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TOPOLOGICAL CENTERS OF THE $N-TH$ DUAL OF MODULE ACTIONS

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Abstract. In this paper, we will study the topological centers of $n-th$ dual of Banach $A-module$ and we extend some propositions from Lau and Ülger into $n-th$ dual of Banach $A-modules$ where $n\geq 0$ is even number. Let B be a Banach $A-bimodule$. By using some new conditions, we show that ${Z^\ell}_{A^{(n)}}(B^{(n)})=B^{(n)}$ and ${Z^\ell}_{B^{(n)}}(A^{(n)})=A^{(n)}$. We also have some conclusions in group algebras.

1. Introduction

Throughout this paper, A is a Banach algebra and A^*, A^{**}, respectively, are the first and second dual of A. Recall that a left approximate identity $(=LAI)$ [resp. right approximate identity $(=RAI)$] in Banach algebra A is a net $(e_\alpha)_{\alpha\in I}$ in A such that $e_\alpha a\longrightarrow a$ [resp. $ae_\alpha\longrightarrow a$]. We say that a net $(e_\alpha)_{\alpha\in I}\subseteq A$ is a approximate identity $(=AI)$ for A if it is LAI and RAI for A. If $(e_\alpha)_{\alpha\in I}$ in A is bounded and AI for A, then we say that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity $(=BAI)$ for A. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $\langle a'a,b\rangle=\langle a',ab\rangle=a'(ab)$ and $\langle aa',b\rangle=\langle a',ba\rangle=a'(ba)$ for all $b\in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a,a'\rangle$ - $\langle a',a\rangle$ for every $a\in A$ and $a'\in A^*$. We denote

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the set $\{a'a : \ a \in A \ and \ a' \in A^*\}$ and $\{aa' : \ a \in A \ and \ a' \in A^*\}$ by A^*A and AA^*, respectively, clearly these two sets are subsets of A^*.
Let A have a BAI. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides.
The extension of bilinear maps on normed space and the concept of regularity of bilinear maps were studied by [1, 2, 3, 6, 8, 14]. We start by recalling these definitions as follows.
Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following
1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.
The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is \ weak^* - to - weak^* \ continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is \ weak^* - to - weak^* \ continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [18].
The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.
Let now B be a Banach $A-bimodule$, and let

$$\pi_\ell :\ A \times B \to B \ and\ \pi_r :\ B \times A \to B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach $A^{**}-bimodule$ with the following module actions where A^{**} is equipped with the left Arens product

$$\pi_\ell^{***} :\ A^{**} \times B^{**} \to B^{**} \ and\ \pi_r^{***} :\ B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with the following module actions where A^{**} is equipped with the right Arens product

$$\pi_\ell^{t***t} :\ A^{**} \times B^{**} \to B^{**} \ and\ \pi_r^{t***t} :\ B^{**} \times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$\begin{aligned} Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} :\ & the\ map\ b'' \to \pi_\ell^{***}(a'', b'')\ : \\ & B^{**} \to B^{**}\ is\ weak^* - to - weak^*\ continuous\} \\ Z^t_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} :\ & the\ map\ b'' \to \pi_r^{t***}(a'', b'')\ : \\ & B^{**} \to B^{**}\ is\ weak^* - to - weak^*\ continuous\} \\ Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} :\ & the\ map\ a'' \to \pi_r^{***}(b'', a'')\ : \\ & A^{**} \to B^{**}\ is\ weak^* - to - weak^*\ continuous\} \\ Z^t_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} :\ & the\ map\ a'' \to \pi_\ell^{t***}(b'', a'')\ : \\ & A^{**} \to B^{**}\ is\ weak^* - to - weak^*\ continuous\}. \end{aligned}$$

We note also that if B is a left(resp. right) Banach $A-module$ and $\pi_\ell :\ A \times B \to B$ (resp. $\pi_r :\ B \times A \to B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A-module$.
We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1a_2, b) = \pi_\ell(a_1, a_2b)$,
$\pi_r(b, a_1a_2) = \pi_r(ba_1, a_2)$, $\pi_\ell^*(a_1b', a_2) = \pi_\ell^*(b', a_2a_1)$,
$\pi_r^*(b'a, b) = \pi_r^*(b', ab)$, for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there

is no confusion.
Regarding A as a Banach $A-bimodule$, the operation $\pi : A\times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**}\times A^{**}$. These extensions are known, respectively, as the first(left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (left) Arens product of $a'', b''\in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b\in A$ and $a'\in A^*$. Similarly, the second (right) Arens product of $a'', b''\in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle.$$

for all $a, b\in A$ and $a'\in A^*$.
The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstine's* Theorem [4, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^*-\lim_\alpha a_\alpha$ and $b'' = weak^*-\lim_\beta b_\beta$. So it is easy to see that for all $a'\in A^*$,

$$\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [14, 18].
We find the usual first and second topological center of A^{**}, which are

$$Z_{A^{**}}(A^{**}) = Z(\pi) = \{a''\in A^{**} : b''\to a''b'' \ is\ weak^*-to-weak^*\ continuous\},$$
$$Z^t_{A^{**}}(A^{**}) = Z(\pi^t) = \{a''\in A^{**} : a''\to a''ob'' \ is\ weak^*-to-weak^*\ continuous\}.$$

