Some Inequalities for Nilpotent Multipliers of Powerful p-Groups

Behrooz Mashayekhy*and Fahimeh Mohammadzadeh

Department of Mathematics,

Center of Excellence in Analysis on Algebraic Stuctures,

Ferdowsi University of Mashhad,

P. O. Box 1159-91775, Mashhad, Iran.

Abstract

In this paper we present some inequalities for the order, the exponent, and the number of generators of the c-nilpotent multiplier (the Baer invariant with respect to the variety of nilpotent groups of class at most $c \geq 1$) of a powerful p-group. Our results extend some of Lubotzky and Mann's (Journal of Algebra, 105 (1987), 484-505.) to nilpotent multipliers. Also, we give some explicit examples showing the tightness of our results and improvement some of the previous inequalities.

A.M.S. Classification 2000: 20C25, 20D15, 20E10, 20F12.

Keywords: Nilpotent multiplier, Powerful p-group.

---

*Corresponding author: mashaf@math.um.ac.ir

1. Introduction and Motivation

Let G be a group with a free presentation F/R. The abelian group

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, {}_cF]}$$

is said to be the c-nilpotent multiplier of G (the Baer invariant of G, after R. Baer [1], with respect to the variety of nilpotent groups of class at most $c \geq 1$). The group $M(G) = M^{(1)}(G)$ is more known as the Schur multiplier of G. When G is finite, $M(G)$ is isomorphic to the second cohomology group $H^2(G, \mathbf{C}^*)$ [8].

It was conjectured for some time that the exponent of the Schur multiplier of a finite p-group is a divisor of the exponent of the group itself. I. D. Macdonald, J. W. Wamsley, and others [2] have constructed an example of a group of exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. In 2007 P. Moravec [15] proved that if G is a group of exponent 4, then $\exp(M(G))$ divides 8. In 1973 Jones [7] proved that the exponent of the Schur multiplier of a finite p-group of class $c \geq 2$ and exponent p^e is at most $p^{e(c-1)}$. A result of G. Ellis [4] shows that if G is a p-group of class $k \geq 2$ and exponent p^e, then $\exp(M^{(c)}(G)) \leq p^{e\lceil k/2 \rceil}$, where $\lceil k/2 \rceil$ denotes the smallest integer n such that $n \geq k/2$. For $c = 1$ P. Moravec [15] showed that $\lceil k/2 \rceil$ can be replaced by $2\lfloor \log_2 k \rfloor$ which is an improvement if $k \geq 11$. Also he proved that if G is a metabelian group of exponent p, then $\exp(M(G))$ divides p. S. Kayvanfar and M.A. Sanati [9] proved that $\exp(M(G)) \leq \exp(G)$ when G is a finite p-group of class 3, 4 or 5 under some arithmetical conditions on p and the exponent of G. On the other hand, the authors in a joint paper [13] proved that if G is a finite p-group of class k with $p > k$, then $\exp(M^{(c)}(G)) | \exp(G)$. In 1972 Jones [6] showed that the order of the Schur multiplier of a finite p-group of order p^n with center of exponent p^k is bounded by $p^{(n-k)(n+k-1)/2}$. In particular,

$|G'||M(G)| \leq p^{\frac{n(n-1)}{2}}$. In 1973 Jones [7] gave a bound for the number of generators of the Schur multiplier of a finite p-group of class c and special rank r. Recently the authors in a joint paper [13] have extended this result to the c-nilpotent multipliers. In 1987 Lubotzky and Mann [10] presented some inequalities for the Schur multiplier of a powerful p-group. They gave a bound for the order, the exponent and the number of generators of the Schur multiplier of a powerful p-group. Their results improve the previous inequalities for powerful p-groups. In this paper we will extend some results of Lubotzky and Mann [10] to the nilpotent multipliers and give some upper bounds for the order, the exponent and the number of generators of the c-nilpotent multiplier of a d-generator powerful p-group G as follows:

$$d(M^{(c)}(G)) \leq \chi_{c+1}(d),\ \exp(M^{(c)}(G)) | \exp(G),$$

$$\text{and } |M^{(c)}(G)| \leq p^{\chi_{c+1}(d)} \exp(G),$$

where $\chi_{c+1}(d)$ is the number of basic commutators of weight $c+1$ on d letters [5]. Our method is similar to that of [10]. Finally, by giving some examples of groups and computing the number of generators, the order and the exponent of their c-nilpotent multipliers explicitly, we compare these numbers with the bounds obtained and show that our results improve some of the previously mentioned inequalities.

