finite meets. For $\mathcal{Q} = \mathbf{2}$, one verifies that $\psi : X \looparrowright 1$ is flat if and only if $\psi : X^{\mathrm{op}} \to \mathbf{2}$ is the characteristic map of a directed down-set. For $\mathcal{Q} = [0, \infty]$ with $\otimes = +$, Theorem 7.15 of [27] states that flat modules are the same as **FSW**-ideals, therefore this example unifies Examples 2.3, 2.4 as well. However, as we will show in Subsection 4.3, flat modules and **FSW**-ideals are in general different.

Example 2.8. For any $\mathcal{Q}$, put JX to be the set of all modules $\psi : X \looparrowright 1$ of the form $\psi = u \cdot x^*$ where $x \in X$ and $u \in Q$. Here we think of $u \in Q$ as a module $1 \looparrowright 1$. Spelled out, for $y \in X$ one has $\psi(y) = X(y, x) \otimes u$. Note that $\psi(y) = \bot$ whenever $u = \bot$, independently of $x \in X$. A $\mathcal{Q}$-category X is J-cocomplete if it admits "tensoring" with elements of $\mathcal{Q}$ in the following sense: for any $x \in X$ and $u \in Q$, there exists a (necessarily unique up to equivalence) element $z \in X$ with

$$X(z, y) = \mathcal{Q}(u, X(x, y))$$

for all $y \in X$, and one denotes z as $u \otimes x$.

2.5. ***J*-continuous *J*-cocomplete $\mathcal{Q}$-categories.** J-continuity for $\mathcal{Q}$-categories, introduced in [12], allows for a unified treatment of many structures that play a major role in theoretical computer science, e.g. continuous domains, complete metric spaces, or completely distributive complete lattices.

Definition 2.9. A J-cocomplete $\mathcal{Q}$-category X is *J-continuous* if the supremum $\mathsf{S}_X\colon JX \to X$ has a left adjoint.

Note that any $\mathcal{Q}$-functor of type $X \to JX$ corresponds to a certain module $X \looparrowright X$ belonging to J. Hence, X is J-continuous if and only if there exists a module $\Downarrow_X\colon X \looparrowright X$ in J with $\ulcorner\Downarrow_X\urcorner \dashv \mathsf{S}_X$. It is not difficult to see that $\mathsf{S}_X^* \cdot \Downarrow_X \leqslant \mathsf{y}_{X*}$, and $\Downarrow_X$ is the largest module that satisfies this inequality; hence we have identified $\Downarrow_X\colon X \looparrowright X$ as the lifting $\Downarrow_X = \mathsf{S}_X^* \multimap \mathsf{y}_{X*}$. In fact, module $\Downarrow_X := \mathsf{S}_X^* \multimap \mathsf{y}_{X*}$ exists for any J-cocomplete $\mathcal{Q}$-category, and we refer to it as the *way-below* module. It is worth noting that JX is J-continuous for every $\mathcal{Q}$-category X. In this case, the way-below module is given by

$$\Downarrow(\psi, \psi') = \bigvee_{x\in X} \psi'(x) \otimes [\psi, x^*]. \tag{2.2}$$

In the simplest case, $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$, the module $\Downarrow_X$ is indeed the (characteristic map of the) way-below relation on X. In the case of metric spaces, as a consequence of symmetry, $\Downarrow_X\colon X \looparrowright X$ is the same as the structure $X\colon X \looparrowright X$.

We call a module $v\colon X \looparrowright X$ *auxiliary*, if $v \leqslant X$; *interpolative*, if $v \leqslant v \cdot v$; *approximating*, if $v \in J$ and $X \multimapdotinv v = X$; *J-cocontinuous*, if $\mathsf{S}_X^* \cdot v = \mathsf{y}_{X*} \cdot v$. In a J-continuous J-cocomplete $\mathcal{Q}$-category, the way-below module is auxiliary, interpolative, approximating and J-cocontinuous. In fact, we show [12] that a J-cocomplete $\mathcal{Q}$-category is J-continuous iff the way-below module is approximating.

Consider some examples: **FSW**-continuous **FSW**-cocomplete **2**-categories are precisely continuous domains; cocontinuous cocomplete **2**-categories are completely distributive complete lattices (there the way-below module becomes the 'totally-below' relation associated with complete distributivity of the underlying lattice); $[0,\infty]$ considered with the generalised metric structure $[0,\infty](x,y) = \max\{y-x, 0\}$ is an **FSW**-continuous **FSW**-complete $[0,\infty]$-category; complete metric spaces are **FSW**-continuous **FSW**-cocomplete $[0,\infty]$-categories.

2.6. **Open modules.** Let J-$\mathbf{Cocts}(X,Y)$ denote the set of all J-cocontinuous $\mathcal{Q}$-functors from X to Y, and we view J-$\mathbf{Cocts}(X,\mathcal{Q})$ as a sub-$\mathcal{Q}$-category of $\mathcal{Q}^X$.

