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Generalized property (ω') of finite rank perturbations of operators*

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Abstract We define the generalized property (ω'), a variant of Weyl's theorem. By means of the new spectrum defined in view of the property of consistency in Fredholm and index, we consider the preservation of generalized property (ω') under a finite rank perturbation commuting with T, whenever T is a-isoloid. The theory is illustrated in the case of some special classes of operators.

Key words generalized property (ω'), consistent Fredholm and index operators, spectrum, Weyl's theorem

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Weyl[1] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has been considered by many authors. Variants have been discussed by Harte and Lee[2] and Rakočević[3-4]. We have established for a bounded linear operator T defined on a Hilbert space the sufficient and necessary conditions for which the generalized property (ω') holds in Ref. [5]. In this paper, we continue to show how generalized property (ω') follows from properties of the variant (σ_1) and the spectrum defined in view of the property of consistency in Fredholm and index (defined in section 1). We consider the preservation of generalized property (ω') under a finite rank perturbation commuting with T, and give the sufficient and necessary condition for which the generalized property (ω') holds for $T+F$, whenever $T \in B(H)$ satisfies generalized property (ω') and $F \in B(H)$ is a finite rank operator commuting with T. Moreover, the theory is applied to several classes of operators.

Throughout this paper, let $B(H)$ denote the algebra of bounded linear operators acting on an infinite-dimensional complex Hilbert space H. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and the range of T; $\sigma(T)$ for the spectrum of T. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range with finite dimensional null space and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator, let $n(T) = \dim N(T)$ and $d(T) =$

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dim $H/R(T)$ = codim $R(T)$. The index of a Fredholm operator $T \in B(H)$ is given by $\text{ind}(T) = n(T) - d(T)$. The ascent of T, $\text{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\text{dsc}(T)$, is the least non-negative integer n such that $R(T^n) = R(T^{n+1})$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. And $T \in B(H)$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ of $T \in B(H)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}, \tag{1}$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}. \tag{2}$$

We denote by $\sigma_{SF}(T)$ the semi-Fredholm spectrum of T defined as the set of all λ in $\mathbb{C}$ for which $T - \lambda I$ is not a semi-Fredholm operator, and let $\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{SF}(T)$. $SF_+^-(H)$ is the set of all $T \in B(H)$ which are upper semi-Fredholm operators of $\text{ind}(T) \leq 0$. The approximate point spectrum is defined by $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$, where an operator is said to be bounded below if it is injective and has closed range. Let $\rho_a(T) = \mathbb{C} \setminus \sigma_a(T)$.

For each nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If for some n, $R(T^n)$ is closed and T_n is an upper (respectively lower) semi-Fredholm operator then T is called an upper (respectively lower) semi-B-Fredholm operator. A semi-B-Fredholm operator is an upper or lower semi-B-Fredholm operator. If moreover, T_n is a Fredholm operator then T is called a B-Fredholm operator. From Proposition 2.1 in Ref. [6] if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$, and $\text{ind}(T_m) = \text{ind}(T_n)$. Then the index of a semi-B-Fredholm operator is defined as the index of the semi-Fredholm operator T_n.

Let $SBF_+(H)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(H)$ the class of all $T \in SBF_+(H)$ such that $\text{ind}(T) \leq 0$, and for any $T \in B(H)$, let $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(H)\}$. An operator $T \in B(H)$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}. \tag{3}$$

An operator $T \in B(H)$ is called Drazin invertible if it has a finite ascent or descent. The Drazin spectrum $\sigma_D(T)$ of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$. It is known that the Drazin spectrum satisfies the spectral mapping theorem and $\sigma_{BW}(T) \subseteq \sigma_D(T)$. We have that if 0 is isolated in the spectrum $\sigma(T)$, then T is a B-Weyl operator if and only if T is Drazin invertible, for a proof see Theorem 4.2 in Ref. [7]. Actually, according to the perturbation theorem of semi-B-Fredholm operators, we can also get that if 0 is isolated in $\sigma_a(T)$, then T is a B-Weyl operator if and only if T is Drazin invertible.

The generalized property (ω') which we will define has close relations with Weyl's theorem. The plan of this paper is as follows. In section 1, by defining two new spectrums, we give the definition of generalized property (ω') and show the preservation of generalized property (ω') under a finite rank perturbation commuting with T. In section 2, the theory is exemplified in the case of some special classes of operators.