2. Notation and Preliminaries

Here we will give some definitions and theorems that will be used in our work. Throughout this paper $\mho_i(G)$ denotes the subgroup of G generated by all p^ith powers, $P_i(G)$ is defined by: $P_1(G) = G$, and $P_{i+1}(G) = [P_i(G), G]\mho_1(P_i(G))$. Finally $d(G)$, $cl(G)$, $l(G)$, $sr(G)$ denote respectively, the minimal number of generators, the nilpotency class, the derived length and the special rank of G, while $e(G)$ is defined by $\exp(G) = p^{e(G)}$.

Theorem 2.1 (M. Hall [5]). *Let F be a free group on $\{x_1, x_2, ..., x_d\}$. Then for all $1 \leq i \leq n$,*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}$$

is a free abelian group freely generated by the basic commutators of weights $n, n+1, ..., n+i-1$ on the letters $\{x_1, x_2, ..., x_d\}$ (for a definition of basic commutators see [5]).

Lemma 2.2 (R. R. Struik [16]). *Let α be a fixed integer and G be a nilpotent group of class at most n. If $b_j \in G$ and $r < n$, then*

$$[b_1, .., b_{i-1}, b_i^{\alpha}, b_{i+1}, ..., b_r] = [b_1, ..., b_r]^{\alpha} c_1^{f_1(\alpha)} c_2^{f_2(\alpha)} ...,$$

where the c_k are commutators in $b_1, ..., b_r$ of weight strictly greater than r, and every b_j, $1 \leq j \leq r$, appears in each commutator c_k, the c_k listed in ascending order. The f_i are of the following form:

$$f_i(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + ... + a_{w_i} \binom{n}{w_i},$$

with $a_j \in \mathbf{Z}$, and w_i is the weight of c_i (in the b_i) minus $(r-1)$.

Powerful p-groups were formally introduced in [10]. They have played a role in the proofs of many important results in p-groups. We will discuss some of them in this section. A p-group G is called *powerful* if p is odd and $G' \leq \mho_1(G)$ or $p = 2$, and $G' \leq \mho_2(G)$. There is a related notion that is often used to find properties of powerful p-groups. If G is a p-group and $H \leq G$, then H is said to be *powerfully embedded* in G if $[G, H] \leq \mho_1(H)$ ($[G, H] \leq \mho_2(H)$ for $p = 2$). Any powerfully embedded subgroup is itself a powerful p-group and must be normal in the whole group. Also a p-group is powerful exactly when it is powerfully embedded in itself. While it is obvious that factor groups and direct products of powerful p-groups are powerful, this property is not subgroup-inherited [10].

We will require some standard properties of powerful p-groups. For the sake of convenience we collect them here.

Theorem 2.3 ([10]). *The following statements hold for a powerful p-group G.*

i) $\gamma_i(G), G^i, \mho_i(G), \Phi(G)$ *are powerfully embedded in* G.

ii) $P_{i+1}(G) = \mho_i(G)$ *and* $\mho_i(\mho_j(G)) = \mho_{i+j}(G)$.

iii) Each element of $\mho_i(G)$ *can be written as* a^{p^i}, *for some* $a \in G$ *and hence* $\mho_i(G) = \{g^{p^i} : g \in G\}$.

iv) If $G = \langle a_1, a_2, ..., a_d \rangle$, *then* $\mho_i(G) = \langle a_1^{p^i}, a_2^{p^i}, ..., a_d^{p^i} \rangle$.

v) If $H \subseteq G$, *then* $d(H) \leq d(G)$.