Lemma 2.10. *J-$\mathbf{Cocts}(X,\mathcal{Q})$ is closed under arbitrary suprema in $\mathcal{Q}^X$. Hence, J-$\mathbf{Cocts}(X,\mathcal{Q})$ is cocomplete.*

Proof. Just observe that $\bigvee\colon \mathcal{Q}^I \to \mathcal{Q}$ is a $\mathcal{Q}$-functor left adjoint to the diagonal $\Delta\colon \mathcal{Q} \to \mathcal{Q}^I$, for any set I; and $u \otimes -\colon \mathcal{Q} \to \mathcal{Q}$ is a $\mathcal{Q}$-functor left adjoint to $\mathcal{Q}(u,-)\colon \mathcal{Q} \to \mathcal{Q}$. □

From the lemma above we deduce that the inclusion functor J-$\mathbf{Cocts}(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$ has a right adjoint $v\colon \mathcal{Q}^X \to J$-$\mathbf{Cocts}(X,\mathcal{Q})$.

If X is J-cocomplete and J-continuous, this right adjoint has a simple description. In fact, since $\Downarrow_X \dashv \mathsf{S}_X$ and $\mathsf{S}_X \dashv \mathsf{y}_X$, the map $\mathcal{Q}^X \to J$-$\mathbf{Cocts}(X,\mathcal{Q})$, $f \mapsto f_L \cdot \Downarrow_X$ (where f_L is left Kan extension of f) is right adjoint to J-$\mathbf{Cocts}(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$ in **Ord**, hence it underlies v. Hence in this case we can write v as the corestriction of the composite of left adjoints

$$\mathcal{Q}^X \longrightarrow J\text{-}\mathbf{Cocts}(JX,\mathcal{Q}) \hookrightarrow \mathcal{Q}^{JX} \xrightarrow{-\cdot\Downarrow_X} \mathcal{Q}^X$$

to J-$\mathbf{Cocts}(X,\mathcal{Q})$, hence v is itself left-adjoint.

Lemma 2.11. *If X is J-cocomplete and J-continuous, then J-$\mathbf{Cocts}(X,\mathcal{Q})$ is totally continuous.*

Proof. $\mathcal{Q}^X$ is totally continuous, and J-$\mathbf{Cocts}(X,\mathcal{Q})$ inherits this property since $v\colon \mathcal{Q}^X \to J$-$\mathbf{Cocts}(X,\mathcal{Q})$ is a left and a right adjoint. □

We put now $FX := J$-$\mathbf{Cocts}(X,\mathcal{Q}) \cap J(X^{\mathrm{op}})$ and call $\alpha \in FX$ an *open module*. More precisely, FX is defined via the pullback in J-**Cocts** of two inclusions: J-$\mathbf{Cocts}(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$, $J(X^{\mathrm{op}}) \hookrightarrow \mathcal{Q}^X$, which tells us that:

- FX is J-cocomplete,
- both inclusion maps $FX \hookrightarrow J(X^{\mathrm{op}})$ and $FX \hookrightarrow J$-$\mathbf{Cocts}(X,\mathcal{Q})$ preserve J-suprema.

Definition 2.12. We say that a J-continuous $\mathcal{Q}$-category X is *open module determined* if for all $x,y \in X$:

$$\Downarrow_X(x,y) = \bigvee_{\alpha\in FX} (\alpha(y) \otimes [\alpha, \lambda_X(x)]). \tag{2.3}$$

Note that, for all $\alpha \in FX$ and $x,y \in X$,

$$\alpha(y) \otimes [\alpha,\lambda_X(x)] = \bigvee_{z\in X} (\alpha(z) \otimes \Downarrow_X(z,y) \otimes [\alpha, X(x,-)]) \leqslant \bigvee_{z\in X} X(x,z) \otimes \Downarrow_X(z,y) = \Downarrow_X(x,y),$$

hence (2.3) is equivalent to

$$\Downarrow_X(x,y) \leqslant \bigvee_{\alpha\in FX} (\alpha(y)\otimes[\alpha,\lambda_X(x)]).$$

Furthermore, (2.3) is equivalent to

$$\Downarrow_X(x,y) = \bigvee_{\alpha\in FX} (\alpha(y)\otimes[\alpha,\Downarrow_X(x,-)])$$

since $\Downarrow_X(x,-)\leqslant\lambda_X(x)$ and

$$\begin{aligned}
\Downarrow_X(x,y) &= \bigvee_{z\in X} \Downarrow_X(x,z)\otimes\Downarrow_X(z,y)\\
&= \bigvee_{z\in X} \Downarrow_X(x,z)\otimes \bigvee_{\alpha\in FX} (\alpha(y)\otimes[\alpha,\lambda_X(z)])\\
&= \bigvee_{\alpha\in FX} \alpha(y)\otimes \bigvee_{z\in X} (\Downarrow_X(x,z)\otimes[\alpha,\lambda_X(z)])\\
&\leqslant \bigvee_{\alpha\in FX} \alpha(y)\otimes [\alpha, \bigvee_{z\in X} \Downarrow_X(x,z)\otimes X(z,-)]\\
&= \bigvee_{\alpha\in FX} (\alpha(y)\otimes[\alpha,\Downarrow_X(x,-)]).
\end{aligned}$$

3. The duality

In this section we assume that a class Φ of limit weights $\phi: 1 \to\!\!\!\!\circ\, I$ is given, and we consider the corresponding class J of modules as described in Example 2.1. Furthermore, let X be a J-cocomplete, J-continuous and open module determined $\mathcal{Q}$-category.

