Some Properties of the Schur Multiplier with Algebraic Topological Approach

Behrooz Mashayekhy [*] **and Hanieh Mirebrahimi**

Department of Mathematics,

Center of Excellence in Analysis on Algebraic Structures,

Ferdowsi University of Mashhad,

P. O. Box 1159-91775, Mashhad, Iran

Abstract

In this paper, using a relationship between the Schur multiplier of a group G, the fundamental group, and the second homology group of the Eilenberg-MacLane space of G, we present new proofs for some famous properties of the Schur multiplier and its structure for the free product and the direct product with an algebraic topological approach. Also, we try to find the Schur multiplier of a free amalgamated product in terms of Schur multipliers of its factors, which is a new result.

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[*]Correspondence: mashaf@math.um.ac.ir

1. Introduction and Preliminaries

Let F/R be a free presentation of a group G, then the Schur multiplier of G is defined to be

$$M(G) = (R \cap F')/[R, F]$$

(see [2] for further details). There are some well-known facts about $M(G)$. By a theorem of Schur [2], with respect to the presentation of G, there exists an upper bound for the number of the generators of $M(G)$. In particular, if G is a cyclic group, $M(G)$ is trivial [2]. A theorem of C. Miller [2], says that

$$M(G_1 * G_2) \cong M(G_1) \oplus M(G_2).$$

Also Schur Theorem [2] asserts that

$$M(G_1 \times G_2) \cong M(G_1) \oplus M(G_2) \oplus G_1 \otimes G_2.$$

In this paper, we are going to give some new proofs for the above facts with algebraic topological methods. Also, as a new result, we try to find the Schur multiplier of a free amalgamated product in terms of Schur multipliers of its factors. We suppose that the reader is familiar with some well-known notions in algebraic topology such as homotopy groups, homology groups, CW-spaces, and basic notions in group theory.

In order to find a suitable relationship between the Schur multiplier and some famous notions of algebraic topology, we mention the following notes.

Theorem 1.1 (Hurewicz Theorem [4]). For any path connected space X, the Hurewicz map induces an isomorphism called Hurewicz isomorphism, as follows:

$$\frac{\pi_1(X, x_0)}{(\pi_1(X, x_0))'} \cong H_1(G).$$

Theorem 1.2 (Hopf Formula [1]). If K is a CW-complex with $\pi_1(K) = G$ and F/R is a presentation of G then there exists the following isomorphism

along the Hurewicz map:

$$\frac{H_2(K)}{h_2(\pi_2(K))} \cong \frac{R \cap F'}{[R,F]},$$

where $H_2(K)$ is the second homology group of K and h_2 is the corresponding Hurewicz map.

Theorem 1.3 ([6]). For any group G, there exists a CW-complex X with

$$\pi_1(X) \cong G \text{ and } \pi_n(X) = 1 \text{ for all } n \geq 2.$$

The space X is called Eilenberg-MacLane space of G.

Corollary 1.4. For any group G and its Eilenberg-MacLane space, K say, we have

$$\pi_1(K) \cong G \text{ and } H_2(K) \cong M(G).$$

Remark 1.5 ([4,6]). For any CW-complex K with k_i many i-cells, $d(H_i(K)) \leq k_i$. In addition, the number of 2-cells in an Eilenberg-MacLane space, obtained from a presentation of a group G, is equal to the number of it's relators. (See the proof of [6, Note 6.44])

Theorem 1.6 ([3,5]). For any numbers $n, m \in N$, there exists a CW-complex $L(n,m)$, called Lens space, with

$$\pi_1(L(n,m)) \cong Z_n \text{ and } H_2(L(n,m)) = 1.$$

Theorem 1.7 (Mayer-Vietoris Sequence [4]). For any two subcomplexes X_1 and X_2 of a CW-complex X, with $X = X_1 \cup X_2$, there is an exact sequence as follows

$$\cdots \to H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \to H_{n-1}(X_1 \cap X_2) \to \cdots.$$

Theorem 1.8 (Kunneth Formula [4]). For any pair of spaces X and Y and for every integer $n \geq 0$, we have the following relation between their

homology groups

$$H_n(X \times Y) \cong \sum_{i+j=n} H_i(X) \otimes H_i(Y) \oplus \sum_{p+q=n-1} Tor(H_p(X), H_q(Y)).$$

2. Main Results

In this section, first, using the previous relationship between the Schur multiplier, homology and homotopy groups, we try to give algebraic topological proofs for some well-known facts about the Schur multiplier.

Theorem 2.1. *For any k-relator group G, there exists an upper bound for the number of the generators of its Schur multiplier, as follows*

$$d(M(G)) \leq k.$$

Proof. First, suppose that K is the Eilenberg-MacLane space of G, then using Remark 1.5, the number of 2-cells in the complex K equals exactly to k and consequently $d(H_2(K)) \leq k$. Finally by Hopf Formula for the Eilenberg-MacLane space K, $M(G) \cong H_2(K)$ and so the result holds. □

Theorem 2.2. *The Schur multiplier of any cyclic group G is trivial.*

Proof. If G is an infinite cyclic group, we can consider a circle S^1 as it's Eilenberg-MacLane space (S^1 is a CW-complex whose fundamental group is infinite cyclic and it's higher homotopy groups are trivial [4]). Hence using the Hopf isomorphism $M(G) \cong H_2(S^1)$ and the fact of S^1 whose second homology group is trivial, in this case, the result is satisfied.