Proposition 2.4 ([10]). *Let N be a powerfully embedded subgroup of G. If N is the normal closure of some subset of G, then N is actually generated by this subset.*

Lemma 2.5. *Let H, K be normal subgroups of G and $H \leq K[H, G]$. Then $H \leq K[H,\ _l\ G]$ for any $l \geq 1$. In particular, if G is nilpotent, then $H \leq K$.*

Proof. An easy exercise. □

Lemma 2.6. *Let G be a finite p-group and $N \trianglelefteq G$. Then N is powerfully embedded in G if and only if $N/[N, G, G]$ is powerfully embedded in $G/[N, G, G]$.*

Proof. See a remark in the proof of Theorem 1.1 in [10]. □

Remark 2.7. To prove that a normal subgroup N is powerfully embedded in G we can assume that

i) $[N, G, G] = 1$ by the above lemma.

ii) $\mho_1(N) = 1$ ($\mho_2(N) = 1$ for $p = 2$) and try to show that $[N, G] = 1$.

iii) $[N, G]^2 = 1$ whenever $p = 2$, since if we assume that $N/[N, G]^2$ is powerfully embedded in $G/[N, G]^2$, then N is powerfully embedded in G. This follows from the proof of Theorem 4.1.1 in [10].

3. Main Results

In order to prove the main results we need the following theorem.

Theorem 3.1. *Let F/R be a free presentation of a powerful d-generator p-group G. Let $Z = R/[R, {}_cF]$ and $H = F/[R, {}_cF]$, so that $G \cong H/Z$. Then $\gamma_{c+1}(H)$ is powerfully embedded in H and $d(\gamma_{c+1}(H)) \leq \chi_{c+1}(d)$.*

Proof. First let p an odd prime. We may assume that $\mho_1(\gamma_{c+1}(H)) = 1$ and try to show that $[(\gamma_{c+1}(H)), H] = 1$ by Remark 2.7(ii). Also we may assume that $\gamma_{c+3}(H) = 1$ by Remark 2.7(i). Let $a, b_1, b_2, ..., b_c \in H$. Then by Lemma 2.2,

$$[a^p, b_1, ..., b_c] = [a, b_1, ..., b_c]^p c_1^{f_1(p)} c_2^{f_2(p)} ... \quad .$$

Since $\gamma_{c+3}(H) = 1$ and $\mho_1(\gamma_{c+1}(H)) = 1$ we have $[a, b_1, ..., b_c]^p = 1, c_i^{f_i(p)} = 1$, for all $i \geq 2$. Also $p > 2$ implies that $p|f_1(p)$, and hence $c_1^{f_1(p)} = 1$ so $a^p \in Z_c(H)$ and $\mho_1(H) \subseteq Z_c(H)$. The powerfulness of G yields $H' \leq \mho_1(H)Z \leq Z_c(H)$. Therefore $[H', {}_cH] = 1$, as desired. Since H/Z is generated by d elements and $Z \leq Z_c(H)$, $\gamma_{c+1}(H)$ is the normal closure of the commutators of weight $c+1$ on d elements. Hence Proposition 2.4 completes the proof, for $p > 2$.

If $p = 2$, then the proof is similar, so we leave out the details, but note that in this case

$$[a^4, b_1, ..., b_c] = [a, b_1, ..., b_c]^4 c_1^{f_1(4)} c_2^{f_2(4)} ... \quad .$$

By Remark 2.7 we can assume $\gamma_{c+3}(H) = \mho_2(\gamma_{c+1}(H)) = ([\gamma_{c+1}(H), H])^2 = 1$. Hence we have $[a^4, b_1, ..., b_c] = 1$ ($c_1^{f_1(4)} = 1$, since $2|f_1(4)$) so $\mho_2(H) \subseteq Z_c(H)$. □

An interesting corollary of this theorem is as follows.

Corollary 3.2. *Let G be powerful p-group with $d(G) = d$. Then $d(M^{(c)}(G)) \leq \chi_{c+1}(d)$.*

Proof. Let F/R be a free presentation of G with $Z = R/[R,_cF]$, so that $G \cong H/Z$, where $H = F/[R,_cF]$. Then the above result and Theorem 2.3(v) implies that
$$d(\frac{R \cap \gamma_{c+1}(F)}{[R,_cF]}) \le d(\frac{\gamma_{c+1}(F)}{[R,_cF]}) \le \chi_{c+1}(d).$$
Hence the result follows. □

Note that by a similar method we can prove Corollary 2.2 of [10] without using the concept of covering group for G.