Each $x\in X$ defines:

$$\begin{aligned}
\mathrm{ev}_x\colon FX &\to \mathcal{Q}\\
\alpha &\mapsto \alpha(x).
\end{aligned}$$

Lemma 3.1. *For any $x\in X$, the map ev_x is an open module on FX.*

Proof. Certainly, ev_x is J-continuous, since it is the restriction of

$$-\cdot x_*\colon J(X^{\mathrm{op}})\to\mathcal{Q} \qquad\qquad (\text{here } x\in X^{\mathrm{op}} \text{ and therefore } x_*\colon 1 \to\!\!\!\!\circ\, X^{\mathrm{op}})$$

to FX. We show now that $\mathrm{ev}_x\in J(FX^{\mathrm{op}})$, that is,

$$C_x := \mathrm{ev}_x\cdot -\colon \mathcal{Q}\text{-}\mathbf{Mod}(FX,1)\to\mathcal{Q},\ \Psi\mapsto \bigvee_{\alpha\in FX}\Psi(\alpha)\otimes\alpha(x)$$

preserves Φ-weighted limits. Note that $\mathcal{Q}\text{-}\mathbf{Mod}(FX,1)\cong\mathcal{Q}\text{-}\mathbf{Mod}(1,FX^{\mathrm{op}})$. Furthermore, since $\alpha\in FX$ is J-cocontinuous, $C_x=\bigvee_{y\in X} C_y\otimes\Downarrow_X(y,x)$. Let $\phi: 1 \to\!\!\!\!\circ\, I$ be in Φ and

$\Psi_{-}\colon I \to \mathcal{Q}\text{-}\mathbf{Mod}(FX,1)$, $i \mapsto \Psi_i$ be a $\mathcal{Q}$-functor. Then

$$\begin{aligned}
\bigwedge_{i\in I} \mathcal{Q}(\phi(i), C_x(\Psi_i)) &= \bigwedge_{i\in I} \mathcal{Q}(\phi(i), \bigvee_{y\in X} C_y(\Psi_i)\otimes \Downarrow_X(y,x)) \\
&= \bigvee_{y\in X}\left(\bigwedge_{i\in I} \mathcal{Q}(\phi(i), C_y(\Psi_i))\right)\otimes \Downarrow_X(y,x) \qquad\qquad (\Downarrow(-,x) \text{ is in } J)\\
&\leqslant \bigvee_{\alpha\in FX} \alpha(x)\otimes \bigvee_{y\in X}\bigwedge_{i\in I} \mathcal{Q}(\phi(i), C_y(\Psi_i)\otimes[\alpha,\lambda_X y])\\
&\leqslant \bigvee_{\alpha\in FX} \alpha(x)\otimes \bigwedge_{i\in I} \mathcal{Q}(\phi(i), \Psi_i(\alpha))
\end{aligned}$$

since

$$C_y(\Psi_i)\otimes[\alpha,\lambda_X y] = \bigvee_{\beta\in FX} \Psi_i(\beta)\otimes[\alpha,\lambda_X y]\otimes[\lambda_X y,\beta] \leqslant \bigvee_{\beta\in FX}\Psi_i(\beta)\otimes[\alpha,\beta] = \Psi_i(\alpha). \qquad \square$$

We further obtain a map $\eta_X\colon X\to FFX$ given by:

$$x \mapsto \mathrm{ev}_x. \tag{3.1}$$

This is indeed a $\mathcal{Q}$-functor, since for any $y,z\in X$ we have:

$$[\eta_X(y),\eta_X(z)] = \bigwedge_{\alpha\in FX} \mathcal{Q}(\alpha(y),\alpha(z)) \geqslant X(y,z).$$

Lemma 3.2. *FX is J-continuous with the way-below module $\Downarrow_{FX}\colon FX \,{-\!\!\circ\!\!\rightarrow}\, FX$ given by:*

$$\Downarrow_{FX}(\beta,\alpha) = \bigvee_{x\in X}(\alpha(x)\otimes[\beta,\lambda_X(x)]). \tag{3.2}$$

Proof. Note that (3.2) states that the way-below module on FX is the restriction of the way-below module on $J(X^{\mathrm{op}})$ (see (2.2)). First we wish to show that

$$\Downarrow_{FX}(-,\alpha) := \bigvee_{x\in X}(\alpha(x)\otimes[-,\lambda_X(x)])$$

is a J-module of type $FX \,{-\!\!\circ\!\!\rightarrow}\, 1$, for every $\alpha\in FX$. To this end, we consider a diagram

$$1 \overset{\phi}{-\!\!\circ\!\!\rightarrow} A \overset{h}{\rightarrow} \mathcal{Q}^{FX}$$

where ϕ belongs to Φ. We calculate:

$$\begin{aligned}
&\bigwedge_{a\in A} \mathcal{Q}(\phi(a), \bigvee_{\beta\in FX} (\Downarrow_{FX}(\beta,\alpha)\otimes h(a,\beta)))\\
&= \bigwedge_{a\in A} \mathcal{Q}(\phi(a), \bigvee_{x\in X} (\alpha(x)\otimes(\bigvee_{\beta\in FX} ([\beta,\lambda_X(x)]\otimes h(a,\beta)))))\\
&\{\text{put } k(a,x) := \bigvee_{\beta\in FX} ([\beta,\lambda_X(x)]\otimes h(a,\beta)) \text{ where } k\colon A\to \mathcal{Q}^{X^{\mathrm{op}}}\}\\
&= \bigvee_{x\in X} (\alpha(x)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), k(a,x))))\\
&= \bigvee_{x,y\in X} ((\alpha(y)\otimes \Downarrow_X(y,x))\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), k(a,x))))\\
&= \bigvee_{\gamma\in FX}\bigvee_{x,y\in X} ((\gamma(x)\otimes\alpha(y)\otimes[\gamma,\lambda_X(y)])\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), k(a,x))))\\
&= \bigvee_{\gamma\in FX}\bigvee_{y\in X} (\alpha(y)\otimes[\gamma,\lambda_X(y)]\otimes(\bigvee_{x\in X}(\gamma(x)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), k(a,x))))))\\
&= \bigvee_{\gamma\in FX} (\Downarrow_{FX}(\gamma,\alpha)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), \bigvee_{x\in X}(\gamma(x)\otimes k(a,x)))))\\
&= \bigvee_{\gamma\in FX} (\Downarrow_{FX}(\gamma,\alpha)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), \bigvee_{\beta\in FX}\bigvee_{x\in X}(\gamma(x)\otimes[\beta,\lambda_X(x)]\otimes h(a,\beta)))))\\
&= \bigvee_{\gamma\in FX} (\Downarrow_{FX}(\gamma,\alpha)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), \bigvee_{\beta\in FX}([\beta,\gamma]\otimes h(a,\beta)))))\\
&\leqslant \bigvee_{\gamma\in FX} (\Downarrow_{FX}(\gamma,\alpha)\otimes \bigwedge_{a\in A} (\mathcal{Q}(\phi(a), h(a,\beta)))),
\end{aligned}$$

as required (recall that the other inequality we get for free). Furthermore, we calculate:

$$\begin{aligned}
\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha))(x) &= \bigvee_{\beta\in FX} (\Downarrow_{FX}(\beta,\alpha)\otimes\beta(x))\\
&= \bigvee_{\beta\in FX}\bigvee_{y\in X} (\alpha(y)\otimes[\beta,\lambda_X(y)]\otimes\beta(x))\\
&= \bigvee_{y\in X} (\alpha(y)\otimes \bigvee_{\beta\in FX} ([\beta,\lambda_X(y)]\otimes\beta(x)))\\
&= \bigvee_{y\in X} (\alpha(y)\otimes \bigvee_{\beta\in FX} ([\beta,\lambda_X(y)]\otimes[\lambda_X(x),\beta]))\\
&= \bigvee_{y\in X} (\alpha(y)\otimes \Downarrow_X(y,x))\\
&= \alpha(x),
\end{aligned}$$

hence $\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha)) = \alpha$. Finally, to conclude that $\ulcorner\Downarrow_{FX}\urcorner \dashv \mathsf{y}_{FX}$, let $\psi : FX \relbar\joinrel\mapsto 1$ in J. Let i denote the inclusion $\mathcal{Q}$-functor $FX \hookrightarrow J(X^{\mathrm{op}})$ and $\Downarrow_{J(X^{\mathrm{op}})}$ the way-below module on $J(X^{\mathrm{op}})$. We observed already that $\Downarrow_{FX} = i^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_*$. Hence,

$$\begin{aligned}
\ulcorner\Downarrow_{FX}\urcorner\cdot\mathsf{S}_{FX}(\psi) = (\mathsf{S}_{FX}(\psi))^*\cdot\Downarrow_{FX} &= (\mathsf{S}_{FX}(\psi))^*\cdot i^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_*\\
&= (\mathsf{S}_{J(X^{\mathrm{op}})}(\psi\cdot i^*))^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_* \leqslant \psi\cdot i^*\cdot i_* = \psi. \quad\square
\end{aligned}$$

Lemma 3.3. *FX is open module determined.*

Proof. For all $\alpha, \beta \in FX$:

$$\begin{aligned}\Downarrow_{FX}(\beta,\alpha) = \bigvee_{z\in X}(\alpha(z)\otimes[\beta,\lambda_X(z)]) &= \bigvee_{z\in X}(\mathrm{ev}_z(\alpha)\otimes[\lambda_X(z)_*,\beta_*]) \\ = \bigvee_{z\in X}(\mathrm{ev}_z(\alpha)\otimes[\mathrm{ev}_z,\lambda_{FX}(\beta)]) &= \bigvee_{\mathcal{A}\in FFX}(\mathcal{A}(\alpha)\otimes[\mathcal{A},\lambda_{FX}(\beta)]) \qquad \square\end{aligned}$$

By the discussion in Section 2.6 and Lemmata 3.2, 3.3 we obtain:

Theorem 3.4. *If X is a J-continuous, J-cocomplete and open module determined $\mathcal{Q}$-category, then so is FX.*

Our next aim is to show that $\eta_X\colon : X \to FFX$ is an isomorphism. To do so, let now $\mathcal{A}\colon FX \to \mathcal{Q}$ be an open module on FX. We define:

$$\psi_{\mathcal{A}}(x) := \bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes[\alpha,\lambda_X(x)]).$$

Such defined $\psi_{\mathcal{A}}$ is a module $X \relbar\!\!\!\circ\!\!\!\rightarrow 1$, since it is the composite:

$$X \xrightarrow{\lambda_{X*}} J(X^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{i^*} FX^{\mathrm{op}} \xrightarrow{\mathcal{A}} 1.$$

We also need to have:

Lemma 3.5. *For every $\mathcal{A} \in FFX$, we have $\psi_{\mathcal{A}} \in JX$.*

Proof. In order to check that $\psi_{\mathcal{A}}\colon X \relbar\!\!\!\circ\!\!\!\rightarrow 1$ belongs to JX, we need to check whether $\psi_{\mathcal{A}}\cdot - \colon \mathcal{Q}^X \to \mathcal{Q}$ preserves Φ-weighted limits. Let

$$1 \xrightarrow{\phi} A \xrightarrow{h} \mathcal{Q}^X$$

be a limit diagram with ϕ in Φ. Spelled out, we have to show that

$$\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),h(y,x)))) \geqslant \bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes h(y,x)))).$$

To this end, we calculate:

$$\begin{aligned}
&\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes h(y,x))))\\
&=\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes[\alpha,\lambda_X(x)]\otimes h(y,x))))\\
&=\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes\Downarrow_{FX}(\alpha,h(y)))))\quad\{\text{since }\mathcal{A}^{\mathrm{op}}\in J\}\\
&=\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{\alpha,\beta\in FX}((\mathcal{A}(\beta)\otimes\Downarrow_{FX}(\beta,\alpha))\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{\alpha,\beta\in FX}\bigvee_{x\in X}((\mathcal{A}(\beta)\otimes\alpha(x)\otimes[\beta,\lambda_X(x)])\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{x\in X}\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes[\beta,\lambda_X(x)])\otimes\bigvee_{\alpha\in FX}\mathrm{ev}_x(\alpha)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y))))\\
&\{\mathrm{ev}_x\text{ is a filter}\}\\
&=\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),\bigvee_{\alpha\in FX}(\alpha(x)\otimes\Downarrow_{FX}(\alpha,h(y)))))\\
&\leqslant\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),\alpha(x)\otimes[\alpha,h(y)]))\\
&\leqslant\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),h(y,x))),
\end{aligned}$$

which proves $\psi_{\mathcal{A}}\in JX$. □

Lemma 3.6. *For any $\alpha\in FX$, we have $\mathcal{A}(\alpha)=\alpha(\mathsf{S}_X(\psi_{\mathcal{A}}))$.*

Proof.

$$\begin{aligned}
\alpha(\mathsf{S}_X(\psi_{\mathcal{A}}))&=\mathrm{colim}(\alpha,\psi_{\mathcal{A}})\\
&=\bigvee_{x\in X}(\alpha(x)\otimes\psi_{\mathcal{A}}(x))\\
&=\bigvee_{x\in X}(\alpha(x)\otimes\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes[\beta,\lambda_X(x)]))\\
&=\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes\bigvee_{x\in X}(\alpha(x)\otimes[\beta,\lambda_X(x)]))\\
&=\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes\Downarrow_{FX}(\beta,\alpha))\\
&=\mathrm{colim}(\mathcal{A},\Downarrow_{FX}(-,\alpha))\\
&=\mathcal{A}(\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha)))\\
&=\mathcal{A}(\alpha).
\end{aligned}$$

□

Definition 3.7. We say that a $\mathcal{Q}$-functor $f\colon X\to Y$ between $\mathcal{Q}$-categories *reflects open modules* if $\alpha\cdot f\in FX$ for every $\alpha\in FY$. Let $(J,\mathcal{Q})$-**Dom** be the category of J-cocomplete, J-continuous and open module determined $\mathcal{Q}$-categories together with open module reflecting maps.

Lemma 3.8. *The pair of operations*

$$\begin{array}{rcl} X & \mapsto & FX \\ f\colon X \to Y & \mapsto & - \cdot f\colon FY \to FX \end{array}$$

defines a contravariant functor, i.e. $F\colon (J,\mathcal{Q})\text{-}\mathbf{Dom}^{\mathrm{op}} \to (J,\mathcal{Q})\text{-}\mathbf{Dom}$.