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a''\in A^{**}$, $a''e''=e''oa''=a''$. By [4, p.146], an element e'' of A^{**} is mixed unit if and only if it is a *weak** cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.
A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a\rightarrow a'a$ from A into A^* is weakly compact. Pym in [18] showed that this definition is equivalent to the following condition
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a\in A: \parallel a\parallel\leq 1\}$, we have

$$\lim_\alpha\lim_\beta\langle a',a_\alpha b_\beta\rangle=\lim_\beta\lim_\alpha\langle a',a_\alpha b_\beta\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a'\in wap(A)$ if and only if $\langle a''b'',a'\rangle=\langle a''ob'',a'\rangle$ for every $a'',\ b''\in A^{**}$.
This paper is organized as follows:
a) Let B be a Banach $A-bimodule$ and $\phi\in U_{n,r}$ for even number $n\geq 0$ and $0\leq r\leq\frac{n}{2}$ whenever $U_{n,r}=A^{(n-r)}A^{(r)})^{(r)}$ or $U_{n,r}=A^{(n-r)}A^{(r-1)})^{(r)}$. Then $\phi\in {Z^\ell}_{B^{(n)}}(U_{n,r})$ if and only if $b^{(n-1)}\phi\in B^{(n-1)}$ for all $b^{(n-1)}\in B^{(n-1)}$.
b) Let B be a Banach $A-bimodule$. Then we have the following assertions.

1. (1) $b^{(n)}\in {Z^\ell}_{A^{(n)}}(B^{(n)})$ if and only if $b^{(n-1)}b^{(n)}\in A^{(n-1)}$ for all $b^{(n-1)}\in B^{(n-1)}$.
2. (2) If $\phi\in {Z^\ell}_{B^{(n)}}(U_{n,r})$, then $a^{(n-2)}\phi\in {Z^\ell}_{B^{(n)}}(A^{(n)})$ for all $a^{(n-2)}\in A^{(n-2)}$.

c) Let B be a Banach space such that $B^{(n)}$ is weakly compact. Then for Banach $A-bimodule$ B, we have the following assertions.

(1) Suppose that $(e_\alpha^{(n)})_\alpha\subseteq A^{(n)}$ is a $BLAI$ for $B^{(n)}$ such that

$$e_\alpha^{(n)}B^{(n+2)}\subseteq B^{(n)},$$

for every α. Then B is reflexive.

(2) Suppose that $(e_\alpha^{(n)})_\alpha\subseteq A^{(n)}$ is a $BRAI$ for $B^{(n)}$ and

$$Z^\ell_{e^{(n+2)}}(B^{(n+2)})=B^{(n+2)},$$

where $e_\alpha^{(n)}\stackrel{w^*}{\rightarrow}e^{(n+2)}$ on $A^{(n)}$. If $B^{(n+2)}e_\alpha^{(n)}\subseteq B^{(n)}$ for every α, then $Z^\ell_{A^{(n+2)}}(B^{(n+2)})=B^{(n+2)}$.

d) Assume that B is a Banach $A-bimodule$. Then we have the following assertions.

(1) $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$ if and only if
$$A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)}).$$

(2) If $A^{(n)}A^{(n+2)} \subseteq A^{(n)}{Z^\ell}_{B^{(n+2)}}(A^{(n+2)})$, then
$$A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)}).$$

e) Let B be a left Banach $A-bimodule$ and $n \geq 0$ be a even. Suppose that $b_0^{(n+1)} \in B^{(n+1)}$. Then $b_0^{(n+1)} \in wap_\ell(B^{(n)})$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^* - to - weak$ continuous.

f) Let B be a left Banach $A-bimodule$. Then for $n \geq 2$, we have the following assertions.

(1) If $A^{(n)} = a_0^{(n-2)}A^{(n)}$ [resp. $A^{(n)} = A^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in A^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [resp. Lw^*w- property] with respect to $B^{(n)}$, then $Z_{B^{(n)}}(A^{(n)}) = A^{(n)}$.

(2) If $B^{(n)} = a_0^{(n-2)}B^{(n)}$ [resp. $B^{(n)} = B^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in A^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [resp. Lw^*w- property] with respect to $B^{(n)}$, then $Z_{A^{(n)}}(B^{(n)}) = B^{(n)}$.