The authors in a joint paper [12] have proved that if G is a finite d-generator p-group of special rank r and nilpotency class t, then $d(M^{(c)}(G)) \le \chi_{c+1}(d)+r^{c+1}(t-1)$. Clearly Corollary 3.2 improves this bound for nonabelian powerful p-groups.

Theorem 3.3. *Let G be powerful p-group. Then $e(M^{(c)}(G)) \le e(G)$.*

Proof. Let $p > 2$ and F/R be a free presentation of G with $Z = R/[R,_cF]$ and $H = F/[R,_cF]$, so that $G \cong H/Z$. Since $e(R\cap\gamma_{c+1}(F)/[R,_cF]) \le e(\gamma_{c+1}(H))$ and $e(H/Z_c(H)) \le e(G)$ it is enough to show that $e(\gamma_{c+1}(H)) = e(H/Z_c(H))$. We will establish by induction on k the equality
$$\mho_k(\gamma_{c+1}(H)) = [\mho_k(H),_cH], \tag{1}$$
which implies the above claim.

If $k = 0$, then (1) holds. Now Assume that (1) holds for some k. Since $\gamma_{c+1}(H)$ is powerfully embedded in H by Theorem 3.1, we have $\mho_{k+1}(\gamma_{c+1}(H)) = \mho_1(\mho_k(\gamma_{c+1}(H)))$, by Theorem 2.3(ii). Similarly $\mho_{k+1}(G)) = \mho_1(\mho_k(G)))$. Since $G \cong H/Z$ we have $\mho_{k+1}(H)Z/Z = \mho_1(\mho_k(H)Z)Z/Z$. Therefore
$$[\mho_{k+1}(H),_cH] = [\mho_{k+1}(H)Z,_cH] = [\mho_1(\mho_k(H)Z)Z,_cH]$$
$$= [\mho_1(\mho_k(H)Z),_cH].$$
This implies that
$$[\mho_{k+1}(H),_cH] = [\mho_1(\mho_k(H)Z),_cH]. \tag{2}$$

Thus (1) for $k+1$ is equivalent to $\mho_1(\mho_k(\gamma_{c+1}(H))) = [\mho_1(\mho_k(H)Z), {}_cH]$. Since $\mho_k(\gamma_{c+1}(H))$ is powerfully embedded in H by Theorem 2.3(i), this implies, by (1) and Lemma 2.2,

$$\begin{aligned}
[\mho_1(\mho_k(H)Z), {}_cH] &\leq \mho_1([\mho_k(H)Z, {}_cH])[\mho_k(H)Z, {}_cH, H] \\
&\leq \mho_1([\mho_k(H), {}_cH])[\mho_k(H), {}_cH, H] \\
&\leq \mho_1(\mho_k(\gamma_{c+1}(H)))[\mho_k(\gamma_{c+1}(H)), H] \\
&\leq \mho_1(\mho_k(\gamma_{c+1}(H))).
\end{aligned}$$

For the reverse inclusion note that since $\mho_1(\mho_k(\gamma_{c+1}(H))) = \mho_1([\mho_k(H),\ {}_cH])$ it is enough to show that

$$\mho_1([\mho_k(H), {}_cH]) \equiv 1 \pmod{[\mho_1(\mho_k(H)Z), {}_cH]}.$$

By Theorem 2.3(i), $\mho_k(H/Z)$ is powerfully embedded in H/Z so that

$$[\frac{\mho_k(H)Z}{Z}, \frac{H}{Z}] \leq \frac{\mho_1(\mho_k(H)Z)Z}{Z}. \tag{3}$$

Also (2) implies that $\mho_1(\mho_k(H)Z) \leq Z_c(H) \pmod{[\mho_{k+1}(H),\ {}_cH]}$. Now (2), (3) and the last inequality imply that

$$[\mho_k(H)Z, H] \leq \mho_1(\mho_k(H)Z)Z \leq Z_c(H) \pmod{[\mho_{k+1}(H),\ {}_cH]}.$$

Hence by Lemma 2.2

$$\begin{aligned}
\mho_1([\mho_k(H),\ {}_cH]) &\equiv \mho_1([\mho_k(H)Z, {}_cH]) \\
&\equiv [\mho_1(\mho_k(H)Z), {}_cH] \\
&\equiv 1 \pmod{[\mho_1(\mho_k(H)Z), {}_cH]},
\end{aligned}$$

as desired.