Proof. Functoriality is trivial; we only need to show that $F(f)$ reflect open modules. Let $\mathcal{A} \in FFX$. By Lemma 3.6 there exists $x \in X$ such that $\mathcal{A} = \mathrm{ev}_x$, namely $x = \mathsf{S}_X\psi_{\mathcal{A}}$. Then, for any $\alpha \in FY$, we have $(\mathcal{A} \cdot F(f))(\alpha) = \mathcal{A}(\alpha \cdot f) = \alpha(f(x)) = \mathrm{ev}_{f(x)}(\alpha)$. Hence $\mathcal{A} \cdot F(f) = \mathrm{ev}_{f(x)}$, i.e. $\mathcal{A} \cdot F(f) \in FFX$. □

Theorem 3.9 (The Duality Theorem)**.** *The category* $(J,\mathcal{Q})$-**Dom** *is self-dual.*

Proof. The natural isomorphism $\eta\colon 1_{(J,\mathcal{Q})\text{-}\mathbf{Dom}} \to FF$ as defined in (3.1) has the converse $\varepsilon\colon FF \to 1_{(J,\mathcal{Q})\text{-}\mathbf{Dom}}$ given by $\varepsilon_X(\mathcal{A}) = \mathsf{S}_X\psi_{\mathcal{A}}$ for every $\mathcal{A} \in FFX$. □

4. Examples of the duality

4.1. **Lawson duality.** The case $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{FSW}$, perhaps the simplest possible, served us as a proof guide throughout the paper. In fact, most of the crucial proof ideas (e.g. Lemma 3.6: any open module on open modules $\mathcal{A}$ is of the form $\mathrm{ev}_{\mathsf{S}_X\psi_{\mathcal{A}}}$ for some J-ideal $\psi_{\mathcal{A}}$) come from an analysis of this simple case. Observe that **FSW**-continuous, **FSW**-cocomplete **2**-categories are continuous dcpos (domains). Furthermore, open modules are nothing else but (the characteristic maps of) Scott-open filters on domains. Recall that in this case any FX is open module determined: the equality (2.3) reduces to

$$\forall x, y \in X \ (x \ll y \ \Rightarrow \ \exists \alpha \in FX \ (y \in \alpha \subseteq \Uparrow x)),$$

and we define such $\alpha \in FX$ by $\alpha := \bigcup_{n\in\omega} \Uparrow x_n$, where the descending chain $(x_n)_{n\in\omega}$ has been obtained by a repeated use of interpolation (see Prop. 3.3 of [11]):

$$x \ll \ldots \ll x_n \ll x_{n-1} \ll \ldots \ll x_2 \ll x_1 \ll x_0 = y.$$

Consequently, the category $(\mathbf{FSW}, \mathbf{2})$–**Dom** is the category of domains with open filter reflecting maps; our Theorem 3.9 reduces to Theorem IV-2.12 of [11] establishing the Lawson duality for domains. It is worth mentioning that the Lawson duality (originally proved in [20]) finds its applications in the theory of locally compact spaces; in particular, the lattice of opens of a locally compact sober space X is Lawson dual to the lattice of compact saturated subsets of X (cf. Hofmann-Mislove theorem).

4.2. **A metric duality.** In the case $\mathcal{Q} = [0,\infty]$ with $\otimes = +$ and J being the class of **FSW**-ideals (or, equivalently, flat modules), our duality works in a certain subcategory of **Met**: its **FSW**-cocomplete objects are known in the literature as Yoneda-complete gmses [4]. The **FSW**-cocomplete and **FSW**-continuous ones form a class not previously discussed in the literature, except in the forthcoming paper [19], where they are shown to be precisely the spaces having continuous and directed-complete formal ball models [8, 2, 23] (this implies, in particular, that their topology and metric structure can be respectively characterized as a subspace Scott topology and a partial metric on a domain).

A proof that objects of $(\mathbf{FSW}, [0,\infty])$–**Dom** are open filter determined can be found in [3]; below we present a sketch of the proof.

We abbreviate $\Downarrow_X$ to $\Downarrow$ and customarily use $+$ instead of $\otimes$, inf instead of $\bigvee$, etc. In order to show (2.3) it is enough to find a family of open filters $(\alpha_{e,b})_{e,b>0}$, such that $e > \Downarrow(x,y)$ implies

$$e + b \geqslant \alpha_{e,b}(y) + [\alpha_{e,b}, \Downarrow(x,-)] \geqslant \inf_{\alpha\in FX}(\alpha(y) + [\alpha, \Downarrow(x,-)]),$$

which, by complete distributivity of $([0,\infty],\geqslant)$, allows us to draw the desired conclusion. Take an arbitrary $e > \Downarrow(x,y)$ and $b > 0$, and choose a chain $(e_n)_{n\in\omega}$ in $([0,\infty],\geqslant)$ such that:

$$\begin{aligned} &b > e_0 + e_0,\\ &e_0 > e_1 > e_2 > \ldots > e_n > \ldots > 0,\\ &e_n \geqslant e_{n+1} + e_{n+2} + \ldots,\\ &\inf_{n\in\omega} e_n = 0. \end{aligned} \tag{4.1}$$