2. **Topological centers of module actions**

Suppose that A is a Banach algebra and B is a Banach $A-bimodule$. According to [5, pp.27 and 28], B^{**} is a Banach $A^{**}-bimodule$, where A^{**} is equipped with the first Arens product. So we recalled the topological centers of module actions of A^{**} on B^{**} as in the following.

$$Z^\ell_{A^{**}}(B^{**}) = \{b'' \in B^{**} : \ the\ map\ a'' \to b''a'' \ : \ A^{**} \to B^{**}$$
$$is\ weak^* - to - weak^*\ continuous\}$$
$$Z^\ell_{B^{**}}(A^{**}) = \{a'' \in A^{**} : \ the\ map\ b'' \to a''b'' \ : \ B^{**} \to B^{**}$$
$$is\ weak^* - to - weak^*\ continuous\}.$$

Let $A^{(n)}$ and $B^{(n)}$ be $n-th$ *dual* of A and B, respectively. By [25, page 4132-4134], if $n\geq 0$ is an even number, then $B^{(n)}$ is a Banach $A^{(n)}-bimodule$. Then for $n\geq 2$, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $b^{(n)}\in B^{(n)}$, $b^{(n-1)}\in B^{(n-1)}$ and $a^{(n-2)}\in A^{(n-2)}$ we define

$$\langle b^{(n)}b^{(n-1)},a^{(n-2)}\rangle=\langle b^{(n)},b^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $b^{(n)}\in B^{(n)}$, $b^{(n-1)}\in B^{(n-1)}$ and $a^{(n-1)}\in A^{(n-1)}$, we define

$$<b^{(n)}b^{(n-1)},a^{(n-1)}>=<b^{(n)},b^{(n-1)}a^{(n-1)}>.$$

If $n=0$, we take $A^{(0)}=A$ and $B^{(0)}=B$.
We also define the topological centers of module actions of $A^{(n)}$ on $B^{(n)}$ as follows

$$Z^{\ell}_{A^{(n)}}(B^{(n)})=\{b^{(n)}\in B^{(n)}:\ \textit{the map } a^{(n)}\rightarrow b^{(n)}a^{(n)}\ :\ A^{(n)}\rightarrow B^{(n)}$$
$$\textit{is weak}^*-\textit{to}-\textit{weak}^*\ \textit{continuous}\}$$
$$Z^{\ell}_{B^{(n)}}(A^{(n)})=\{a^{(n)}\in A^{(n)}:\ \textit{the map } b^{(n)}\rightarrow a^{(n)}b^{(n)}\ :\ B^{(n)}\rightarrow B^{(n)}$$
$$\textit{is weak}^*-\textit{to}-\textit{weak}^*\ \textit{continuous}\}.$$

Let A be a Banach algebra and let $A^{(n)}$ and $A^{(m)}$ be $n-th$ *dual* and $m-th$ *dual* of A, respectively. Suppose that at least one of n or m is an even number. Then we define the set $A^{(n)}A^{(m)}$ as a linear space that generated by the following set

$$\{a^{(n)}a^{(m)}:\ a^{(n)}\in A^{(n)}\ \textit{and}\ a^{(m)}\in A^{(m)}\}.$$

Where the production of $a^{(n)}a^{(m)}$ is defined with respect to the first Arens product. If $n\geq m$, then $A^{(n)}A^{(m)}$ is a subspace of $A^{(n)}$. $A^{(n)}A^{(m)}$ is Banach algebra whenever n and m are even numbers , but if one of them is odd number, then $A^{(n)}A^{(m)}$ is in general not a Banach algebra. Let $n\geq 0$ be an even number and $0\leq r\leq \frac{n}{2}$. For a Banach algebra A, we define a new Banach algebra $U_{n,r}$ with respect to the first Arens product as following.
If r is an even (resp. odd) number, then we write $U_{n,r}=(A^{(n-r)}A^{(r)})^{(r)}$ (resp. $U_{n,r}=(A^{(n-r)}A^{(r-1)})^{(r)}$). It is clear that $U_{n,r}$ is a subalgebra of $A^{(n)}$. For example, if we take $n=2$ and $r=1$, then $U_{2,1}=(A^*A)^*$ is a subalgebra of A^{**} with respect to the first Arens product.
Now if B is a Banach $A-bimodule$, then it is clear that $B^{(n)}$ is a Banach $U_{n,r}-bimodule$ with respect to the first Arens product, for detail

see [25], and so we can define the topological centers of module actions $U_{n,r}$ on $B^{(n)}$ as $Z^{\ell}_{B^{(n)}}(U_{n,r})$ and $Z^{\ell}_{U_{n,r}}(B^{(n)})$ similarly to the preceding definitions.
In every parts of this paper, $n \geq 0$ is even number.