If $p = 2$, then the proof is similar to the previous case. This completes the proof. □

Note that G. Ellis [4], using the nonabelian tensor products of groups, showed that $\exp(M^{(c)}(G))$ divides $\exp(G)$ for all $c \geq 1$ and all p-groups satisfying $[[G^{pi-1}, G], G] \subseteq G^{p^i}$ for $1 \leq i \leq e$, where $\exp(G) = p^e$. Note that the results of [10] imply that every powerful p-group G satisfies the latter commutator condition.

Lubotzky and Mann [10] found bounds for $cl(G), l(G), |G|$ and $|M(G)|$ of a powerful d-generator p-group G of exponent p^e as follows:

$$cl(G) \leq e,\ l(G) \leq \log_2 e + 1,\ |G| \leq p^{de} \text{ and } |M(G)| \leq p^{(d(d-1)/2)e}.$$

In the following proposition we find an upper bound for the order of c-nilpotent multiplier of G.

Proposition 3.4. *Let G be a powerful p-group, with $d(G) = d$ and $e(G) = e$. Then $|M^{(c)}(G)| \leq p^{\chi_{c+1}(d)e}$.*

Proof. It is obtained by combining Corollary 3.2 and Theorem 3.3. □

4. Some Examples

In this final section we are going to give some explicit examples of p-groups and calculate their c-nilpotent multipliers in order to compare our new bounds with the exact values. This will show tightness of our results and improvement some of the previously mentioned inequalities.

Example 4.1. Let G be a finite abelian p-group. Clearly G is a powerful p-group and by the fundamental theorem of finitely generated abelian groups G has the following structure

$$G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \ldots \oplus \mathbf{Z}_{p^{\alpha_d}}$$

for some positive integers $\alpha_1, \alpha_2, \ldots, \alpha_d$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d$. By [11] the c-nilpotent multiplier of G can be calculated explicitly as follows:

$$M^{(c)}(G) \cong \mathbf{Z}_{p^{\alpha_2}}^{(b_2)} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_{p^{\alpha_d}}^{(b_d-b_{d-1})},$$

where $b_i = \chi_{c+1}(i)$ and $\mathbf{Z}_n^{(m)}$ denotes the direct sum of m copies of the cyclic group $\mathbf{Z}_n$. Now it is easy to see that

(*i*) $d(M^{(c)}(G)) = \chi_{c+1}(d)$, where $d = d(G)$. Hence the bound of Corollary 3.2 is attained and the best one in the abelian case.

(*ii*) $e(M^{(c)}(G)) = \alpha_2$, whereas $e(G) = \alpha_1$. Hence the bound of Theorem 3.3 is attained when $\alpha_1 = \alpha_2$ and it is the best one in the abelian case.

(*iii*) $|M^{(c)}(G)| = p^{\alpha_2 b_2 + \sum_{i=3}^{d} \alpha_i (b_i - b_{i-1})} \leq p^{\alpha_1} \chi_{c+1}(d)$. Hence the bound of Proposition 3.4 is attained if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_d$.

