THE STABILITY OF STOCHASTIC SHELL MODELS

H. BESSAIH, R. KAPICA, AND T. SZAREK

Abstract. We formulate and prove a new criterion for stability of e–processes introduced by A. Lasota and T. Szarek [J. Differential Equations **231** (2006), 513–533]. In particular we prove that that any e–process which is averagely bounded and concentrating is asymptotically stable. This general result is applied to stochastic equations corresponding to some shell models (the GOY and the Sabra model) driven by an additive noise. For these equations the e–process property is proved to not depend on the number of modes of the noise. Hence the e-process property holds even for degenerate noises. As a consequence, we also obtain the uniqueness of the invariant measure of the stochastic shell model when the noise is affecting only finitely many modes, a result conjectured in Barbato *et al.* in [3].

1. Introduction

In this paper we will present a new criterion for stability of Markov semigroups and apply it to some stochastic shell models, proving the existence of a unique invariant measure and its stability. An invariant measure is roughly speaking a statistical stationary solution and it is a good candidate to represent the asymptotic behavior of the system. If the invariant measure is unique, there are chances that the law of the process solution will converge to it. One of the principle of the statistical approach to fluid dynamics is that every regular observable defined over the phase space (a space of velocity fields) and for every initial velocity field (except for a set of initial fields that is negligible in some sense), the time average of the observable tends, as time goes to infinity, to the mean value of the observable with respect to the unique invariant measure. Such a result has been proved for the 2D stochastic Navier-Stokes equations in [9] when the noise is non-degenerate. Similar results have been proved in [7] with less restrictive assumptions on the noise, see also the references therein. The approach used in those papers is based on the strong Feller property of the transition semigroup. In many dissipative systems, including the stochastic Navier-Stokes equations, only finitely many modes are unstable. Conceivably, these systems are ergodic even if the noise is transmitted only to those unstable modes rather than to the whole system. In recent years, this topic has attracted a great interest and has been an intense

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subject of study. A main contribution to the 2D stochastic Navier-Stokes equations with a degenerate noise has been made by M. Hairer and J. Mattingly in [12]. They define a weaker property satisfied by the system, the *asymptotic strong Feller property*, and show that it is sufficient to give ergodicity. Recently A. Lasota and T. Szarek, see [15], have introduced the *e-process property* that is a more general concept than the asymptotic strong Feller property (see also [24]). Using this property T. Komorowski, S. Peszat and T. Szarek [13] have proved ergodicity for the process solution of the passive tracer equation.

In this paper we are dealing with some stochastic shell models of turbulence, the GOY and Sabra model. These are very popular examples of simplified phenomenological models of turbulence. Although they are not based on conservations laws, they capture some essential statistical properties and features of turbulent flows, like the energy and the enstrophy cascade and the power law decay of the structure functions in some range of wave numbers, the inertial range. We refer the reader to [22], [1], [4], [10] and [11] and the references therein and to [5], [8] and [3] for some rigorous results. We are interested in a noise where only finitely many modes are nonzero and we prove the e-process property. Moreover, we are able to prove asymptotic stability that will lead to the uniqueness of an invariant measure.

More specifically, we prove that any averagely bounded Markov semigroup with the e-property concentrating at some point admits a unique invariant measure. Moreover the semigroup is stable. The proof is based on the lower bound technique that has been introduced by A. Lasota and J. Yorke see [16], where the authors showed the existence of an absolutely continuous invariant measure for the Frobenius–Perron operator corresponding to piecewise monotonic transformations. Since then, the technique has been generalized first for Markov semigroups acting on densities (see [14]), subsequently for general Markov semigroups defined on arbitrary Borel measures in finite dimensions (see [17]) and finally it has been extended to infinite dimensional spaces (see [21]). Generally speaking the method relies on an easy observation that two regular trajectories starting at different measures which visit some small set with positive, bounded from below, probability converge in the weak topology. Additionally, if we assume that every neighborhood of some point is visited infinitely many times, then we may show that the process admits an invariant measure.

Let us mention that the proof of the e-property for processes corresponding to the shell models is very challenging. In fact, we introduce here a new and promising method. Namely we approximate the shell model equation with degenerate noise by a similar one with a non-degenerate noise and prove that the e-property is preserved by passing to the limit on the coefficient of approximation. The e–property for the nondegenerate equation is proved by the Malliavin calculus. Boundedness and concentrating property, in turn, easily follow from standard estimates for shell models. Their proofs are rather straightforward. The main result of our paper answers to the conjecture posed by Barbato *et al.* (see [3]) who anticipated that in the case when the number of modes to which we add the noise is large enough, using the techniques developed by M. Hairer and J. Mattingly in [12], it would be possible to prove the uniquness of an invariant measure. In our proof we make no assumption on the number of modes affected by noise.

The paper is organized as follows. In section 2, we introduce the concepts on e-property, averagely bounded and concentrating at a point. We also prove (Proposition 1) the main result about asymptotic stability for Markov processes. In Section 3, we introduce the GOY and Sabra models and give general results about their well posedness. In Section 4, we apply the results of Section 2 to the shell models and prove the e-property, the average boundedness and the concentrating property and state our main result for the uniqueness of the invariant measure for the stochastic shell model with a degenerate noise. In the Appendix, some estimates needed to prove the e-property for shell models are given and proved.

2. Criterion on Stability

Let (X,ρ) be a Polish space. By $B_b(X)$ we denote the space of all bounded Borel–measurable functions equipped with the supremum norm. Let $(P_t)_{t\geq 0}$ be the *Markovian semigroup* defined on $B_b(X)$. For each $t \geq 0$ we have $P_t\mathbf{1}_X = \mathbf{1}_X$ and $P_t\psi \geq 0$ if $\psi \geq 0$. Throughout this paper we shall assume that the semigroup is *Feller*, i.e. $P_t(C_b(X)) \subset C_b(X)$ for all $t > 0$. Here and in the sequel $C_b(X)$ is the subspace of all bounded continuous functions with the supremum norm $\|\cdot\|_\infty$. By $L_b(X)$ we will denote the subspace of all bounded Lipschitz functions. We shall also assume that $(P_t)_{t\geq 0}$ is *stochastically continuous*, which implies that $\lim_{t\to 0^+} P_t\psi(x) = \psi(x)$ for all $x \in X$ and $\psi \in C_b(X)$.

Let $\mathcal{M}_1$ stand for the space of all Borel probability measures on X. Denote by $\mathcal{M}_1^W$, $W \subset X$, the subspace of all Borel probability measures supported in W, i.e. $\{x \in X : \mu(B(x,r)) > 0 \text{ for any } r > 0\} \subset W$, where $B(x,r)$ denotes the ball in X with center at x and radius r.. For $\varphi \in B_b(X)$ and $\mu \in \mathcal{M}_1$ we will use the notation $\langle\varphi,\mu\rangle = \int_X \varphi(x)\mu(\mathrm{d}x)$. Recall that the *total variation norm* of a finite signed measure $\mu \in \mathcal{M}_1 - \mathcal{M}_1$ is given by $\|\mu\|_{TV} = \mu^+(X) + \mu^-(X)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

We say that $\mu_* \in \mathcal{M}_1$ is *invariant* for $(P_t)_{t\geq 0}$ if $\langle P_t\psi, \mu_*\rangle = \langle\psi,\mu_*\rangle$ for every $\psi \in B_b(X)$ and $t \geq 0$. Alternatively, we can say that $P_t^*\mu_* = \mu_*$ for all $t \geq 0$, where $(P_t^*)_{t\geq 0}$ denotes the semigroup dual to $(P_t)_{t\geq 0}$, i.e. for a given Borel measure μ, Borel subset A of X, and $t \geq 0$ we set

$$P_t^*\mu(A) := \langle P_t\mathbf{1}_A, \mu\rangle.$$

A semigroup $(P_t)_{t\geq 0}$ is said to be *asymptotically stable* if there exists an invariant measure $\mu_* \in \mathcal{M}_1$ such that $P_t^*\mu$ converges weakly to μ_* as $t \to +\infty$ for every $\mu \in \mathcal{M}_1$. Obviously μ_* is unique.