Theorem 2.1. *Let B be a Banach $A-$bimodule and $\phi \in U_{n,r}$. Then $\phi \in {Z^{\ell}}_{B^{(n)}}(U_{n,r})$ if and only if $b^{(n-1)}\phi \in B^{(n-1)}$ for all $b^{(n-1)} \in B^{(n-1)}$.*

Proof. Let $\phi \in {Z^{\ell}}_{B^{(n)}}(U_{n,r})$. Suppose that $(b^{(n)}_{\alpha})_{\alpha} \subseteq B^{(n)}$ such that $b^{(n)}_{\alpha} \stackrel{w^*}{\rightarrow} b^{(n)}$ on $B^{(n)}$. Then, for every $b^{(n-1)} \in B^{(n-1)}$, we have

$$\begin{aligned}\langle b^{(n-1)}\phi, b^{(n)}_{\alpha}\rangle = \langle b^{(n-1)}, \phi b^{(n)}_{\alpha}\rangle = \langle \phi b^{(n)}_{\alpha}, b^{(n-1)}\rangle \rightarrow \langle \phi b^{(n)}, b^{(n-1)}\rangle \\ = \langle b^{(n-1)}\phi, b^{(n)}\rangle.\end{aligned}$$

It follows that $b^{(n-1)}\phi \in (B^{(n+1)}, weak^*)^* = B^{(n-1)}$.
Conversely, let $b^{(n-1)}\phi \in B^{(n-1)}$ for every $b^{(n-1)} \in B^{(n-1)}$ and suppose that $(b^{(n)}_{\alpha})_{\alpha} \subseteq B^{(n)}$ such that $b^{(n)}_{\alpha} \stackrel{w^*}{\rightarrow} b^{(n)}$ on $B^{(n)}$. Then

$$\begin{aligned}\langle \phi b^{(n)}_{\alpha}, b^{(n-1)}\rangle = \langle \phi, b^{(n)}_{\alpha} b^{(n-1)}\rangle = \langle b^{(n)}_{\alpha} b^{(n-1)}, \phi\rangle = \langle b^{(n)}_{\alpha}, b^{(n-1)}\phi\rangle \\ \rightarrow \langle b^{(n)}, b^{(n-1)}\phi\rangle = \langle \phi b^{(n)}, b^{(n-1)}\rangle.\end{aligned}$$

It follows that $\phi b^{(n)}_{\alpha} \stackrel{w^*}{\rightarrow} \phi b^{(n)}$, and so $\phi \in {Z^{\ell}}_{B^{(n)}}(U_{n,r})$. $\square$

In Theorem 2.1, if we take $B = A$, $n = 2$ and $r = 1$, we obtain Lemma 3.1 (b) from [14].

Theorem 2.2. *Let B be a Banach $A-$bimodule and $b^{(n)} \in B^{(n)}$. Then we have the following assertions.*

(1) *$b^{(n)} \in {Z^{\ell}}_{A^{(n)}}(B^{(n)})$ if and only if $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ for all $b^{(n-1)} \in B^{(n-1)}$.*
(2) *If $\phi \in {Z^{\ell}}_{B^{(n)}}(U_{n,r})$, then $a^{(n-2)}\phi \in {Z^{\ell}}_{B^{(n)}}(A^{(n)})$ for all $a^{(n-2)} \in A^{(n-2)}$.*

Proof.

(1) Let $b^{(n)} \in {Z^{\ell}}_{A^{(n)}}(B^{(n)})$. We show that $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ where $b^{(n-1)} \in B^{(n-1)}$. Suppose that $(a^{(n)}_{\alpha})_{\alpha} \subseteq A^{(n)}$ and $a^{(n)}_{\alpha} \stackrel{w^*}{\rightarrow} a^{(n)}$ on $A^{(n)}$. Then we have

$$\langle b^{(n-1)}b^{(n)}, a^{(n)}_{\alpha}\rangle = \langle b^{(n-1)}, b^{(n)}a^{(n)}_{\alpha}\rangle = \langle b^{(n)}a^{(n)}_{\alpha}, b^{(n-1)}\rangle$$

$$\to \langle b^{(n)}a^{(n)}, b^{(n-1)}\rangle = \langle b^{(n-1)}b^{(n)}, a^{(n)}\rangle.$$

Consequently $b^{(n-1)}b^{(n)} \in (A^{(n+1)}, weak^*)^* = A^{(n-1)}$. It follows that
$b^{(n-1)}b^{(n)} \in A^{(n-1)}$.
Conversely, let $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ for each $b^{(n-1)} \in B^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ on $A^{(n)}$. Then we have

$$\langle b^{(n)}a_\alpha^{(n)}, b^{(n-1)}\rangle = \langle b^{(n)}, a_\alpha^{(n)}b^{(n-1)}\rangle = \langle a_\alpha^{(n)}b^{(n-1)}, b^{(n)}\rangle$$

$$= \langle a_\alpha^{(n)}, b^{(n-1)}b^{(n)}\rangle \to \langle a^{(n)}, b^{(n-1)}b^{(n)}\rangle = \langle b^{(n)}a^{(n)}, b^{(n-1)}\rangle.$$

It follows that $b^{(n)}a_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}a^{(n)}$, and so $b^{(n)} \in {Z^\ell}_{A^{(n)}}(B^{(n)})$.