Example 4.2. Let p be any odd prime number and $s,\ t$ be positive integers with $s \geq t$. Consider the following finite d-generator p-group with nilpotency class 2:

$$P_{s,t} = \langle y_1, \ldots, y_d : y_i^{p^s} = [y_j, y_k]^{p^t} = [[y_j, y_k], y_i] = 1,\ 1 \leq i, j, k \leq d,\ j \neq k \rangle.$$

One can see that $P_{s,t}$ is not a powerful p-group (clearly $\mho_1(P_{1,1}) = 1$). By [14] the c-nilpotent multiplier of $P_{s,t}$ is as follows:

$$M^{(c)}(P_{s,t}) \cong \mathbf{Z}_{p^s}^{(\chi_{c+1}(d))} \oplus \mathbf{Z}_{p^t}^{(\chi_{c+2}(d))}.$$

Therefore we have

(*i*) $d(M^{(c)}(P_{s,t})) = \chi_{c+1}(d) + \chi_{c+2}(d) > \chi_{c+1}(d)$. Hence the condition of being powerful cannot be omitted from Corollary 3.2.

(*ii*) $|M^{(c)}(P_{s,t}))| = p^{s\chi_{c+1}(d) + t\chi_{c+2}(d)} > p^{s\chi_{c+1}(d)}$. Hence powerfulness is also a necessary condition for the bound of Proposition 3.4. Note that here we have $e(M^{(c)}(P_{s,t})) = se(P_{s,t})$.

The authors in a joint paper [13] have proved that $\exp(M^{(c)}(G)) | \exp(G)$, when G is a nilpotent p-group of class k, and $k < p$. In the following example we find a powerful p-group of class $k \geq p$ such that $\exp(M^{(c)}(G))$ divides $\exp(G)$.

Example 4.3 ([17]). We work in $\mathrm{GL}(\mathbf{Z}_{p^{l+2}})$, the 2×2 invertible matrices over the ring of integers modulo p^{l+2}. In this ring any integer not divisible by p is invertible. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1-p \end{bmatrix}, Y = \begin{bmatrix} 1/(1-p) & p/(1-p) \\ 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}.$$

One quickly calculates that $[X,Y] = Z^p$, $[X,Z] = Z^p$, $[Y,Z] = Z^p$ and

$$[Z^p, {}_k X] = \begin{bmatrix} 1 & (-1)^{k+2}p^{k+2} \\ 0 & 1 \end{bmatrix}. \tag{4}$$

Notice also that $X^{p^{l+1}} = Y^{p^{l+1}} = Z^{p^{l+1}} = 1$. We claim that $P = \langle X, Y, Z\rangle$ is a powerful p-group. By the above relations we can express every word in P as a product $X^aY^bZ^c$ for some $0 \le a, b, c < p^{l+1}$. Also

$$X^aY^bZ^c = \begin{bmatrix} \frac{1}{(1-p)^b} & \frac{1+pc-(1-p)^b}{(1-p)^b} \\ 0 & (1-p)^a \end{bmatrix}$$

and hence all of these elements are distinct. Therefore the order of P is $p^{3(l+1)}$ and hence P is a p-group and the relations imply that $P' \le \mho_1(P)$. Therefore P is a powerful p-group. The exponent of P is p^{l+1}, and (4) implies that P has nilpotency class $l+1$. By Theorem 3.3 $\exp(M^{(c)}(P))$ divides $\exp(P)$. Note that the nilpotency class of P is $l+1$ which is greater than or equal to p.

Let G be a finite d-generator p-group of order p^n where p is any prime. By [11] we have

$$p^{\chi_{c+1}(d)} \le |M^{(c)}(G)||\gamma_{c+1}(G)| \le p^{\chi_{c+1}(n)}.$$

Now if we put $l = 2$ in the above example, then P is 3-generator powerful p-group of order p^9 with nilpotency class 3. Thus by the above bounds we have

$$p^{18} = p^{\chi_4(3)} \le |M^{(3)}(P)||\gamma_4(P)| = |M^{(3)}(P)| \le p^{\chi_4(9)} = p^{1620}.$$

But by Proposition 3.4 $|M^{(3)}(P)| \le p^{3\chi_4(3)} = p^{54}$. Hence this example and also Example 4.1 show that Proposition 3.4 improves the above bound for powerful p-groups.

Example 4.4. Using the list of nonabelian groups of order at most 30 with their c-nilpotent multipliers for $c = 1, 2$ in the table of Fig.2 in [3], we are going to give two nonabelian powerful p-groups in order to compute explicitly the number of generators, the order and the exponent of their 2-nilpotent multipliers and then compare these numbers with bounds obtained.