1 CFI operators and generalized property (ω')

We begin with a definition and a lemma derived from Ref. [8]:

Definition 1.1[8] We say $T \in B(H)$ is consistent in Fredholm and index (abbrev. a CFI operator), if for each $B \in B(H)$, one of the cases occurs:

1) TB and BT are Fredholm together and $\text{ind}(TB) = \text{ind}(BT) = \text{ind}(B)$;

2) TB and BT are not Fredholm.

Lemma 1.1[8] $T \in B(H)$ is a CFI operator if and only if one of the following three mutually disjoint cases occurs:

1) T is Weyl;

2) $R(T)$ is not closed;

3) $R(T)$ is closed and $\dim N(T) = \operatorname{codim} R(T) = \infty$.

Let
$$\rho_2(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is a CFI operator}\}, \tag{4}$$
and let $\sigma_2(T) = \mathbb{C} \setminus \rho_2(T)$. Clearly, $\lambda_0 \in \sigma_2(T)$ if and only if $T - \lambda_0 I$ is a semi-Fredholm operator but ind $(T - \lambda_0 I) \neq 0$. By perturbation theorem of semi-Fredholm operator, $\sigma_2(T)$ is an open set in the spectrum $\sigma(T)$ of operator T. Let $H(T)$ be the class of complex-valued functions which are analytic in a neighborhood of $\sigma(T)$ and are not constant on any neighbourhood of any component of $\sigma(T)$.

Berkani and Koliha[9] have discussed generalized Weyl's theorem. In the following, we consider a variant of Weyl's theorem called generalized property (ω').

Recall that the generalized Weyl's theorem holds for $T \in B(H)$ if there is equality
$$\sigma(T) \setminus \sigma_{BW}(T) = E(T), \tag{5}$$
where $E(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues. The generalized a-Weyl's theorem holds for $T \in B(H)$ if there is equality
$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T), \tag{6}$$
where $E^a(T)$ for the isolated points of $\sigma_a(T)$ which are eigenvalues.

Definition 1.2[5] $T \in B(H)$ is said to satisfy generalized property (ω') if
$$\sigma(T) \setminus \sigma_{BW}(T) = E^a(T). \tag{7}$$

It is easy to prove that generalized property (ω') implies generalized Weyl's theorem, but the converse is not true.

Let
$$\rho_1(T) = \{\lambda \in \mathbb{C} : \text{there exists } \varepsilon > 0 \text{ such that } T - \mu I \in SF_+^-(H) \text{ and}$$
$$N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n] \text{ if } 0 < |\mu - \lambda| < \varepsilon\}, \tag{8}$$
and let $\sigma_1(T) = \mathbb{C} \setminus \rho_1(T)$. Clearly, $\sigma_1(T) \subseteq \operatorname{acc} \sigma(T) \subseteq \sigma_D(T)$. T is called a-isoloid if $\lambda \in \operatorname{iso} \sigma_a(T) \Rightarrow N(T - \lambda I) \neq \{0\}$. The following Theorem 1.1 and Theorem 1.2 give the relations between generalized property (ω') and property of consistency in Fredholm and index (see Ref. [5] for the proof).

Theorem 1.1[5] $T \in B(H)$ satisfies generalized property (ω') if and only if
$$\sigma_D(T) = \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \operatorname{acc} \sigma_a(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}. \tag{9}$$

Corollary 1.1[5] $T \in B(H)$ is a-isoloid and generalized property (ω') holds for T if and only if
$$\sigma_D(T) = \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \operatorname{acc} \sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]. \tag{10}$$

Remark 1.1[5] 1) If $\sigma_D(T) = \sigma_1(T)$, then T is a-isoloid and generalized property (ω') holds for T.

If $\sigma_2(T) \subseteq \sigma_a(T)$, then for any $\lambda \notin \sigma_a(T)$, $T - \lambda I$ is a bounded below operator and is also a CFI operator, which means that $T - \lambda I$ is invertible by Lemma 1.1. Then $\lambda \notin \sigma(T)$. Hence, if $\sigma_2(T) \subseteq \sigma_a(T)$, then $\sigma(T) = \sigma_a(T)$.

Theorem 1.2[5] Suppose $T \in B(H)$ is a-isoloid and generalized property (ω') holds for T. If $\sigma_2(T) \subseteq \sigma_a(T)$, and for any $f \in H(T)$, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$, then for any $f \in H(T)$, $f(T)$ is a-isoloid and generalized property (ω') holds for $f(T)$.