Definition 2.1. *We say that a semigroup $(P_t)_{t\geq 0}$ has* the e–property *if the family of functions $(P_t\psi)_{t\geq 0}$ is equicontinuous at every point x of X for any bounded and Lipschitz function ψ, i.e.*

$$\forall\,\psi \in L_b(X),\, x \in X,\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in B(x,\delta),\, t \geq 0 : \, |P_t\psi(x) - P_t\psi(z)| < \varepsilon.$$

Remark. One can show (see [13]) that to obtain the e–property in the case when X is a Hilbert space, it is enough to verify the above condition for every function with bounded Fréchet derivative.

Definition 2.2. *A semigroup $(P_t)_{t\geq 0}$ is called* averagely bounded *if for any $\varepsilon > 0$ and bounded set $A \subset X$ there is a bounded Borel set $B \subset X$ such that*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B)\mathrm{d}s > 1-\varepsilon \qquad \text{for } \mu \in \mathcal{M}_1^A.$$

Definition 2.3. *A semigroup $(P_t)_{t\geq 0}$ is* concentrating *at z if for any $\varepsilon > 0$ and bounded set $A \subset X$ there exists $\alpha > 0$ such that for any two measures $\mu_1, \mu_2 \in \mathcal{M}_1^A$ holds*

$$P_t^*\mu_i(B(z,\varepsilon)) \geq \alpha \text{ for } i = 1,2 \text{ and some } t > 0.$$

Proposition 1. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then for any $\varphi \in L_b(X)$ and $\mu_1, \mu_2 \in \mathcal{M}_1$ we have*

$$\lim_{t\to\infty} |\langle \varphi, P_t^*\mu_1\rangle - \langle \varphi, P_t^*\mu_2\rangle| = 0. \tag{2.1}$$

Proof. First observe that to finish the proof it is enough to show that condition (2.1) holds for arbitrary Borel probability measures with bounded support. Indeed, the set of all probability measures with bounded support is dense in the space $(\mathcal{M}_1, \|\cdot\|_{TV})$. Moreover, P_t^*, $t \geq 0$, is nonexpansive with respect to the total variation norm.

Fix $\varphi \in L_b(X)$, $x_0 \in X$ and $\varepsilon \in (0, 1/2)$. Let $\mu_1, \mu_2 \in \mathcal{M}_1^{B(x_0,r_0)}$ for some $r_0 > 0$. Choose $\delta > 0$ such that

$$\sup_{t\geq 0} |P_t\varphi(x) - P_t\varphi(y)| < \varepsilon/2 \tag{2.2}$$

for $x, y \in B(z, \delta)$, by the e-property.

Since $(P_t)_{t\geq 0}$ is averagely bounded we may find $R_0 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B(x_0,R_0))\mathrm{d}s > 1 - \varepsilon^2/(4\|\varphi\|_\infty) \tag{2.3}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,r_0)}$. Let $R > \max\{R_0, r_0\}$ satisfy

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B(x_0,R))\mathrm{d}s > 3/4 \tag{2.4}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,R_0)}$. Since $(P_t)_{t\geq 0}$ is concentrating at z we may choose $\alpha > 0$ such that for any $\nu_1, \nu_2 \in \mathcal{M}_1^{B(x_0,R)}$ there exists $t > 0$ and the condition

$$P_t^*\nu_i(B(z,\delta)) \geq \alpha \qquad \text{for } i = 1,2 \tag{2.5}$$

holds.

Set $\gamma := \alpha\varepsilon/2 > 0$. Let k be the minimal integer such that $4(1-\gamma)^k\|\varphi\|_\infty \leq \varepsilon$.

We will show by induction that for every $l \leq k$, $l \in \mathbb{N}$, there exist $t_1, \dots, t_l > 0$ and $\nu_1^i, \dots, \nu_l^i, \mu_l^i \in \mathcal{M}_1$ such that $\nu_j^i \in \mathcal{M}_1^{B(z,\delta)}$ for $j = 1, \dots, l$ and

$$\begin{aligned} P_{t_1+\dots+t_l}^*\mu_i = {} & \gamma P_{t_2+\dots+t_l}^*\nu_1^i + \gamma(1-\gamma)P_{t_3+\dots+t_l}^*\nu_2^i \\ & + \dots + \gamma(1-\gamma)^{l-1}\nu_l^i + (1-\gamma)^l\mu_l^i \qquad \text{for } i = 1,2. \end{aligned} \tag{2.6}$$

Indeed, let $t_1 > 0$ be such that

$$P^*_{t_1}\mu_i(B(z,\delta)) \geq \alpha > \gamma \qquad \text{for } i = 1,2.$$

Set

$$\nu^i_1 = \frac{P^*_{t_1}\mu_i(\,\cdot \cap B(z,\delta))}{P^*_{t_1}\mu_i(B(z,\delta))}, \tag{2.7}$$

$$\mu^i_1 = (1-\gamma)^{-1}(P^*_{t_1}\mu_i - \gamma\nu^i_1) \qquad \text{for } i = 1,2$$

and observe that $\mu^i_1 \in \mathcal{M}_1$ and $\nu^i_1 \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1,2$. Then condition (2.6) holds for $l = 1$.

Now assume that we have done it for some l and $4(1-\gamma)^l\|\varphi\|_\infty > \varepsilon$. Then there exist $s_i > 0$ for $i = 1,2$ such that

$$P^*_{t_1+\cdots+t_l+s_i}\mu_i(X \setminus B(x_0,R_0)) < \varepsilon^2/(4\|\varphi\|_\infty)$$

for $i = 1,2$, by (2.3). Since $(1-\gamma)^l > \varepsilon/(4\|\varphi\|_\infty)$, from the linearity of $P^*_{s_i}$ we obtain that

$$P^*_{s_i}\mu^i_l(B(x_0,R_0)) > \varepsilon \qquad \text{for } i = 1,2.$$

Thus we may find two measures $\tilde{\mu}^1_l, \tilde{\mu}^2_l \in \mathcal{M}_1^{B(x_0,R_0)}$ such that

$$P^*_{s_i}\mu^i_l \geq \varepsilon\tilde{\mu}^i_l. \tag{2.8}$$

These measures may be defined as restriction of $P^*_{s_i}\mu^i_l$ to $B(x_0,R_0)$ respectively normed (see formula (2.7)). Further, from (2.4) it follows that

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\int_0^T [P^*_{s+s_2}(\tilde{\mu}^1_l/2)(B(x_0,R)) + P^*_{s+s_1}(\tilde{\mu}^2_l/2)(B(x_0,R))]\mathrm{d}s \\
&= \limsup_{T\to\infty}\frac{1}{T}\int_0^T P^*_s(\tilde{\mu}^1_l/2 + \tilde{\mu}^2_l/2)(B(x_0,R))\mathrm{d}s > 3/4,
\end{aligned}$$

by the fact that $\tilde{\mu}^1_l/2 + \tilde{\mu}^2_l/2 \in \mathcal{M}_1^{B(x_0,R_0)}$. Consequently, for some $s > 0$ we have

$$P^*_{s+s_2}\tilde{\mu}^1_l(B(x_0,R)) \geq 1/2 \quad \text{and} \quad P^*_{s+s_1}\tilde{\mu}^2_l(B(x_0,R)) \geq 1/2.$$