(2) Let $\phi \in {Z^\ell}_{B^{(n)}}(U_{n,r})$ and $a^{(n-2)} \in A^{(n-2)}$. Assume that $(b_\alpha^{(n)})_\alpha \subseteq B^{(n)}$ and $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ on $B^{(n)}$. Then for all $b^{(n-1)} \in B^{(n-1)}$, we have

$$\langle (a^{(n-2)}\phi)b_\alpha^{(n)}, b^{(n-1)}\rangle = \langle \phi b_\alpha^{(n)}, b^{(n-1)}a^{(n-2)}\rangle \to \langle \phi b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle$$

$$= \langle (a^{(n-2)}\phi)b^{(n)}, b^{(n-1)}\rangle.$$

It follows that $(a^{(n-2)}\phi)b_\alpha^{(n)} \xrightarrow{w^*} (a^{(n-2)}\phi)b^{(n)}$, and so $a^{(n-2)}\phi \in {Z^\ell}_{B^{(n)}}(A^{(n)})$. $\square$

In the preceding theorem, part (1), if we take $B = A$ and $n = 2$, we conclude Lemma 3.1 (a) from [14]. In part (2) of this theorem, if we take $B = A$, $n = 2$ and $r = 1$, we also obtain Lemma 3.1 (c) from [14].

Definition. Let B be a Banach $A-bimodule$ and suppose that $a'' \in A^{**}$. Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. If for every $b'' \in B^{**}$, we have $a''_\alpha b'' \xrightarrow{w^*} a''b''$, then we say that $a'' \to b''a''$ is $weak^* - to - weak^*$ point continuous.
Suppose that B is a Banach $A-bimodule$. Assume that $a'' \in A^{**}$. Then we define the locally topological center of a'' on B^{**} as follows

$$Z^\ell_{a''}(B^{**}) = \{b'' \in B^{**} :\ a'' \to b''a''\ is\ weak^* - to - weak^*\ point$$

$$continuous\}.$$

The definition of $Z^{\ell}_{b''}(A^{**})$ where $b'' \in B^{**}$ are similar.
It is clear that

$$\bigcap_{a'' \in A^{**}} Z^{\ell}_{a''}(B^{**}) = Z^{\ell}_{A^{**}}(B^{**}),$$

$$\bigcap_{b'' \in B^{**}} Z^{\ell}_{b''}(A^{**}) = Z^{\ell}_{B^{**}}(A^{**}).$$

Let B be a Banach space. Then $K \subseteq B$ is recalled weakly compact, if K is compact with respect to weak topology on B. By [7], we know that K is weakly compact if and only if K is weakly limit point compact.

Theorem 2.3. *Assume that B is a Banach $A-$bimodule such that $B^{(n)}$ is weakly compact. Then we have the following assertions.*

(1) *Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BLAI$ for $B^{(n)}$ such that*

$$e_\alpha^{(n)} B^{(n+2)} \subseteq B^{(n)},$$

for every α. Then B is reflexive.

(2) *Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BRAI$ for $B^{(n)}$ and*

$$Z^{\ell}_{e^{(n+2)}}(B^{(n+2)}) = B^{(n+2)},$$

where $e_\alpha^{(n)} \stackrel{w^*}{\to} e^{(n+2)}$ on $A^{(n)}$. If $B^{(n+2)} e_\alpha^{(n)} \subseteq B^{(n)}$ for every α, then $Z^{\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$.

Proof.

(1) Let $b^{n+2} \in B^{n+2}$. Since $(e_\alpha^{(n)})_\alpha$ is a $BLAI$ for $B^{(n)}$, without loss generality, there is left unit $e^{(n+2)} \in A^{n+2}$ for B^{n+2} such that $e_\alpha^{(n)} \stackrel{w^*}{\to} e^{(n+2)}$ on $A^{(n+2)}$, see [10]. Then we have $e_\alpha^{(n)} b^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$ on $B^{(n+2)}$. Since $e_\alpha^{(n)} b^{(n+2)} \in B^{(n)}$, we have $e_\alpha^{(n)} b^{(n+2)} \stackrel{w}{\to} b^{(n+2)}$ on $B^{(n)}$. We conclude that $b^{n+2} \in B^n$ of course B^n is weakly compact.

(2) Suppose that $b^{(n+2)} \in Z^{\ell}_{A^{(n+2)}}(B^{(n+2)})$ and $e_\alpha^{(n)} \stackrel{w^*}{\to} e^{(n+2)}$ on $A^{(n)}$ such that $e^{(n+2)}$ is right unit for $B^{(n+2)}$, see [10]. Then we have $b^{(n+2)} e_\alpha^{(n)} \stackrel{w^*}{\to} b^{(n+2)}$ on $B^{(n+2)}$. Since $B^{(n+2)} e_\alpha^{(n)} \subseteq B^{(n)}$ for every α, $b^{(n+2)} e_\alpha^{(n)} \stackrel{w}{\to} b^{(n+2)}$ on $B^{(n)}$ and since $B^{(n)}$ is weakly compact,

$b^{(n+2)} \in B^{(n)}$. It follows that $Z^{\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$.