Remark 1.2 1) It is easy to prove that $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for any $f \in H(T)$ if and only if for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_e(T)$, $\operatorname{ind}(T - \lambda I) \operatorname{ind}(T - \mu I) \geq 0$.

2) " T is a-isoloid" is essential in Theorem 1.2. For example, let $A \in B(l^2)$ be an injective

quasinilpotent operator, and define on $H \oplus H$ the operator T by $T = (A - I) \oplus I$. It is easy to check that

$$\sigma(T) = \sigma_a(T) = \{-1, 1\},\ \sigma_{BW}(T) = \{-1\}. \tag{11}$$

Since $n(T-I) = \infty$ and $n(T+I) = 0$, then $E^a(T) = \{1\}$. Hence T satisfies generalized property (ω'), but T is not a-isoloid. Moreover, $\sigma(T) = \sigma_a(T)$, $\sigma_e(T) = \sigma_w(T) = \{-1, 1\}$, so $\sigma_2(T) \subseteq \sigma_a(T)$ and for any $f \in H(T)$, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$.

On the other hand, let $f(T) = T^2$. Then

$$\sigma(f(T)) = \sigma_a(f(T)) = \sigma_{BW}(f(T)) = \{1\}. \tag{12}$$

Since $f(T) - I = (T-I)(T+I)$ and $n(T-I) = \infty$, then $E^a(f(T)) = \{1\}$. This shows that generalized property (ω') does not hold for $f(T)$.

Oberai[10] had examples which show that the Weyl's theorem for T is not sufficient for the Weyl's theorem for $T+F$ with finite rank F. For generalized property (ω'), it has the same case. See the example in 1) of Remark 1.3.

Theorem 1.3 Suppose that $T \in B(H)$ is a-isoloid and generalized property (ω') holds for T. If $F \in B(H)$ is a finite rank operator commuting with T, then $\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$ if and only if $T+F$ is a-isoloid and generalized property (ω') holds for $T+F$.

proof Suppose that $\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$. By Corollary 1.1, we only need to prove that $\sigma_D(T+F) \subseteq \sigma_1(T+F) \cup [\overline{\sigma_2(T+F)} \cap \text{acc}\,\sigma_a(T+F)] \cup [\rho_a(T+F) \cap \sigma(T+F)]$. Let $\lambda_0 \notin \sigma_1(T+F) \cup [\overline{\sigma_2(T+F)} \cap \text{acc}\,\sigma_a(T+F)] \cup [\rho_a(T+F) \cap \sigma(T+F)]$, without loss of generality, we suppose that $\lambda_0 \in \sigma_a(T+F)$. Then there exists $\varepsilon > 0$, such that $T+F-\lambda I \in SF_+^-(H)$ and

$$N(T+F-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T+F-\lambda I)^n]. \tag{13}$$

if $0 < |\lambda - \lambda_0| < \varepsilon$. Also $\lambda_0 \notin \overline{\sigma_2(T+F)} \cap \text{acc}\,\sigma_a(T+F)$, then we claim that $\lambda_0 \in \text{iso}\,\sigma_a(T) \cup \rho_a(T)$. In fact, if $\lambda_0 \notin \overline{\sigma_2(T+F)}$, we can prove that $T+F-\lambda I$ is Weyl if $0 < |\lambda - \lambda_0|$ is small enough, then $T-\lambda I$ is Weyl. Since generalized property (ω') holds for T, we know that $T-\lambda I$ is Browder. This shows that $T+F-\lambda I$ is Browder, then

$$N(T+F-\lambda I) = N(T+F-\lambda I) \cap \bigcap_{n=1}^{\infty} R[(T+F-\lambda I)^n] = \{0\} \tag{14}$$

(Lemma 3.4 in Ref. [11]) which means that $T+F-\lambda I$ is invertible, that is, $\lambda_0 \in \text{iso}\,\sigma(T+F)$. If $\lambda_0 \notin \text{acc}\,\sigma_a(T+F)$, again, we get that $\lambda_0 \in \text{iso}\,\sigma_a(T+F)$. So $\lambda_0 \in \text{iso}\,\sigma_a(T) \cup \rho_a(T)$ (Corollary 2.4 in Ref. [12]). Thus if $\lambda_0 \in \text{iso}\sigma_a(T)$, then $\lambda_0 \in \sigma(T) \backslash \sigma_{BW}(T)$ since T is a-isoloid and generalized property (ω') holds for T, which implies that $T-\lambda_0 I$ is Drazin invertible. Then $T+F-\lambda_0 I$ is Drazin invertible (Theorem 2.7 in Ref. [13]), that is, $\lambda_0 \notin \sigma_D(T+F)$. If $\lambda_0 \in \rho_a(T)$, then $\lambda_0 \notin \sigma_a(T) \cup \rho_a(T+F)$. The fact that $\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$ tells us that $T-\lambda_0 I$ is invertible. Thus $T+F-\lambda_0 I$ is Drazin invertible, that is, $\lambda_0 \notin \sigma_D(T+F)$. □