Comparing (2.8) and the above we obtain

$$P^*_{s+s_1+s_2}\mu^i_l \geq (\varepsilon/2)\hat{\mu}^i_l$$

for some $\hat{\mu}^i_l \in \mathcal{M}_1^{B(x_0,R)}$, $i = 1,2$, by argument similar to that in (2.8). Using it once again and taking into consideration (2.5) we obtain that there exists $t > 0$ such that

$$P^*_{t+s+s_1+s_2}\mu^i_l \geq (\alpha\varepsilon/2)\nu^i_{l+1} = \gamma\nu^i_{l+1}$$

for some $\nu^i_{l+1} \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1,2$. Therefore, setting $t_{l+1} = t + s + s_1 + s_2$ we obtain

$$\begin{aligned}
P^*_{t_1+\cdots+t_l+t_{l+1}}\mu_i &= \gamma P^*_{t_2+\cdots+t_{l+1}}\nu^i_1 + \gamma(1-\gamma)P^*_{t_3+\cdots+t_{l+1}}\nu^i_2 \\
&\quad + \cdots + \gamma(1-\gamma)^{l-1}P^*_{t_{l+1}}\nu^i_l + \gamma(1-\gamma)^l\nu^i_{l+1} + (1-\gamma)^{l+1}\mu^i_{l+1},
\end{aligned}$$

where

$$\mu^i_{l+1} = (1-\gamma)^{-1}(P^*_{t_{l+1}}\mu^i_l - \gamma\nu^i_{l+1}) \qquad \text{for } i = 1,2.$$

This completes the proof of condition (2.6). In turn, this and (2.2) give for $t \geq t_1 + \cdots + t_k$

$$\begin{aligned}|\langle \varphi, P_t^*\mu_1\rangle - \langle \varphi, P_t^*\mu_2\rangle| &= |\langle P_{t-(t_1+\cdots t_k}\varphi, P^*_{t_1+\cdots t_k}\mu_1\rangle - \langle P_{t-(t_1+\cdots t_k}\varphi, P^*_{t_1+\cdots t_k}\mu_2\rangle| \\ &\leq \gamma|\langle P_{t-t_1}\varphi, \nu_1^1 - \nu_1^2\rangle| + \gamma(1-\gamma)|\langle P_{t-(t_1+t_2)}\varphi, \nu_2^1 - \nu_2^2\rangle| + \cdots \\ &+ \gamma(1-\gamma)^{k-1}|\langle P_{t-(t_1+\cdots+t_k)}\varphi, \nu_k^1 - \nu_k^2\rangle| + 2(1-\gamma)^k\|\varphi\|_\infty \\ &\leq (\gamma + \gamma(1-\gamma) + \cdots + \gamma(1-\gamma)^{k-1}) \sup_{t\geq 0,\, x,y\in B(z,\delta)} |P_t\varphi(x) - P_t\varphi(y)| \\ &+ \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

Proposition 2. *Assume that there exists $z \in X$ such that for any $\varepsilon > 0$*

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P_s^*\mu(B(z,\varepsilon))\mathrm{d}s > 0. \tag{2.9}$$

If $(P_t)_{t\geq 0}$ satisfies the e–property, then it admits an invariant measure.

Proof. Assume, contrary to our claim, that $(P_t)_{t\geq 0}$ does not possess any invariant measure. From Step I of Theorem 3.1 in [15] it follows that then there exists an $\varepsilon > 0$, a sequence of compact sets $(K_i)_{i\geq 1}$, and an increasing sequence of positive reals $(q_i)_{i\geq 1}$, $q_i \to \infty$, satisfying

$$P^*_{q_i}\delta_z(K_i) \geq \varepsilon \qquad \text{for } i \in \mathbb{N}$$

and

$$\min\{\rho(x,y) : x \in K_i, y \in K_j\} \geq \varepsilon \qquad \text{for } i \neq j,\ i,j \in \mathbb{N}.$$

We will show that for every open neighborhood U of z and every $i_0 \in \mathbb{N}$ there exists $y \in U$ and $i \geq i_0$, $i \in \mathbb{N}$, such that

$$P^*_{q_i}\delta_y\left(K_i^{\varepsilon/3}\right) < \varepsilon/2,$$

where $K_i^{\varepsilon/3} = \{y \in X : \inf_{v\in K_i}\rho(y,v) < \varepsilon/3\}$.

On the contrary, suppose that there exists an open neighbourhood U of z and $i_0 \in \mathbb{N}$ such that

$$\inf\left\{P^*_{q_i}\delta_y\left(K_i^{\varepsilon/3}\right) : y \in U, i \geq i_0\right\} \geq \varepsilon/2. \tag{2.10}$$

Clearly

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s > \alpha \tag{2.11}$$

for some $\alpha > 0$. Further, let $N \in \mathbb{N}$ satisfy $(N - i_0 + 1)\alpha\varepsilon > 2$. Choose $\gamma \in (0, \alpha\varepsilon/2)$ such that

$$(N - i_0 + 1)(\alpha\varepsilon - 2\gamma) > 2.$$

It easily follows that there exists $T_0 > 0$ such that for any $\mu \in \mathcal{M}_1$ and $T \geq T_0$ we have

$$\max_{i\leq N}\left\|\frac{1}{T}\int_0^T P_s^*\mu\,\mathrm{d}s - \frac{1}{T}\int_0^T P^*_{s+q_i}\mu\,\mathrm{d}s\right\|_{TV} < \gamma.$$

Choose $T \geq T_0$ and $\mu \in \mathcal{M}_1$ such that

$$\frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s \geq \alpha, \tag{2.12}$$

by (2.11). From (2.10) and the Markov property it follows that

$$P_{s+q_i}^*\mu\left(K_i^{\varepsilon/3}\right) = \int_X P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right)P_s^*(\mathrm{d}y) \geq \int_U P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right)P_s^*(\mathrm{d}y) \geq \frac{\varepsilon}{2}P_s^*\mu(U)$$

for $i \geq i_0$ and $s \geq 0$. Consequently, we have for $i_0 \leq i \leq N$

$$\begin{aligned}\frac{1}{T}\int_0^T P_s^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s &\geq \frac{1}{T}\int_0^T P_{s+q_i}^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s - \gamma \\ &\geq \frac{\varepsilon}{2}\,\frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s - \gamma \geq \frac{\varepsilon}{2}\alpha - \gamma,\end{aligned}$$

by (2.12). From this and the fact that $K_i^{\varepsilon/3} \cap K_j^{\varepsilon/3} = \emptyset$ for $i \neq j$ we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T P_s^*\mu\left(\bigcup_{i=i_0}^N K_i^{\varepsilon/3}\right)\mathrm{d}s &= \sum_{i=i_0}^N \frac{1}{T}\int_0^T P_s^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s \\ &\geq (N - i_0 + 1)(\varepsilon\alpha - 2\gamma)/2 > 1,\end{aligned}$$

which is impossible.