Also for a finite cyclic group G of order n, using Theorem 1.2, we have the Lens space $L(n, 1)$ as an Eilenberg-MacLane space of G. So by a similar argument to the above and Theorem 1.2, the proof is completed. □

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Theorem 2.3. *For any two groups G_1 and G_2, we have the isomorphism*

$$M(G_1 * G_2) \cong M(G_1) \oplus M(G_1).$$

Proof. First, using Theorem 1.1, let K_1 and K_2 be the Eilenberg-MacLane spaces of G_1 and G_2, respectively. By Van-Kampen Theorem for the fundamental group of wedge space, $\pi_1(K_1 \vee K_2) \cong \pi_1(K_1) * \pi_1(K_2)$. Also using definitions of π_n and wedge space, $\pi_n(K_1 \vee K_2) = 1$, for all $n \geq 2$. Hence the wedge spaces $K_1 \vee K_2$ can be considered as an Eilenberg-MacLane spaces of $G_1 * G_2$ and with respect to the Hopf Theorem, we have

$$M(G_1) \cong H_2(K_1) \ , \ \ M(G_2) \cong H_2(K_2)$$

$$\& \ \ M(G_1 * G_2) \cong H_2(K_1 \vee K_2).$$

Finally, using the Mayer-Vietories sequence for the wedge space $K_1 \vee K_2$, we conclude the following isomorphism which completes the proof

$$H_2(K_1 \vee K_2) \cong H_2(K_1) \oplus H_2(K_2). \ \square$$

Theorem 2.4. *For any two groups G_1 and G_2, the following isomorphism holds:*

$$M(G_1 \times G_2) \cong M(G_1) \oplus M(G_2) \oplus (G_1)_{ab} \otimes (G_2)_{ab}.$$

Proof. Similar to the previous proof, suppose that K_1 and K_2 are the Eilenberg-MacLane spaces of G_1 and G_2, respectively. Using one of the properties of the homotopy functor π_n to preserve the direct product, we have $\pi_1(K_1 \times K_2) \cong \pi_1(K_1) \times \pi_1(K_2)$ and $\pi_n(K_1 \times K_2) = 1$, for all $n \geq 1$.

So the space $K_1 \times K_2$ is an Eilenberg-MacLane spaces of $G_1 \times G_2$. Hence by Hopf isomorphism, we have $M(G_i) \cong H_2(K_i)$, for $i = 1, 2$, and

$$M(G_1 \times G_2) \cong H_2(K_1 \times K_2).$$

Also, using Kunneth Formula, the properties of the functor Tor and tensor product, and the fact of $H_0(X)$ which is isomorphic to the infinite cyclic group Z where X is a path connected space (note that Eilenberg-MacLane spaces are path connected), we have the following relation between the first and the second homology groups,

$$H_2(K_1 \times K_2) \cong H_2(K_1) \oplus H_2(K_2) \oplus H_1(K_1) \otimes H_1(K_2).$$

Finally, by Hurewicz isomorphisms $H_1(K_i) \cong (\pi_1(K_i))_{ab} = (G_i)_{ab}$, for $i = 1, 2$, we conclude the result of the theorem. $\square$

Now, as a new result, we try to find the structure of the Schur multiplier of a free amalgamated product in terms of the Schur multipliers of its factors.

Theorem 2.5. *Suppose that G is the free amalgamated product of its two subgroups G_1 and G_1 over a subgroup H. Then we have the following exact sequence, which is induced by the Mayer-Vietories sequence along the Hurewicz map:*

$$\cdots \to M(H) \to M(G_1) \oplus M(G_2) \to M(G)$$

$$\to H_{ab} \to G_{1ab} \oplus G_{2ab} \to G_{ab} \to \cdots.$$

Proof. First, we consider an Eilenberg-Maclane space corresponding to the presentation of H, Y say, and note that we can extend the presentation of H to a presentation for G_1 and a presentation for G_2.

Also, by joining some 1-cells and attaching 2-cells via the relations, similar to the method of [4, Theorem 7.45] and [6, Note 6.44], we extend Y to Eilenberg-Maclane spaces X_1 and X_2 corresponding to the above presentations of G_1 and G_2, respectively. Note that the construction is considered in such a way that Y is a deformation retract of the space $X_1 \cap X_2$.

Now using the van-Kampen theorem, the fundamental group $\pi_1(X_1 \cup X_2)$ is the free amalgamated product of two groups $\pi_1(X_1) \cong G_1$ and $\pi_1(X_2) \cong G_2$ over the subgroup $\pi_1(Y) \cong \pi_1(X_1 \cap X_2) \cong H$ [5].

Hence by uniqueness of the free amalgamated product up to isomorphism, we conclude that

$$G \cong \pi_1(X_1 \cup X_2).$$

Also, we have the Mayer-Vietoris Sequence for CW-complexes X_1 and X_2 [4], as follow:

$$\cdots \to H_2(X_1 \cap X_2) \to H_2(X_1) \oplus H_2(X_2) \to H_2(X_1 \cup X_2)$$

$$\to H_1(X_1 \cap X_2) \to H_1(X_1) \oplus H_1(X_2) \to \to H_2(X_1 \cup X_2) \to 1.$$

Finally, using Hurewicz maps and so Hopf Formula (See the proof of [1, Theorem(Hopf)]), we get the result. □

As some corollaries of the recent theorem, we are going to present the following notes about the structure of the Schur multiplier of a free amalgamated product in some special cases.

Remark 2.6. Let G be the free amalgamated product of its two subgroups G_1 and G_2 over a subgroup H. Then by the exact sequence in previous theorem, the following hold.

i) If the Schur multiplier of H is trivial, then $M(G)$ contains a subgroup isomorphic to the group $M(G_1) \oplus M(G_2)$.

ii) If H is a perfect group with trivial Schur multiplier, then $M(G)$ is isomorphic to the group $M(G_1) \oplus M(G_2)$.

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