(i) Consider the finite 2-group $G = \langle a, b : a^2 = 1, aba = b^{-3}\rangle$. It is easy to see that G is a powerful 2-group and $|G| = 16$, $d(G) = 2$, $\exp(G) = 8$. By [3, Fig.2, ♯ 13] $M^{(2)}(G) \cong \mathbf{Z}_2^{(2)}$ and hence $|M^{(2)}(G)| = 4$, $d(M^{(2)}(G)) = 2$, $\exp(M^{(2)}(G)) = 2$. It is seen that the bound of Corollary 3.2 is attained.

(ii) Consider the finite 3-group $G = \langle a, b : a^3 = 1, a^{-1}ba = b^{-2}\rangle$. It is easy to see that G is a powerful 3-group and $|G| = 27$, $d(G) = 2$, $\exp(G) = 9$. By [3, Fig.2, ♯ 40] $M^{(2)}(G) \cong \mathbf{Z}_3^{(2)}$ and hence $|M^{(2)}(G)| = 9$, $d(M^{(2)}(G)) = 2$, $\exp(M^{(2)}(G)) = 3$. It is also seen that the bound of Corollary 3.2 is attained.

Acknowledgements

This research was in part supported by a grant from Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad.

The authors are grateful to the referee for useful comments and careful corrections.

References

[1] R. Baer, Representations of Groups as Quotient Groups I,II,III, *Trans. Amer. Math. Soc.* **58** (1945), 295-419.

[2] A. J. Bayes, J. Kautsky, J. W. Wamsley, Computation in Nilpotent Groups (application), in: Proc. 2nd Internat. Conf. Theory of Groups, *in: Lecture Notes in Math. Vol.* **372**, Springer-Verlag (1973), 82-89.

[3] J. Burns and G. Ellis, On the Nilpotent Multipliers of a Group, *Math. Z.* **226** (1997), 405-423.

[4] G. Ellis, On the Relation Between Upper Central Quotients and Lower Central Series of a Group, *Trans. Amer. Soc.* **353** (2001), 4219-4234.

[5] M. Hall, *The Theory of Groups*, MacMillan Company: NewYork, 1959.

[6] M. R. Jones, Multiplicators of p-Groups, *Math. Z.* **127** (1972), 165-166.

[7] M. R. Jones, Some Inequalities for the Multiplicator of a Finite Group, *Proc. Amer. Math. Soc.* **39** (1973), 450-456.

[8] G. Karpilovsky, *The Schur Multiplier*, London Math. Soc. Monographs, New Series No. 2 (1987).

[9] S. Kayvanfar and M. A. Sanati, A Bound for the Exponent of the Schur Multiplier of a some Finite p-Group, *Bull. Iranian Math. Soc. Vol.* **26**, No. 2 (2000), 89-95.

[10] A. Lubotzky and A. Mann, Powerful p-Groups. I. Finite Groups., *J. Algebra* **105** (1987), 484-505.

[11] B. Mashayekhy and M. R. R. Moghaddam, Higher Schur Multiplicator of a Finite Abelian Group, *Algebra Colloquium* **4:3** (1997), 317-322.

[12] B. Mashayekhy, F. Mohammadzadeh and A. Hokmabadi, Some Inequalities for Nilpotent Multipliers of Finite Groups, *submitted*.

[13] B. Mashayekhy, F. Mohammadzadeh and A. Hokmabadi, On a Conjecture of a Bound for the Exponent of the Schur Multiplier of Finite p-Groups, *submitted.*

[14] M. R. R. Moghaddam, Calculation of the Baer Invariant of Certain Groups, *Mh. Math.* **97** (1984), 191-206.

[15] P. Moravec, Schur Multipliers and Power Endomorphisms of groups, *J. Algebra* **308** (2007), 12-25.

[16] R. R. Struik, On Nilpotent Product of Cyclic Groups, *Canad. J. Math.* **12** (1960), 447-462.

[17] L. E. Wilson, *Powerful Groups of Prime Power Order*, Ph.D. Thesis, University of Chicago (2002).