Now analogously as in the proof of Theorem 3.1 in [15], Step III, we define a sequence of Lipschitzian functions $(f_n)_{n\geq 1}$, a sequence of points $(y_n)_{n\geq 1}$, $y_n \to z$ as $n \to \infty$, two increasing sequences of integers $(i_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$, $i_n < k_n < i_{n+1}$ for $n \in \mathbb{N}$, and a sequence of reals $(p_n)_{n\geq 1}$ such that

$$f_n|_{K_{i_n}} = 1, \qquad 0 \leq f_n \leq \mathbf{1}_{K_{i_n}^{\varepsilon/3}}, \qquad \mathrm{Lip}\ f_n \leq 3/\varepsilon, \tag{2.13}$$

$$\left|P_{p_n}\left(\sum_{i=1}^n f_i\right)(z) - P_{p_n}\left(\sum_{i=1}^n f_i\right)(y_n)\right| > \frac{\varepsilon}{4}, \tag{2.14}$$

$$P_{p_n}^*\delta_u\left(\bigcup_{i=k_n}^\infty K_i^{\varepsilon/3}\right) < \frac{\varepsilon}{16} \quad \text{for } u \in \{z, y_n\} \tag{2.15}$$

for every $n \in \mathbb{N}$. From (2.13)-(2.15) it follows (see the proof of Theorem 3.1 in [15], Step III, once again) that

$$|P_{p_n}f(z) - P_{p_n}f(y_n)| > \frac{\varepsilon}{8}$$

for $n \in \mathbb{N}$ and $f := \sum_{n=1}^\infty f_n \in L_b(X)$. Since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that the family $\{P_t f : t \geq 0\}$ is equicontinuous in z. The proof is complete. □

Theorem 1. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then it is asymptotically stable.*

Proof. Fix $x \in X$. Since $(P_t)_{t\geq 0}$ is averagely bounded there is $R > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(x,R))\mathrm{d}s > \frac{1}{2}.$$

Let $(T_n)_{n\geq 1}$ be an increasing sequence of reals such that $T_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{T_n}\int_0^{T_n} P_s^*\delta_x(B(x,R))\mathrm{d}s > \frac{1}{2} \qquad \text{for } n \in \mathbb{N}.$$

Set $\mu_n = \frac{1}{T_n}\int_0^{T_n} P_s^*\delta_x \,\mathrm{d}s$, $n \in \mathbb{N}$, and observe that there are $\mu_n^R \in \mathcal{M}_1^{B(x,R)}$ such that

$$\mu_n \geq \frac{1}{2}\mu_n^R \qquad \text{for } n \in \mathbb{N}.$$

Indeed, we may define μ_n^R by the formula $\mu_n^R = \mu_n(\,\cdot \cap B(x,R))/\mu_n(B(x,R))$ for $n \in \mathbb{N}$. Further, observe that, by concentrating at z, for fixed $\varepsilon > 0$ there is $\alpha > 0$ such that we have

$$P_{s_n}^*\mu_n^R(B(z,\varepsilon)) \geq \alpha$$

for some $s_n > 0$, $n \in \mathbb{N}$. Hence

$$P_{s_n}^*\mu_n(B(z,\varepsilon)) \geq \frac{1}{2}\alpha \qquad \text{for } n \in \mathbb{N},$$

by linearity of $(P_t^*)_{t\geq 0}$. Consequently,

$$\frac{1}{T_n}\int_0^{T_n} P_s^*(P_{s_n}^*\delta_x)(B(z,\varepsilon))\mathrm{d}s \geq \frac{1}{2}\alpha \qquad \text{for } n \in \mathbb{N},$$

and condition (2.9) in Proposition 2 is satisfied. Now Proposition 2 implies the existence of an invariant measure. Further, from Proposition 1 it follows that for any $f \in L_b(X)$ and $\mu \in \mathcal{M}_1$

$$\langle \varphi, P_t^*\mu\rangle \to \langle \varphi, \mu_*\rangle$$

as t tends to $+\infty$. Application of the Alexandrov theorem finishes the proof (see [2]). $\square$

3. The models

3.1. GOY and Sabra shell models and functional setting. Let $u = (u_{-1}, u_0, u_1, \ldots)$ be an infinite sequence of complex valued functions on $[0,\infty)$ satisfying the following equations for $n = 1, 2, \ldots$

$$\mathrm{d}u_n(t) + \nu k_n^2 \nu_n(t)\mathrm{d}t + [B(u,u)]_n\mathrm{d}t = \sigma_n \mathrm{d}w_n \tag{3.1}$$

with the initial conditions

$$u_{-1}(t) = u_0(t) = 0 \quad \text{and} \quad u_n(0) = \xi_n.$$

Here $k_n = k_0 2^n$, $k_0 > 1$ and $\nu > 0$. Moreover $(w_n(t))_{n\geq 1}$ denotes a sequence of independent Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that $\sigma_n \in \mathbb{C}$ and there is $n_0 \in \mathbb{N}$ such that $\sigma_n = 0$ for $n \geq n_0$. Further B is a bilinear operator which will be defined later on.

Let H be the set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider H as a *real* Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$ of the form

$$(u,v) = \mathrm{Re} \sum_{n\geq 1} u_n v_n^*, \quad |u|^2 = \sum_{n\geq 1} |u_n|^2, \tag{3.2}$$

where v_n^* denotes the complex conjugate of v_n. The space H is separable. Let $A : \mathrm{D}(A) \subset H \to H$ be the non-bounded linear operator defined by

$$(Au)_n = k_n^2 u_n, \quad n = 1, 2, \ldots, \qquad \mathrm{D}(A) = \Big\{ u \in H \, : \, \sum_{n\geq 1} k_n^4 |u_n|^2 < \infty \Big\}.$$

The operator A is clearly self-adjoint, strictly positive definite since $(Au, u) \geq k_0^2 |u|^2$ for $u \in \mathrm{D}(A)$. For any $\alpha > 0$, set

$$\mathcal{H}_\alpha = \mathrm{D}(A^\alpha) = \{ u \in H \, : \, \sum_{n\geq 1} k_n^{4\alpha} |u_n|^2 < +\infty \}, \; \|u\|_\alpha^2 = \sum_{n\geq 1} k_n^{4\alpha} |u_n|^2 \text{ for } \; u \in \mathcal{H}_\alpha.$$

Obviously $\mathcal{H}_0 = H$. Define

$$V := \mathrm{D}(A^{\frac12}) = \Big\{ u \in H \, : \, \sum_{n\geq 1} k_n^2 |u_n|^2 < +\infty \Big\}$$

and set

$$\mathcal{H} = \mathcal{H}_{\frac14}, \; \|u\|_{\mathcal{H}} = \|u\|_{\frac14}.$$

Then V is a Hilbert space for the scalar product $(u,v)_V = \mathrm{Re}(\sum_n k_n^2 \, u_n \, v_n^*)$, $u, v \in V$ and the associated norm is denoted by

$$\|u\|^2 = \sum_{n\geq 1} k_n^2 \, |u_n|^2.$$

The adjoint of V with respect to the H scalar product is $V' = \{ (u_n) \in \mathbb{C}^{\mathbb{N}} \, : \, \sum_{n\geq 1} k_n^{-2} \, |u_n|^2 < +\infty \}$ and $V \subset H \subset V'$ is a Gelfand triple. Let $\langle u \, , \, v \rangle_{V',V} = \mathrm{Re}\left(\sum_{n\geq 1} u_n \, v_n^* \right)$ denote the duality between $u \in V'$ and $v \in V$.

Set $u_{-1} = u_0 = 0$, let a, b be real numbers and let $B : H \times V \to H$ (or $B : V \times H \to H$) denote the bilinear operator defined by

$$[B(u,v)]_n = i \left(a k_{n+1} u_{n+1}^* v_{n+2}^* + b k_n u_{n-1}^* v_{n+1}^* - a k_{n-1} u_{n-1}^* v_{n-2}^* - b k_{n-1} u_{n-2}^* v_{n-1}^* \right)$$

for $n = 1, 2, \ldots$ in the GOY shell model (see, e.g. [22]) or

$$[B(u,v)]_n = i \left(a k_{n+1} u_{n+1}^* \, v_{n+2} + b k_n u_{n-1}^* v_{n+1} + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} u_{n-2} v_{n-1} \right),$$

in the Sabra shell model introduced in [18].