Now, by interpolation, we can find a sequence $(x_n)_{n\in\omega}$ such that:

$$\begin{array}{ll} e > \Downarrow(x,x_0) + \Downarrow(x_0,y) & \text{and } e_0 > \Downarrow(x_0,y),\\ e > \Downarrow(x,x_1) + \Downarrow(x_1,x_0) + \Downarrow(x_0,y), & \text{and } e_1 > \Downarrow(x_1,x_0),\\ e > \Downarrow(x,x_2) + \Downarrow(x_2,x_1) + \Downarrow(x_1,x_0) + \Downarrow(x_0,y) & \text{and } e_2 > \Downarrow(x_2,x_1),\\ \ldots & \\ e > \Downarrow(x,x_n) + \Downarrow(x_n,x_{n-1}) + \cdots + \Downarrow(x_1,x_0) + \Downarrow(x_0,y) & \text{and } e_n > \Downarrow(x_n,x_{n-1}),\\ \ldots & \end{array}$$

Define $\alpha_{e,b}\colon X \to [0,\infty]$ as $\alpha_{e,b}(z) := \inf_{n\in\omega}\sup_{k\geq n} X(x_k,z)$; this map is an open module on X. In order to conclude (2.3), it is now enough to verify that

$$e + b \geqslant \alpha_{e,b}(y) + [\alpha_{e,b}, \Downarrow(x,-)]. \tag{4.2}$$

However

$$\begin{aligned} \alpha_{e,b}(y) &= \inf_{n\in\omega}\sup_{k\geq n} X(x_k,y)\\ &\leqslant \sup_{k\geq 1}\,(X(x_k,x_{k-1}) + \cdots + X(x_1,x_0) + X(x_0,y))\\ &\leqslant \sup_{k\geq 1}\,(\Downarrow(x_k,x_{k-1}) + \cdots + \Downarrow(x_1,x_0) + \Downarrow(x_0,y)) \quad \{\text{by (4.1)}\}\\ &\leqslant e_0 + e_0\\ &< b. \end{aligned}$$

and

$$\begin{aligned} [\alpha_{e,b}, \Downarrow(x,-)] &= \sup_{z\in X}(\Downarrow(x,z) - \alpha_{e,b}(z))\\ &\leqslant \sup_{z\in X}(\Downarrow(x,z) - (\inf_{n\in\omega}\sup_{k\geq n} X(x_k,z)))\\ &\leqslant \sup_{z\in X}(\inf_{n\in\omega}\sup_{k\geq n}(\Downarrow(x,z) - X(x_k,z)))\\ &\leqslant \sup_{n\in\omega}\sup_{k\geq n}\Downarrow(x,x_k)\\ &\leqslant e. \end{aligned}$$

so (4.2), and therefore also (2.3) are now verified.

4.3. **An ultrametric duality.** For the quantale $\mathcal{Q} = [0, \infty]$ with $\otimes = \max$, $\mathcal{Q}$-**Cat** is the category **UMet** of ultrametric spaces and contraction maps. As above, we can choose J to be the class of all flat modules (see Example 2.7), and obtain that the corresponding category $(J, \mathcal{Q})$-**Dom** is self-dual. However, in ultrametric spaces flat modules are not, in general, **FSW**-ideals, as the following example shows.

Example 4.1. Consider the set $\mathbb{N}$ of natural numbers with the distance

$$\mathbb{N}(n,m) = \begin{cases} 0 & \text{if } n = m, \\ \max(n,m) & \text{otherwise.} \end{cases}$$

This distance is a symmetric, separable ultrametric. Take

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Trivially, ϕ preserves the empty meet. Now, observe that the proof of (the equivalence of (1) and (2) of) Proposition 7.9 in [27] holds verbatim for $\otimes = \max$, hence it is enough to show that $(\phi \cdot -)$ preserves meets of modules of the form $\max(\mathbb{N}(-,x), c)$ for some $c \in [0, \infty]$. Suppose $A := \max(\mathbb{N}(-,a), c_1)$ and $B := \max(\mathbb{N}(-,b), c_2)$ for $c_1, c_2 \in [0, \infty]$; we are heading to prove:

$$(*) \qquad \inf_{z\in\mathbb{N}} \max(Az, Bz, \phi z) = \max(\inf_{s\in\mathbb{N}}(\max(As, \phi s)), \inf_{r\in\mathbb{N}}(\max(Br, \phi r))).$$

We have

$$\begin{aligned} \inf_{z\in\mathbb{N}} \max(Az, Bz, \phi z) &= \inf_{z\in\mathbb{N}} \max(z, a, b, c_1, c_2, \phi z) = \max(a, b, c_1, c_2), \\ \inf_{s\in\mathbb{N}} \max(As, \phi s) &= \inf_{s\in\mathbb{N}} \max(s, a, c_1, \phi s) = \max(a, c_1), \\ \inf_{r\in\mathbb{N}} \max(Br, \phi r) &= \inf_{r\in\mathbb{N}} \max(r, b, c_2, \phi r) = \max(b, c_2) \end{aligned}$$

since all these infima are attained for $z = r = s = 0$. This shows (*), and so $\phi\colon X \dashrightarrow 1$ is a flat module.

On the other hand, ϕ is not an **FSW**-ideal: we have $\phi(2) < 2$ and $\phi(3) < 2$ but there is no $z \in \mathbb{N}$ with $\phi(z) < 1$ and $\mathbb{N}(2,z) < 2$ and $\mathbb{N}(3,z) < 2$.