Conversely, suppose that $T+F$ is a-isoloid and generalized property (ω') holds for $T+F$. Let $\lambda_0 \notin \sigma_a(T) \cup \rho_a(T+F)$, then $T-\lambda_0 I$ is bounded below, and $T+F-\lambda_0 I$ is an upper semi-Fredholm operator of $n(T+F-\lambda_0 I) > 0$ and $\text{asc}(T+F-\lambda_0 I) < \infty$. Thus $\lambda_0 \in \text{iso}\sigma_a(T+F)$, that is, $\lambda_0 \in E^a(T+F)$. Since $T+F$ satisfies generalized property (ω'), we get that $T+F-\lambda_0 I$ is B-Weyl and also Drazin invertible. Hence $T-\lambda_0 I$ is Drazin invertible, then using the fact that $T-\lambda_0 I$ is bounded below, we know that $T-\lambda_0 I$ is invertible. By Lemma 1.1, we have proved that $T-\lambda_0 I$ is a CFI operater, that is, $\lambda_0 \notin \sigma_2(T)$. □

The next result deals with nilpotent perturbations. We first recall two well-known results: if N is a nilpotent operator commuting with $T \in B(H)$, then

$$\sigma(T) = \sigma(T+N) \text{ and } \sigma_a(T) = \sigma_a(T+N). \tag{15}$$

Corollary 1.2　Suppose that $T \in B(H)$ satisfies generalized property (ω'). If $F \in B(H)$ is a finite rank nilpotent operator commuting with T, then generalized property (ω') holds for $T + F$.

Proof　From Theorem 1.1, we only need to prove that $\sigma_D(T+F) \subseteq \sigma_1(T+F) \cup [\overline{\sigma_2(T+F)} \cap \operatorname{acc}\sigma_a(T+F)] \cup \{\lambda \in \sigma(T+F): n(T+F-\lambda I) = 0\}$. Let $\lambda_0 \notin \sigma_1(T+F) \cup [\overline{\sigma_2(T+F)} \cap \operatorname{acc}\sigma_a(T+F)] \cup \{\lambda \in \sigma(T+F): n(T+F-\lambda I) = 0\}$, without loss of generality, suppose $\lambda_0 \in \sigma(T+F)$. Then $n(T+F-\lambda_0 I) > 0$, and from the proof of Theorem 1.3 we know that $\lambda_0 \in \operatorname{iso}\sigma_a(T+F)$. Since F is a nilpotent operator, then $\lambda_0 \in \operatorname{iso}\sigma_a(T)$, and there exists $p \in \mathbb{N}$, such that $F^p = 0$. If $x \in N(T+F)$, $(T+F)^p x = 0$, then

$$T^p x = (-1)^p T^p x = 0, \tag{16}$$

that is, $N(T+F) \subseteq N(T^p)$. Using the fact that $n(T+F-\lambda_0 I) > 0$, we get $n((T-\lambda_0 I)^p) > 0$, which induces that $n(T-\lambda_0 I) > 0$. Therefore $\lambda_0 \in E^a(T)$, and $T - \lambda_0 I$ is Drazin invertible since generalized property (ω') holds for T. Thus $T + F - \lambda_0 I$ is Drazin invertible, that is, $\lambda_0 \notin \sigma_D(T+F)$. □

Remark 1.3　1) "T is a-isoloid" is essential in Theorem 1.3. For example, let $T = A \oplus I$ act on $H \oplus H$ with an injective quasinilpotent operator A. It is clear that T satisfies generalized property (ω'), but T is not a-isoloid. Note that $\sigma_2(T) = \emptyset$. Take any finite rank projection $P \in B(H)$, and let $F = 0 \oplus (-P)$. Then $TF = FT$ and $\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$, but generalized property (ω') fails for $T + F$ because $0 \in E^a(T+F) \cap \sigma_{BW}(T+F)$.