Obviously, there exists $C > 0$ such that

$$|B(u,v)| \leq C \|u\| |v| \qquad \text{for } u \in V \text{ and } v \in H. \tag{3.3}$$

Note that B can be extended as a bilinear operator from $H \times H$ to V' and that there exists a constant $C > 0$ such that given $u, v \in H$ and $w \in V$ we have

$$|\langle B(u,v) \, , \, w \rangle_{V',V}| + |\big(B(u,w) \, , \, v \big)| + |\big(B(w,u) \, , \, v \big)| \leq C \, |u| \, |v| \, \|w\|. \tag{3.4}$$

An easy computation proves that for $u, v \in H$ and $w \in V$ (resp. $v, w \in H$ and $u \in V$),

$$\langle B(u,v)\,,\,w\rangle_{V',V} = -\big(B(u,w)\,,\,v\big) \text{ (resp. } \big(B(u,v)\,,\,w\big) = -\big(B(u,w)\,,\,v\big) \text{).}$$

Hence $(B(v,u),u) = 0$ for $u \in H$ and $v \in V$. Furthermore, $B : V \times V \to V$ and $B : \mathcal{H} \times \mathcal{H} \to H$; indeed, for $u, v \in V$ (resp. $u, v \in \mathcal{H}$) we have

$$\|B(u,v)\|^2 = \sum_{n\geq 1} k_n^2\,|B(u,v)_n|^2 \leq C\,\|u\|^2 \sup_n k_n^2 |v_n|^2 \leq C\,\|u\|^2\,\|v\|^2,$$
$$|B(u,v)| \leq C\,\|u\|_{\mathcal{H}}\,\|v\|_{\mathcal{H}}.$$

3.2. **Well-posedness.** Consider the abstract equation on H of the form

$$\mathrm{d}u(t) = [-\nu A u(t) + B(u(t),u(t))]\,\mathrm{d}t + Q\mathrm{d}W(t), \quad t \geq 0 \tag{3.5}$$

with the initial condition $u(0) = \xi \in H$, where $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ is some matrix with $\mathrm{Tr}(QQ^*) < \infty$ and $W(t) = (w_n(t))_{n\geq 1}$ is a cylindrical Wiener noise on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Definition 3.1. *A stochastic process $u(t,\omega)$ is a generalized solution in $[0,T]$ of the system* (3.5) *if*

$$u(\cdot,\omega) \in C([0,T];H) \cap L^2(0,T;\mathcal{H})$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, u is progressively measurable in these topologies and equation (3.5) *is satisfied in the integral sense*

$$(u(t),\varphi) + \int_0^t \nu(u(s),A\varphi)\mathrm{d}s + \int_0^t \left(B\left(u(s),\varphi\right), u(s)\right)\mathrm{d}s$$
$$= (\xi,\varphi) + (QW(t),\varphi)$$

for all $t \in [0,T]$ and $\varphi \in D(A)$.

Theorem 2. *Let us assume that the initial condition ξ is an $\mathcal{F}_0$-random variable with values in H. Then there exists a unique solution $(u(t))_{t\geq 0}$ to equation* (3.5)*. Moreover, if $\mathbb{E}|\xi|^2 < +\infty$, then*

$$\mathbb{E}|u(t)|^2 + \int_0^t 2\nu\mathbb{E}\|u(s)\|^2\mathrm{d}s = \mathbb{E}|\xi|^2 + \mathrm{Tr}(QQ^*)t \tag{3.6}$$

for any $t \geq 0$.

Proof. We will prove well–posedness using a pathwise argument (for similar results see [3] and the references therein). Let us introduce the Ornstein-Uhlenbeck process solution of

$$\begin{cases} \mathrm{d}z(t) + \nu A z(t)\mathrm{d}t = Q\mathrm{d}W, \\ z(0) = 0. \end{cases} \tag{3.7}$$

The above equation has a unique progressively measurable solution such that $\mathbb{P}$-a.s.

$$z \in C([0,T];\mathcal{H})$$

(for more details see [6]). Set $v = u - z$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v(t) + \nu A v(t) - B(v(t)+z(t), v(t)+z(t)) = 0, \\ v(0) = \xi, \end{cases} \tag{3.8}$$

is a deterministic system. The existence and uniqueness of global weak solutions v follow from the Galerkin approximation procedure and then passing to the limit using the appropriate compactness theorems. We omit the details which can be found in [3] and the references therein. Instead, we present the formal computations which lead to the basic a priori estimates, this is in order to stress the role played by z. Using equation (3.8) and various properties of the nonlinear operator B, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v(t)|^2 + \nu\|v(t)\|^2 &\leq |(B(v(t)+z(t), z(t)), v(t))| \\ &\leq C\|v(t)\||v(t)+z(t)||z(t)| \\ &\leq \frac{\nu}{2}\|v(t)\|^2 + C(\nu)\left(|v(t)|^2|z(t)|^2 + |z(t)|^4\right). \end{aligned}$$

Using Gronwall's Lemma and the fact that $\|z\|_{C([0,T];\mathcal{H})} \leq C(\omega)$, we have

$$\sup_{0\leq t\leq T} |v(t)|^2 \leq C(|\xi|, T, C(\omega)).$$

Again, using the above inequality in the previous estimate, we obtain that

$$\int_0^T \|v(s)\|^2 \mathrm{d}s \leq C(|\xi|, T, C(\omega)).$$

Then, by classical arguments, see [23], $v \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/2}))$. Therefore $u = v + z \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/4}))$ $\mathbb{P}$-a.s.

To finish the proof observe that condition (3.6) follows from Itô's formula. □

The uniqueness of solutions is established in the following theorem.

Theorem 3. *Let $\left(u^{(1)}(t)\right)_{t\geq 0}$, $\left(u^{(2)}(t)\right)_{t\geq 0}$, be two continuous adapted solutions of (3.5) in H, with the initial conditions $u_0^{(1)}$ and $u_0^{(2)}$ as above. Then there is a constant $C(\nu) > 0$, depending only on ν, such that $\mathbb{P}$-a.s.*

$$\left|u^1(t) - u^2(t)\right|^2 \leq \mathrm{e}^{C(\nu)\int_0^t \left|u^1(s)\right|^2 ds} \left|u_0^1 - u_0^2\right|^2 \quad t \geq 0.$$

Proof. Let us put $u(t) = u^1(t) - u^2(t)$. Then u is the solution of the following equation

$$\mathrm{d}u + \nu A u \mathrm{d}t - \left(B(u^1, u^1) - B(u^2, u^2)\right)\mathrm{d}t = 0.$$

Using again the properties of operator B, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 &\leq |(B(u, u^1), u)| \\ &\leq \frac{\nu}{2}\|u\|^2 + C(\nu)|u|^2|u^1|^2. \end{aligned}$$

Hence, by the Gronwall lemma, we obtain that

$$|u(t)|^2 \le |u(0)|^2 \mathrm{e}^{C(\nu)\left(\int_0^T |u^1(s)|^2 ds\right)},$$

which finishes the proof. □

4. Stability of the model

Let a diagonal matrix $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ such that there is $n_0 \in \mathbb{N}$ and $q_{n,n} = 0$ for $n \ge n_0$ be given. For any $\varepsilon \ge 0$ consider the equation on H of the form

$$\mathrm{d}u(t) = [-\nu A u(t) + B(u(t), u(t))]\mathrm{d}t + Q_\varepsilon \mathrm{d}W(t) \quad t \ge 0, \tag{4.1}$$

where $(W(t))_{t\ge 0}$ is a certain cylindrical Wiener process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$. Further, $Q_\varepsilon = (\tilde{q}_{i,j})_{i,j\in\mathbb{N}}$ is the diagonal matrix such that $\tilde{q}_{i,i} = q_{i,i}$ if $q_{i,i} \ne 0$ and $\tilde{q}_{n,n} = (\varepsilon 2^{-n})^{1/2}$ otherwise. Observe that problem (3.1) is equivalent to problem (4.1) with $\varepsilon = 0$.