Definition. Let B be a Banach $A-bimodule$ and the integer $n \geq 0$ be an even number. Then $b^{(n+2)} \in B^{(n+2)}$ is said to be weakly left almost periodic functional if the set

$$\{b^{(n+1)}a^{(n)} : \ a^{(n)} \in A^{(n)}, \ \| a^{(n)} \| \leq 1\},$$

is relatively weakly compact, and $b^{(n+2)} \in B^{(n+2)}$ is said to be weakly right almost periodic functional if the set

$$\{a^{(n)}b^{(n+1)} : \ a^{(n)} \in A^{(n)}, \ \| a^{(n)} \| \leq 1\},$$

is relatively weakly compact. We denote by $wap_{\ell}(B^{(n)})$ [resp. $wap_r(B^{(n)})$] the closed subspace of $B^{(n+1)}$ consisting of all the weakly left [resp. right] almost periodic functionals in $B^{(n+1)}$. By [6, 14, 18], the definition of $wap_{\ell}(B^{(n)})$ and $wap_r(B^{(n)})$, respectively, are equivalent to the following

$$wap_{\ell}(B^{(n)}) = \{b^{(n+1)} \in B^{(n+1)} : \ \langle b^{(n+2)}a_{\alpha}^{(n+2)}, b^{(n+1)}\rangle \rightarrow$$

$$\langle b^{(n+2)}a^{(n+2)}, b^{(n+1)}\rangle \ where \ a_{\alpha}^{(n+2)} \xrightarrow{w^*} a^{(n+2)}\}.$$

and

$$wap_r(B^{(n)}) = \{b^{(n+1)} \in B^{(n+1)} : \ \langle a^{(n+2)}b_{\alpha}^{(n+2)}, b^{(n+1)}\rangle \rightarrow$$

$$\langle a^{(n+2)}b^{(n+2)}, b^{(n+1)}\rangle \ where \ b_{\alpha}^{(n+2)} \xrightarrow{w^*} b^{(n+2)}\}.$$

If we take $A = B$ and $n = 0$, then $wap_{\ell}(A) = wap_r(A) = wap(A)$.

Theorem 2.4. *Assume that B is a Banach $A-bimodule$ and the integer $n \geq 0$ be an even number. Then we have the following assertions.*

(1) $B^{(n+1)}A^{(n)} \subseteq wap_{\ell}(B^{(n)})$ if and only if

$$A^{(n)}A^{(n+2)} \subseteq Z^{\ell}_{B^{(n+2)}}(A^{(n+2)}).$$

(2) If $A^{(n)}A^{(n+2)} \subseteq A^{(n)}{Z^{\ell}}_{B^{(n+2)}}(A^{(n+2)})$, then

$$A^{(n)}A^{(n+2)} \subseteq Z^{\ell}_{B^{(n+2)}}(A^{(n+2)}).$$

Proof.

(1) Suppose that $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$. Let $a^{(n)} \in A^{(n)}$, $a^{(n+2)} \in A^{(n+2)}$ and let $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have

$$\langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle = \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle$$

$$\to \langle a^{(n+2)}b^{(n+2)}, b^{(n+1)}a^{(n)}\rangle = \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle.$$

It follows that $a^{(n)}a^{(n+2)} \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$.

Conversely, let $a^{(n)}a^{(n+2)} \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$ for every $a^{(n)} \in A^{(n)}$, $a^{(n+2)} \in A^{(n+2)}$ and suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have

$$\langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle = \langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle$$

$$\to \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle = \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle.$$

It follows that $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$.

(2) Since $A^{(n)}A^{(n+2)} \subseteq A^{(n)}{Z^\ell}_{B^{(n)}}((A^{(n+2)})$, for every $a^{(n)} \in A^{(n)}$ and $a^{(n+2)} \in A^{(n+2)}$, we have $a^{(n)}a^{(n+2)} \in A^{(n)}{Z^\ell}_{B^{(n+2)}}(A^{(n+2)})$.

Then there are $x^{(n)} \in A^{(n)}$ and $\phi \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$ such that $a^{(n)}a^{(n+2)} = x^{(n)}\phi$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have

$$\langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle = \langle (x^{(n)}\phi)b_\alpha^{(n+2)}, b^{(n+1)}\rangle$$

$$= \langle \phi b_\alpha^{(n+2)}, b^{(n+1)}x^{(n)}\rangle \to \langle \phi b^{(n+2)}, b^{(n+1)}x^{(n)}\rangle$$

$$= \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle.$$

In the preceding theorem, if we take $B = A$ and $n = 0$, we conclude Theorem 3.6 (a) from [14].

Theorem 2.5. *Assume that B is a Banach $A-$bimodule and the integer $n \geq 0$ be an even number. If $A^{(n)}$ is a left ideal in $A^{(n+2)}$, then $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$.*

Proof. Proof is clear.