2) "$\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$" is essential in Theorem 1.3. For example, let $B, P \in B(l^2)$ be defined:

$$B(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots), \tag{17}$$

$$P(x_1, x_2, x_3, \cdots) = (-x_1, 0, 0, \cdots). \tag{18}$$

Define on $H \oplus H$ the operators T and F by $T = B \oplus I$, $F = 0 \oplus P$. Clearly, F is a finite rank operator and $FT = TF$. It is easy to check that

$$\sigma(T) = \sigma_{BW}(T) = \{\lambda \in \{\mathbb{C}\}: |\lambda| \leq 1\}, \ \sigma_a(T) = \{\lambda \in \mathbb{C}: |\lambda| = 1\}, \ E^a(T) = \emptyset. \tag{19}$$

So T is a-isoloid and generalized property (ω') holds for T. Moreover, $\sigma_2(T) = \{\lambda \in \mathbb{C}: |\lambda| < 1\}$.

On the other hand,

$$\sigma(T+F) = \sigma_{BW}(T+F) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\},$$
$$\sigma_a(T+F) = \{\lambda \in \mathbb{C}: |\lambda| = 1\} \cup \{0\}, \ E^a(T+F) = \{0\}. \tag{20}$$

Hence, $\sigma_2(T) \nsubseteq \sigma_a(T) \cup \rho_a(T+F)$ and generalized property (ω') does not hold for $T + F$.

3) "a finite rank operator F commutes with T" is essential in Theorem 1.3. For example, let $T, F \in B(l^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots\right), \tag{21}$$

$$F(x_1, x_2, x_3, \cdots) = (0, -x_1, 0, 0, \cdots). \tag{22}$$

Clearly $TF \neq FT$. It is easy to check that

$$\sigma(T) = \sigma_{BW}(T) = \{0\}, \ E^a(T) = \emptyset. \tag{23}$$

So T is a-isoloid and generalized property (ω') holds for T. Since $\sigma_2(T) = \emptyset$, then $\sigma_2(T) \subseteq \sigma_a(T) \cup \rho_a(T+F)$. On the other hand,

$$\sigma(T+F) = \sigma_{BW}(T+F) = \{0\}, \ E^a(T+F) = \{0\}. \tag{24}$$

It follows that $T + F$ does not satisfy generalized property (ω').

Theorem 1.4　Suppose that $T \in B(H)$ is a-isoloid, generalized property (ω') holds for T, and $F \in B(H)$ is a finite rank operator commuting with T. If $\sigma_2(T) \subseteq \sigma_a(T)$, and for any $f \in H(T)$, $\sigma_{BW}(f(T)) =$

$f(\sigma_{BW}(T))$, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Proof By Theorem 1.2 and Theorem 1.3, we need to prove that $\sigma_2(f(T)) \subseteq \sigma_a(f(T))$. Let $\mu_0 \notin \sigma_a(f(T))$, then $f(T)-\mu_0 I$ is bounded below. Let

$$f(T)-\mu_0 I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_k I)^{n_k} g(T), \tag{25}$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $T-\lambda_i I$ is bounded below, which implies that $T-\lambda_i I$ is invertible since $\sigma_2(T) \subseteq \sigma_a(T)$. Hence $f(T)-\mu_0 I$ is invertible. Then $\mu_0 \notin \sigma_2(f(T))$. □

If $\sigma_D(T) = \sigma_1(T)$, using the perturbation theorem of semi-Fredholm operators, we can prove that $\sigma_2(T) \subseteq \sigma_a(T)$ and $\operatorname{ind}(T-\lambda I) \geqslant 0$ for any $\lambda \in \{\mathbb{C}\}\backslash\sigma_e(T)$. By Corollary 1.1 and Theorem 1.4, we get:

Corollary 1.3 If $\sigma_D(T) = \sigma_1(T)$ and $F \in B(H)$ is a finite rank operator commuting with T, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Corollary 1.4 Suppose that $T \in B(H)$ is a-isoloid, generalized property (ω') holds for T and $F \in B(H)$ is a finite rank operator commuting with T. Then

1) If $\overline{\sigma_2(T)} \subseteq \sigma_1(T)$, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

2) If $\sigma_2(T) \subseteq \sigma_a(T) \cap \sigma_e(T)$, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Proof 1) If $\overline{\sigma_2(T)} \subseteq \sigma_1(T)$, then $\overline{\sigma_2(T)} \cap \operatorname{acc} \sigma_a(T) \subseteq \sigma_1(T)$ and $\rho_a(T) \cap \sigma(T) = \emptyset$. By Theorem 1.1, we get that $\sigma_D(T) = \sigma_1(T)$. Hence $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$ by Corollary 1.3.