Fix an $\varepsilon > 0$. By Theorem 2 for every $x \in H$ there is a unique continuous solution $(u^x_\varepsilon(t))_{t\ge 0}$ in H, hence the transition semigroup is well defined. From Theorem 3 we obtain that the solution satisfies the Feller property, i.e. for any $t \ge 0$ if $x_n \to x$ in H, then $\mathbb{E} f(u^{x_n}_\varepsilon(t)) \to \mathbb{E} f(u^x_\varepsilon(t))$ for any $f \in C_b(H)$. Set

$$P_{\varepsilon,t} f(x) = \mathbb{E} f(u^x_\varepsilon(t)) \quad \text{for any } f \in C_b(H).$$

Obviously $(P_{\varepsilon,t})_{t\ge 0}$ is stochastically continuous. First note that $DP_{\varepsilon,t}f(x)[v]$, the value of the Frechet derivative $DP_{\varepsilon,t}f(x)$ at $v \in H$, is equal to $\mathbb{E}\left\{Df(u^x_\varepsilon(t))[U(t)]\right\}$, where $U(t) := \partial u^x_\varepsilon(t)[v]$ and

$$\partial u^x_\varepsilon(t)[v] := \lim_{\eta\downarrow 0} \frac{1}{\eta}\left(u^{x+\eta v}_\varepsilon(t) - u^x_\varepsilon(t)\right)$$

and the limit is in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ (see [13] also [12]). The process $U = (U(t))_{t\ge 0}$ satisfies the linear evolution equation

$$\begin{aligned} \frac{\mathrm{d}U(t)}{\mathrm{d}t} &= -\nu A U(t) + B(u^x_\varepsilon(t), U(t)) + B(U(t), u^x_\varepsilon(t)), \\ U(0) &= v. \end{aligned} \tag{4.2}$$

Suppose that $\mathcal{X}$ is a certain Hilbert space and $\Phi \colon H \to \mathcal{X}$ a Borel measurable function. Given an $(\mathcal{F}_t)_{t\ge 0}$-adapted process $g \colon [0,\infty)\times\Omega \to H$ satisfying $\mathbb{E}\int_0^t \|g(s)\|^2 \mathrm{d}s < \infty$ for each $t \ge 0$ we denote by $\mathcal{D}_g \Phi(u^x_\varepsilon(t))$ the Malliavin derivative of $\Phi(u^x_\varepsilon(t))$ in the direction of g; that is the $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{X})$-limit, if exists, of

$$\mathcal{D}_g \Phi(u^x_\varepsilon(t)) := \lim_{\eta\downarrow 0} \frac{1}{\eta}\left[\Phi(u^x_{\varepsilon,\eta g}(t)) - \Phi(u^x_\varepsilon(t))\right],$$

where $u^x_{\varepsilon,g}(t)$, $t \ge 0$, solves the equation

$$\mathrm{d}u^x_{\varepsilon,g}(t) = \left[-\nu A u^x_{\varepsilon,g}(t) + B(u^x_{\varepsilon,g}(t), u^x_{\varepsilon,g}(t))\right]\mathrm{d}t + Q_\varepsilon\left(\mathrm{d}W(t) + g(t)\mathrm{d}t\right), \qquad u^x_{\varepsilon,g}(0) = x.$$

In particular, one can easily show that when $\mathcal{X} = H$ and $\Phi = I$, where I is the identity operator, the Malliavin derivative of $u_\varepsilon^x(t)$ exists and the process $D(t) := \mathcal{D}_g u_\varepsilon^x(t)$, $t \geq 0$, solves the linear equation

$$\frac{\mathrm{d}D}{\mathrm{d}t}(t) = -\nu A D(t) + B(u_\varepsilon^x(t), D(t)) + B(D(t), u_\varepsilon^x(t)) + Q_\varepsilon g(t), \qquad D(0) = 0. \tag{4.3}$$

Directly from the definition of the Malliavin derivative we conclude the *chain rule:* suppose that $\Phi \in C_b^1(H; \mathcal{X})$ then

$$\mathcal{D}_g \Phi(u_\varepsilon^x(t)) = D\Phi(u_\varepsilon^x(t))[D(t)].$$

(Here $C_b^1(H;\mathcal{X})$ denotes the space of all bounded continuous functions $\Phi : H \to \mathcal{X}$ with continuous and bounded first derivative with the natural norm. In the case when $\mathcal{X} = \mathbb{R}$ we simply write $C_b^1(H)$.) In addition, the *integration by parts formula* holds, see Lemma 1.2.1, p. 25 of [19]. Indeed, suppose that $\Phi \in C_b^1(H)$. Then

$$\mathbb{E}[\mathcal{D}_g \Phi(u_\varepsilon^x(t))] = \mathbb{E}\left[\Phi(u_\varepsilon^x(t)) \int_0^t (g(s), Q_\varepsilon \mathrm{d}W(s))\right]. \tag{4.4}$$

The crucial role in our consideration is played by the following lemma.

Lemma 1. *Let $(P_{\varepsilon,t})_{t\geq 0}$ for $\varepsilon \geq 0$ correspond to problem (4.1). For any $f \in C_b^1(H)$ and $R > 0$ there exists a constant $C_0 > 0$ such that*

$$\sup_{t\geq 0} \sup_{|x|\leq R} \sup_{|v|\leq 1} |DP_{\varepsilon,t} f(x)[v]| \leq C_0 \|f\|_{C_b^1(H)} \qquad \text{for all } \varepsilon \in (0,1).$$

In particular, the semigroup $(P_{\varepsilon,t})_{t\geq 0}$ for $\varepsilon > 0$ satisfies the e–property.

Proof. Fix $\varepsilon > 0$, $v \in H$ with $|v| \leq 1$ and $f \in C_b^1(H)$. Let u_ε^x be the solution to problem (4.1). Let $\xi_n := \xi_n(v,x)(t)$, $n \in \mathbb{N}$, be the solution of the problem

$$\frac{\mathrm{d}\xi_n}{\mathrm{d}t} = -\nu A \xi_n + B(u_\varepsilon^x, \xi_n) \tag{4.5}$$

with the initial condition $\xi_n(0) = v$. Further, let $f_n(t) = B(\xi_n, \Pi_n u_\varepsilon^x)$, where Π_n denotes the projection $\mathbb{C}^{\mathbb{N}}$ on the subspace $H_n = \{u \in H : u_k = 0 \, \text{for}\, k \geq n\}$. Set $g_n := (Q_\varepsilon)^{-1} f_n$ and observe that $g_n : [0,\infty) \times \Omega \to H$.

Let $\omega_t^n(x) := \mathcal{D}_{g_n} u_\varepsilon^x(t)$ and $\rho_t^n(v,x) := \partial u_\varepsilon^x(t)[v] - \mathcal{D}_{g_n} u_\varepsilon^x(t)$. Then,

$$\begin{aligned} DP_{\varepsilon,t} f(x)[v] &= \mathbb{E}\{Df(u_\varepsilon^x(t))[\omega_t^n(x)]\} + \mathbb{E}\{Df(u_\varepsilon^x(t))[\rho_t^n(v,x)]\} \\ &= \mathbb{E}\{\mathcal{D}_{g_n} f(u_\varepsilon^x(t))\} + \mathbb{E}\{Df(u_\varepsilon^x(t))[\rho_t^n(v,x)]\} \\ &\overset{(4.4)}{=} \mathbb{E}\left\{f(u_\varepsilon^x(t)) \int_0^t (g_n(s), Q_\varepsilon \mathrm{d}W(s))\right\} + \mathbb{E}\{Df(u_\varepsilon^x(t))[\rho_t^n(v,x)]\} \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

We have

$$\left|\mathbb{E}\left\{f(u_\varepsilon^x(t)) \int_0^t (g_n(s), Q_\varepsilon \mathrm{d}W(s))\right\}\right| \leq \|f\|_{L^\infty} \left(\mathbb{E}\int_0^t |Q_\varepsilon g_n(s)|^2 \mathrm{d}s\right)^{1/2}$$

and

$$|\mathbb{E}\,\{Df(u_\varepsilon^x(t))[\rho_t^n(v,x)]\}| \le \|f\|_{C_b^1(H)}\mathbb{E}\,|\rho_t^n(v,x)| \le \|f\|_{C_b^1(H)}(\mathbb{E}\,|\rho_t^n(v,x)|^2)^{1/2}.$$