Theorem 2.6. *Let B be a left Banach $A-bimodule$ and $n\geq 0$ be a even number. Suppose that $b_0^{(n+1)}\in B^{(n+1)}$. Then $b_0^{(n+1)}\in wap_\ell(B^{(n)})$ if and only if the mapping $T:b^{(n+2)}\rightarrow b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^*-to-weak$ continuous.*

Proof. Let $b_0^{(n+1)}\in B^{(n+1)}$ and suppose that $b_\alpha^{(n+2)}\overset{w^*}{\rightarrow} b^{(n+2)}$ on $B^{(n+2)}$. Then for every $a^{(n+2)}\in A^{(n+2)}$, we have

$$\langle a^{(n+2)},b_\alpha^{(n+2)}b_0^{(n+1)}\rangle=\langle a^{(n+2)}b_\alpha^{(n+2)},b_0^{(n+1)}\rangle\rightarrow\langle a^{(n+2)}b^{(n+2)},b_0^{(n+1)}\rangle$$
$$=\langle a^{(n+2)},b^{(n+2)}b_0^{(n+1)}\rangle.$$

It follows that $b_\alpha^{(n+2)}b_0^{(n+1)}\overset{w}{\rightarrow} b^{(n+2)}b_0^{(n+1)}$ on $A^{(n+1)}$.
The proof of the converse is similar of preceding proof.

Corollary 2.7. *Assume that B is a Banach $A-bimodule$. Then $Z^\ell_{A^{(n+2)}}(B^{(n+2)})=B^{(n+2)}$ if and only if the mapping $T:b^{(n+2)}\rightarrow b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^*-to-weak$ continuous for every $b_0^{(n+1)}\in B^{(n+1)}$.*

Corollary 2.8. *Let A be a Banach algebra. Assume that $a'\in A^*$ and $T_{a'}$ is the linear operator from A into A^* defined by $T_{a'}a=a'a$. Then, $a'\in wap(A)$ if and only if the adjoint of $T_{a'}$ is $weak^*-to-weak$ continuous. So A is Arens regular if and only if the adjoint of the mapping $T_{a'}a=a'a$ is $weak^*-to-weak$ continuous for every $a'\in A^*$.*

Definition. Let B be a left Banach $A-bimodule$. We say that $a^{(n)}\in A^{(n)}$ has $Left-weak^*-weak$ property ($=Lw^*w-$ property) with respect to $B^{(n)}$, if for every $(b_\alpha^{(n+1)})_\alpha\subseteq B^{(n+1)}$, $a^{(n)}b_\alpha^{(n+1)}\overset{w^*}{\rightarrow}0$ implies $a^{(n)}b_\alpha^{(n+1)}\overset{w}{\rightarrow}0$. If every $a^{(n)}\in A$ has Lw^*w- property with respect to $B^{(n)}$, then we say that $A^{(n)}$ has Lw^*w- property with respect to $B^{(n)}$. The definition of the $Right-weak^*-weak$ property ($=Rw^*w-$ property) is the same.
We say that $a^{(n)}\in A^{(n)}$ has $weak^*-weak$ property ($=w^*w-$ property) with respect to $B^{(n)}$ if it has Lw^*w- property and Rw^*w- property with respect to $B^{(n)}$.
If $a^{(n)}\in A^{(n)}$ has Lw^*w- property with respect to itself, then we say that $a^{(n)}\in A^{(n)}$ has Lw^*w- property.

Example.

(1) If B is Banach A-bimodule and reflexive, then A has w^*w-property with respect to B.
(2) $L^1(G)$, $M(G)$ and $A(G)$ have w^*w-property when G is finite.
(3) Let G be locally compact group. $L^1(G)$ [resp. $M(G)$] has w^*w-property [resp. Lw^*w- property] with respect to $L^p(G)$ whenever $p>1$.
(4) Suppose that B is a left Banach $A-module$ and e is left unit element of A such that $eb=b$ for all $b\in B$. If e has Lw^*w- property, then B is reflexive.
(5) If S is a compact semigroup, then $C^+(S)=\{f\in C(S):\ f>0\}$ has w^*w-property.

Theorem 2.9. *Let B be a left Banach $A-bimodule$ and the integer $n\geq 2$ be an even number. Then we have the following assertions.*

(1) *If $A^{(n)}=a_0^{(n-2)}A^{(n)}$ [resp. $A^{(n)}=A^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)}\in A^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [resp. Lw^*w- property] with respect to $B^{(n)}$, then $Z_{B^{(n)}}(A^{(n)})=A^{(n)}$.*

(2) *If $B^{(n)}=a_0^{(n-2)}B^{(n)}$ [resp. $B^{(n)}=B^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)}\in A^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [resp. Lw^*w- property] with respect to $B^{(n)}$, then $Z_{A^{(n)}}(B^{(n)})=B^{(n)}$.*

Proof.