4.4. **The absolute case.** For any quantale $\mathcal{Q}$, we can consider Φ being the empty class and therefore $JX = \widehat{X}$ is the collection of all modules of type $X \dashrightarrow 1$. In this case, every cocontinuous $\mathcal{Q}$-functor $\alpha\colon X \to \mathcal{Q}$ is an open module. Furthermore, every totally continuous cocomplete $\mathcal{Q}$-category is open module determined since $\Downarrow_X(x,-)\colon X \to \mathcal{Q}$ is in FX. Finally, a $\mathcal{Q}$-functor $f\colon X \to Y$ reflects open modules if and only if f is left adjoint. Therefore Theorem 3.9 states that the category of totally continuous cocomplete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors is self-dual.

4.5. **A somehow different example.** We consider now $\mathcal{Q} = [0, \infty]$ where $\otimes = +$, with the class J of modules described in Example 2.8. However, for technical reasons we consider the unique module $\varnothing \dashrightarrow 1$ as a formal ball, so that $J\varnothing = 1$. Consequently, the empty space is not J-cocomplete. We will show now that our duality theorem holds in this case too, despite the fact that this class of modules is (to our knowledge) not defined via a class of limit weights.

Let now X be a J-cocomplete and J-continuous metric space. We write $\Downarrow : X \to JX$ for the left adjoint to $\mathsf{S} : JX \to X$. Hence, for any $x \in X$, $\Downarrow(x) \in JX$ is of the form $\Downarrow(x) = X(-,x_1)+u$ for some $x_1 \in X$ and $u \in [0, \infty]$. Note that $u < \infty$ if x is not the bottom element of X. Assume that $\Downarrow(x_1) = X(-,x_2) + u_2$. Then

$$X(-,x_1) + u = \Downarrow(x) = \Downarrow(x_1 + u_1) = \Downarrow(x_1) + u_1 = X(-,x_2) + u_2 + u_1,$$

hence, $X(-,x_1)=X(-,x_2)+u_2$. In particular, $0=X(x_1,x_2)+u_2$, and therefore $u_2=0$ and we obtain $\Downarrow(x_1)=\mathsf{y}(x_1)$. Let A be the equaliser of y and $\Downarrow$, that is, $A=\{x\in X \mid \Downarrow(x)=\mathsf{y}(x)\}$. By the considerations above, $\Downarrow : X\to JX$ factors through the inclusion $JA\hookrightarrow JX$. Moreover, for any $X(-,x)+u$ with $x\in A$, $\Downarrow(x+u)=\Downarrow(x)+u=X(-,x)+u$, which gives $X\cong JA$. We also remark that $x\in A$ if and only if $X(x,-):X\to[0,\infty]$ preserves tensoring. One has $\phi\in FX$ precisely if $\phi=X(x,-)+u$ for some $x\in X$ and $u\in[0,\infty]$ and if, moreover, ϕ preserves tensoring. If $u<\infty$, then also $X(x,-)$ preserves tensoring, hence $x\in A$. Consequently, $FX\cong J(A^{\mathrm{op}})$.

Consider now $f:X\to Y$ with $X\cong JA$ and $Y\cong JB$ as above. Then f is open module reflecting if, and only if, for each $y_0\in B$, there exists some $x_0\in A$ and some $v\in[0,\infty]$ with $Y(y_0,f(-))=X(x_0,-)+v$. We show that f necessarily preserves tensoring. To this end, let $x\in X$ and $u\in[0,\infty]$. Then

$$Y(y_0,f(x+u))=X(x_0,x+u)+v=X(x_0,x)+v+u=Y(y_0,f(x))+u=Y(y_0,f(x)+u)$$

for all $y_0\in B$, hence $f(x+u)=f(x)+u$. Therefore f corresponds to a module $\phi:B{\longrightarrow\!\!\!\!\!\!\circ\,}A$ in the sense that, when identifying X with JA and Y with JB, then $f(\psi)=\psi\cdot\phi$. Hence, for any $x\in A$, $x^*\cdot\phi=\phi(-,x)$ belongs to JB, and the f being open module reflecting translates to $\phi\cdot y_*=\phi(y,-)\in J(A^{\mathrm{op}})$ for all $y\in B$. Recall that for each module $\phi:B{\longrightarrow\!\!\!\!\!\!\circ\,}A$ we have its dual $\phi^{\mathrm{op}}:A^{\mathrm{op}}{\longrightarrow\!\!\!\!\!\!\circ\,}B^{\mathrm{op}}$, $\phi^{\mathrm{op}}(x,y)=\phi(y,x)$, and with this notation the latter condition reads as $y^*\cdot\phi^{\mathrm{op}}\in J(A^{\mathrm{op}})$ for all $y\in B^{\mathrm{op}}$. We conclude that the category of J-cocomplete and J-continuous metric spaces and open module reflecting contraction maps is dually equivalent to the category of all metric spaces with morphisms those modules $\phi:X\to Y$ satisfying

$$\forall y\in Y\,.\,(y^*\cdot\phi\in JX) \qquad\text{and}\qquad \forall x\in X^{\mathrm{op}}\,.\,(x^*\cdot\phi^{\mathrm{op}}\in J(Y^{\mathrm{op}})),$$

and the latter category is obviously self-dual.

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