Adding $f_n(t)$ to both sides of (4.5) we obtain

$$\begin{aligned}\frac{\mathrm{d}\xi_n(t)}{\mathrm{d}t} + f_n(t) &= -\nu A\xi_n(t) + B(u_\varepsilon^x(t),\xi_n(t)) \\ &\quad + B(\xi_n(t),\Pi_n u_\varepsilon^x(t)), \\ \xi_n(0) &= v.\end{aligned}$$

On the other hand, $\partial u_\varepsilon^x(t)[v]$ and $\mathcal{D}_{g_n} u_\varepsilon^x(t)$ obey equations (4.2) and (4.3), respectively. Hence $\rho_t^n := \rho_t^n(v,x)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}\rho_t^n}{\mathrm{d}t} &= -\nu A\rho_t^n + B(u_\varepsilon^x(t),\rho_t^n) + B(\rho_t^n,u_\varepsilon^x(t)) - Q_\varepsilon g_n(t), \\ \rho_0 &= v.\end{aligned}$$

Since, $f_n(t) = Q_\varepsilon g_n(t)$, we conclude that $\theta_n(t) = \rho_t^n(v,x) - \xi_n(v,x)(t)$ solves the equation

$$\begin{aligned}\frac{\mathrm{d}\theta_n(t)}{\mathrm{d}t} &= -\nu A\theta_n(t) + B(u_\varepsilon^x(t),\theta_n(t)) + B(\theta_n(t),u_\varepsilon^x(t)) - B(\xi_n(t),\Pi_n u_\varepsilon^x(t) - u_\varepsilon^x(t)) \\ \theta_n(0) &= 0.\end{aligned}$$

From (4.5) it follows that

$$\frac{1}{2}\frac{\mathrm{d}|\xi_n(v,x)(t)|^2}{\mathrm{d}t} \le -\nu k_0|\xi_n(v,x)(t)|^2, \tag{4.6}$$

which gives that θ_n satisfies the inequality

$$|\theta_n(t)|^2 \le C\left(\int_0^t |\theta_n(s)|^2\|u_\varepsilon^x(s)\|\mathrm{d}s + \int_0^t \|\Pi_n(u_\varepsilon^x(s)) - u_\varepsilon^x(s)\|^2\mathrm{d}s\right) \tag{4.7}$$

for some $C > 0$, by (3.4). From this it follows that $\mathbb{E}|\theta_n(t)|^2 \to 0$ as $n \to \infty$ (see the Appendix). Choose $m := m(\nu,x,t) \in \mathbb{N}$ such that $\mathbb{E}|\theta_m(t)|^2 \le |v|^2\exp(-2\nu k_0 t)/2$. Then $\mathbb{E}|\rho_t^m(v,x)|^2 \le 2\mathbb{E}|\xi_m(v,x)(t)|^2 + 2\mathbb{E}|\theta_m(t)|^2 \le 2\mathbb{E}|\xi_m(v,x)(t)|^2 + |v|^2\exp(-2\nu k_0 t) \le 3|v|^2\exp(-2\nu k_0 t) \le 3|v|^2 < \infty$, by (4.6).

We are in a position to show that

$$\mathbb{E}\int_0^t |Q_\varepsilon g_m(s)|^2\mathrm{d}s \le D(|x|)|v|^2,$$

where $D : [0,+\infty) \to [0,+\infty)$ is the increasing function given by the formula

$$D(p) = 2C^2k_0\int_0^\infty \exp(-2\nu k_0 s)(p^2 + (\operatorname{Tr} Q^2 + 1)s)\mathrm{d}s$$

with $C > 0$ given by (3.3). Indeed, we have

$$\begin{aligned}
&\mathbb{E}\int_0^t |Q_\varepsilon g_m(s)|^2 \mathrm{d}s = \mathbb{E}\int_0^t |f_m(s)|^2 \mathrm{d}s = \mathbb{E}\int_0^t |B(\xi_m(v,x)(s), \Pi_m u_\varepsilon^x(s))|^2 \mathrm{d}s \\
&\le C^2 \mathbb{E}\int_0^t |\xi_m(v,x)(s)|^2 \|u_\varepsilon^x(s)\|^2 \mathrm{d}s \le C^2 \int_0^\infty \mathbb{E}(|\xi_m(v,x)(s)|^2 \|u_\varepsilon^x(s)\|^2) \mathrm{d}s \\
&\le 2C^2 k_0 |v|^2 \int_0^\infty \exp(-2\nu k_0 s) \left(\int_0^s \nu \mathbb{E}\, \|u_\varepsilon^x(r)\|^2 \mathrm{d}r \right) \mathrm{d}s \\
&\le 2C^2 k_0 |v|^2 \int_0^\infty \exp(-2\nu k_0 s)(|x|^2 + (\operatorname{Tr} Q^2 + \varepsilon)s) \mathrm{d}s \\
&\le 2C^2 k_0 |v|^2 \int_0^\infty \exp(-2\nu k_0 s)(|x|^2 + (\operatorname{Tr} Q^2 + 1)s) \mathrm{d}s = D(|x|)|v|^2,
\end{aligned}$$

by (3.6). Putting $C_0 = \sqrt{3} + \sqrt{D(R)}$ completes the proof. □

As a simple consequence of the above lemma we obtain the following theorem.

Theorem 4. *Let $(P_t)_{t\ge 0}$ correspond to problem (3.5). Then $(P_t)_{t\ge 0}$ satisfies the e–property.*

Proof. For any $R > 0$ and $f \in C_b^1(H)$ there exists a general constant $C_0 > 0$ such that for any $\varepsilon \in (0,1)$ and $x, y \in B(0,R)$ we have

$$|\mathbb{E}\, f(u_\varepsilon^x(t)) - \mathbb{E}\, f(u_\varepsilon^y(t))| \le C_0 |x - y| \qquad \text{for any } t \ge 0.$$

On the other hand, by a standard argument (Gronwall's lemma) we may show that for any $t > 0$ and $w \in H$ we have $\sup_{s\in[0,t]} |\mathbb{E}\, f(u_\varepsilon^w(s)) - \mathbb{E}\, f(u^w(s))| \to 0$ as $\varepsilon \to 0$, where u^w denotes the solution of (3.5) with $u(0) = w$ and $\varepsilon = 0$. Therefore, we obtain

$$|\mathbb{E}\, f(u^x(t)) - \mathbb{E}\, f(u^y(t))| \le C_0 |x - y| \qquad \text{for any } t \ge 0$$

and the proof is complete. □

Lemma 2. *(Average boundedness) Let $(P_t)_{t\ge 0}$ correspond to problem (3.5). Then $(P_t)_{t\ge 0}$ is averagely bounded.*

Proof. Fix an $\varepsilon > 0$ and let $r > 0$ be given. If $x \in B(0,r)$, then

$$\begin{aligned}
&\frac{1}{T}\int_0^T P_s^*\delta_x(H \setminus B(0,R)) \mathrm{d}s = \frac{1}{T}\int_0^T \mathbb{P}(|u^x(s)| > R) \mathrm{d}s \le \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\| > R) \mathrm{d}s \\
&= \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\|^2 > R^2) \mathrm{d}s \le \frac{1}{T}\int_0^T \frac{\mathbb{E}\|u^x(s)\|^2}{R^2} \mathrm{d}s \\
&= \frac{1}{\nu R^2}\frac{1}{T}\int_0^T \nu \mathbb{E}\|u^x(s)\|^2 \mathrm{d}s \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + |x|^2/T) \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + r^2/T)
\end{aligned}$$

for arbitrary $R > 0$, by (3.6). Hence there is $R_0 > 0$ such that

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0)) \mathrm{d}s > 1 - \varepsilon.$$