(1) Suppose that $A^{(n)}=a_0^{(n-2)}A^{(n)}$ for some $a_0^{(n-2)}\in A$ and $a_0^{(n-2)}$ has Rw^*w- property. Let $(b_\alpha^{(n)})_\alpha\subseteq B^{(n)}$ such that $b_\alpha^{(n)}\stackrel{w^*}{\to}b^{(n)}$. Then for every $a^{(n-2)}\in A^{(n-2)}$ and $b^{(n-1)}\in B^{(n-1)}$, we have

$$\langle b_\alpha^{(n)}b^{(n-1)},a^{(n-2)}\rangle=\langle b_\alpha^{(n)},b^{(n-1)}a^{(n-2)}\rangle\to\langle b^{(n)},b^{(n-1)}a^{(n-2)}\rangle$$
$$=\langle b^{(n)}b^{(n-1)},a^{(n-2)}\rangle.$$

It follows that $b_\alpha^{(n)}b^{(n-1)}\stackrel{w^*}{\to}b^{(n)}b^{(n-1)}$. Also it is clear that $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w^*}{\to}(b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Since $a_0^{(n-2)}$ has Rw^*w- property, $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w}{\to}(b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Now, let $a^{(n)}\in A^{(n)}$. Since $A^{(n)}=a_0^{(n-2)}A^{(n)}$, there is $x^{(n)}\in A^{(n)}$ such that

$a^{(n)}=a_0^{(n-2)}x^{(n)}$. Thus we have

$$\langle a^{(n)}b_\alpha^{(n)},b^{(n-1)}\rangle=\langle a^{(n)},b_\alpha^{(n)}b^{(n-1)}\rangle=\langle a_0^{(n-2)}x^{(n)},b_\alpha^{(n)}b^{(n-1)}\rangle$$
$$=\langle x^{(n)},(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle\rightarrow\langle x^{(n)},(b^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle$$
$$=\langle a^{(n)}b,b^{(n-1)}\rangle.$$

It follows that $a^{(n)}\in Z_{A^{(n)}}(B^{(n)})$.
Proof of the next part is similar to preceding proof.

(2) Let $B^{(n)}=a_0^{(n-2)}B^{(n)}$ for some $a_0^{(n-2)}\in A$ and $a_0^{(n-2)}$ has Rw^*w- property with respect to $B^{(n)}$. Assume that $(a_\alpha^{(n)})_\alpha\subseteq A^{(n)}$ such that $a_\alpha^{(n)}\stackrel{w^*}{\rightarrow}a^{(n)}$. Then for every $b^{(n-1)}\in B^{(n-1)}$, we have

$$\langle a_\alpha^{(n)}b^{(n-1)},b^{(n-2)}\rangle=\langle a_\alpha^{(n)},b^{(n-1)}b^{(n-2)}\rangle\rightarrow\langle a^{(n)},b^{(n-1)}b^{(n-2)}\rangle$$
$$=\langle a^{(n)}b^{(n-1)},b^{(n-2)}\rangle$$

We conclude that $a_\alpha^{(n)}b^{(n-1)}\stackrel{w^*}{\rightarrow}a^{(n)}b^{(n-1)}$. It is clear that

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w^*}{\rightarrow}(a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Since $a_0^{(n-2)}$ has Rw^*w- property,

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w}{\rightarrow}(a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Suppose that $b^{(n)}\in B^{(n)}$. Since $B^{(n)}=a_0^{(n-2)}B^{(n)}$, there is $y^{(n)}\in B^{(n)}$ such that $b^{(n)}=a_0^{(n-2)}y^{(n)}$. Consequently, we have

$$\langle b^{(n)}a_\alpha^{(n)},b^{(n-1)}\rangle=\langle b^{(n)},a_\alpha^{(n)}b^{(n-1)}\rangle=\langle a_0^{(n-2)}y^{(n)},a_\alpha^{(n)}b^{(n-1)}\rangle$$
$$=\langle y^{(n)},(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle\rightarrow\langle y^{(n)},(a^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle$$
$$=\langle a_0^{(n-2)}y^{(n)},(a^{(n)}b^{(n-1)})\rangle=\langle b^{(n)}a^{(n)},b^{(n-1)}\rangle.$$

Thus $b^{(n)}a_\alpha^{(n)}\stackrel{w}{\rightarrow}b^{(n)}a^{(n)}$. It follows that $b^{(n)}\in Z_{A^{(n)}}(B^{(n)})$.
The proof of the next part similar to the preceding proof.

Example. Let G be a locally compact group. Since $M(G)$ is a Banach $L^1(G)$-bimodule and the unit element of $M(G)^{(n)}$ has not Lw^*w- property or Rw^*w- property, by Theorem 2.9, $Z_{L^1(G)^{(n)}}(M(G)^{(n)})\neq M(G)^{(n)}$.
ii) If G is finite, then by Theorem 2.9, we have $Z_{M(G)^{(n)}}(L^1(G)^{(n)})=$

$L^1(G)^{(n)}$ and $Z_{L^1(G)^{(n)}}(M(G)^{(n)}) = M(G)^{(n)}$.

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