On the other hand, by Fatou's lemma we have

$$\begin{aligned}\liminf_{T\to+\infty}\frac{1}{T}\int_0^T P_s^*\mu(B(0,R_0))\mathrm{d}s &\geq \int_H\left(\liminf_{T\to+\infty}\frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0))\mathrm{d}s\right)\mu(\mathrm{d}x)\\ &\geq \int_H (1-\varepsilon)\mu(\mathrm{d}x) = 1-\varepsilon\end{aligned}$$

for any $\mu \in \mathcal{M}_1^{B(0,r)}$. The proof is complete. □

Lemma 3. *(Concentrating at* 0*) Let* $(P_t)_{t\geq 0}$ *correspond to problem (3.5). Then* $(P_t)_{t\geq 0}$ *is concentrating at* 0*.*

Proof. Consider first the deterministic equation

$$\mathrm{d}v^x(t) = [-\nu A v^x(t) + B(v^x(t), v^x(t))]\mathrm{d}t$$

with the initial condition $v^x(0) = x$. Then

$$\frac{1}{2}\frac{\mathrm{d}|v^x(t)|^2}{\mathrm{d}t} \leq -\nu k_0 |v^x(t)|^2$$

and consequently

$$|v^x(t)|^2 \to 0 \qquad \text{as } t \to +\infty$$

uniformly on bounded sets. Further, fix $\varepsilon > 0$ and $r > 0$. Let $t_0 > 0$ be such that $v^x(t_0) \in B(0, \varepsilon/2)$ for all $x \in B(0,r)$. We may show (see Theorem 8 in [3]) that the process corresponding to the considered model is stochastically stable (see also [13]), i.e. there exists $\eta > 0$ and the set $F_\eta = \{\omega \in \Omega : \sup_{0\leq t\leq t_0} |QW(t)(\omega)| \leq \eta\}$ such that

$$|u^x(t_0)(\omega) - v^x(t_0)| \leq \varepsilon/2 \qquad \text{for any } \omega \in F_\eta.$$

Since the process is degenerate, we have $\alpha := \mathbb{P}(F_\eta) > 0$. Consequently, we obtain

$$P_{t_0}^*\delta_x(B(0,\varepsilon)) \geq \mathbb{P}(\{\omega \in \Omega : u^x(t_0)(\omega) \in B(0,\varepsilon)\}) \geq \mathbb{P}(F_\eta) = \alpha$$

for arbitrary $x \in B(0,r)$. Since

$$P_{t_0}^*\mu(B(0,\varepsilon)) = \int_H P_{t_0}^*\delta_x(B(0,\varepsilon))\mu(\mathrm{d}x),$$

we obtain $P_{t_0}^*\mu(B(0,\varepsilon)) \geq \alpha$ for any $\mu \in \mathcal{M}_1^{B(0,r)}$. But $\varepsilon > 0$ and $r > 0$ were arbitrary and hence the concentrating property follows. □

We may formulate the main theorem of this part of our paper.

Theorem 5. *The semigroup* $(P_t)_{t\geq 0}$ *corresponding to problem (3.5) is asymptotically stable. In particular, it admits a unique invariant measure.*

Proof. From Theorem 4 it follows that the semigroup $(P_t)_{t\geq 0}$ satisfies the e–property. It is also averagely bounded and concentrating at 0, by Lemmas 2 and 3. Application of Theorem 1 finishes the proof. □

5. Appendix

At the beginning we show that there is a constant $\kappa > 0$ such that

$$\mathbb{E}(\exp(\kappa \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s)) < +\infty, \tag{5.1}$$

where is u_ε^x is the unique solution to equation (4.1) with the initial condition $u_\varepsilon^x(0) = x \in H$. Indeed, from Itô's formula, for any $\eta > 0$, we obtain

$$\begin{aligned} &\eta |u_\varepsilon^x(t)|^2 + \eta\nu \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s - \eta(\operatorname{Tr} Q_\varepsilon^2)t - \eta|x|^2 \\ &= \eta \int_0^t (u_\varepsilon^x(s), Q_\varepsilon \mathrm{d}W(s)) - \eta\nu \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s. \end{aligned}$$

Let $M(t) = \eta \int_0^t (u_\varepsilon^x(s), Q_\varepsilon \mathrm{d}W(s))$ and let $N(t) = M(t) - \eta\nu \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s$. Choose $\alpha > 0$ such that $\nu\|u_\varepsilon^x(s)\|^2 > (\alpha/2)|Q_\varepsilon u_\varepsilon^x(s)|^2$. Now observe that $N(t) \le M(t) - (\alpha/2\eta)\langle M\rangle(t)$, where $\langle M\rangle(t)$ denotes the quadratic variation of the continous L^2–martingale M with the filtration generated by the noise. Hence by a standard variation of the Kolmogorov–Doob martingale inequality (see [20]) we have

$$\mathbb{P}(N(t) \ge K) \le \exp(-\alpha K/\eta)$$

and consequently, for $\eta \in (0, \alpha/2)$, we obtain

$$\mathbb{P}(\exp N(t) \ge \exp K) \le \exp(-\alpha K/\eta) \le \exp(-2K)$$

for any $K > 0$. An easy observation that if some positive random variable, say X, satisfies the condition $\mathbb{P}(X \ge C) \le C^{-2}$ for every $C > 0$, then $\mathbb{E}X \le 2$ gives

$$\mathbb{E}(\exp(\eta|u_\varepsilon^x(t)|^2 + \eta\nu \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s - \eta(\operatorname{Tr} Q_\varepsilon^2)t - \eta|x|^2)) \le 2,$$

which, in turn, gives (5.1) with $\kappa = \eta\nu$.

Now we are in a position to show that $\mathbb{E}|\theta_n(t)|^2 \to 0$ as $n \to \infty$. From (4.7) and Gronwall's lemma we have

$$\mathbb{E}|\theta_n(t)|^2 \le C \left(\mathbb{E} \left(\int_0^t \|\Pi_n(u_\varepsilon^x(s)) - u_\varepsilon^x(s)\|^2 \mathrm{d}s \right)^2 \right)^{1/2} \left(\mathbb{E} \exp \left(2 \int_0^t \|u_\varepsilon^x(s)\| \mathrm{d}s \right) \right)^{1/2}.$$

From (5.1) it follows that the second term on the right hand side is finite. On the other hand, we have

$$\begin{aligned} \left(\int_0^t \|\Pi_n(u_\varepsilon^x(s)) - u_\varepsilon^x(s)\|^2 \mathrm{d}s \right)^2 &\le (16/\kappa^2) \left(\kappa \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s \right)^2 \\ &\le (32/\kappa^2) \exp \left(\kappa \int_0^t \|u_\varepsilon^x(s)\|^2 \mathrm{d}s \right). \end{aligned}$$

Condition (5.1) allows us to apply the Lebesgue dominated convergence theorem and we obtain

$$\mathbb{E}\left(\int_0^t \|\Pi_n(u_\varepsilon^x(s)) - u_\varepsilon^x(s)\|^2 \mathrm{d}s\right)^2 \to 0 \qquad \text{as } n \to \infty,$$

by the fact that $\|\Pi_n(u_\varepsilon^x(s)) - u_\varepsilon^x(s)\|^2 \to 0$ as $n \to \infty$. Consequently, we have $\mathbb{E}|\theta_n(t)|^2 \to 0$ as $n \to \infty$. □

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Department of Mathematics, University of Wyoming, Laramie, Wyoming, U.S.A.
E-mail address: Bessaih@uwyo.edu

Department of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
E-mail address: rkapica@math.us.edu.pl

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: szarek@intertele.pl