2) If $\sigma_2(T) \subseteq \sigma_a(T) \cap \sigma_e(T)$, then $\sigma_2(T) \subseteq \sigma_a(T)$ and $\sigma_e(T) = \sigma_w(T)$, which implies that for any $f \in H(T)$, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. By Theorem 1.4, we know that $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$. □

2 The application

In this section the theory is applied to several classes of operators.

For $x \in H$, the orbit of x under T is the set of images of x under successive iterates of T: $\operatorname{Orb}(T, x) = \{x, Tx, T^2x, \cdots\}$. A vector $x \in H$ is supercyclic if the set of scalar multiples of $\operatorname{Orb}(T, x)$ is dense in H, and x is hypercyclic if $\operatorname{Orb}(T, x)$ is dense. A hypercyclic operator is one that has a hypercyclic vector. We similarly define the notion of supercyclic operator. We denote by $HC(H)$ ($SC(H)$) the set of all hypercyclic (supercyclic) operators in $B(H)$ and $\overline{HC(H)}$ ($\overline{SC(H)}$) the norm-closure of the class $HC(H)$ ($SC(H)$). The essential facts for hypercyclic operators and supercyclic operators were described by Herrero[14] in 1991.

If $T \in \overline{HC(H)}$ or $T \in \overline{SC(H)}$, we have that $\operatorname{ind}(T-\lambda I) \geqslant 0$ for any $\lambda \in \rho_{SF}(T)$. It follows that $\sigma_2(T) \subseteq \sigma_a(T)$, and for any $f \in H(T)$, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. By Theorem 1.4, we get:

Theorem 2.1 Suppose that $T \in \overline{HC(H)}$ or $T \in \overline{SC(H)}$. If T is a-isoloid, generalized property (ω') holds for T and $F \in B(H)$ is a finite rank operator commuting with T, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Let weak-$H(p)$ denote the class of all those operates $T \in B(H)$ satisfying the conditions:

1) $\operatorname{asc}(T-\lambda I) < \infty$ for all $\lambda \in \mathbb{C}$;

2) there exists positive integer $p := p(\lambda)$ such that $H_0(T-\lambda I) = N(T-\lambda I)^p$ for all $\lambda \in \operatorname{iso} \sigma(T)$.

Evidently, if $\lambda \in \operatorname{iso} \sigma(T)$, then λ is a pole of the resolvent of T. And weak-$H(p)$ contains the class of operators that satisfy the following property $H(p)$: $H_0(T-\lambda I) = N(T-\lambda I)^p$ for all $\lambda \in \{\mathbb{C}\}$. The class $H(p)$ is large, which contains p-hyponormal operator, M-hyponormal operator, totally $*$-paranormal operator, totally paranormal operator, transaloid operator and so on.

Theorem 2.2 Suppose that $T^* \in$ weak-$H(p)$ and $F \in B(H)$ is a finite rank operator commuting with T, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Proof Suppose $T^* \in$ weak-H(p), we claim that $\sigma_D(T) = \sigma_1(T)$. In fact, we only need to prove $\sigma_D(T) \subseteq \sigma_1(T)$. Let $\lambda_0 \notin \sigma_1(T)$, then there exists $\varepsilon > 0$ such that $T-\lambda I \in SF_+^-(H)$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0<|\lambda-\lambda_0|<\varepsilon$. It follows that $T^*-\bar{\lambda}I$ is lower semi-Fredholm and $\mathrm{ind}(T^*-\bar{\lambda}I) \geq 0$. Thus $\mathrm{ind}(T^*-\bar{\lambda}I)=0$ since $T^* \in$ weak-$H(p)$, which means that $T^*-\bar{\lambda}I$ is Weyl if $0<|\lambda-\lambda_0|<\varepsilon$. By $\mathrm{asc}(T^*-\bar{\lambda}I)<\infty$, we know that $T^*-\bar{\lambda}I$ is Browder, and hence $T-\lambda I$ is Browder too if $0<|\lambda-\lambda_0|<\varepsilon$. Then $N(T-\lambda I) = N(T-\lambda I) \cap \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n] = \{0\}$ (Lemma 3.4 in Ref. [11]), which means that $T-\lambda I$ is invertible. Now we have that $\lambda_0 \in \mathrm{iso}\,\sigma(T) \cup \rho(T)$. Without loss of generality, we suppose that $\lambda_0 \in \mathrm{iso}\,\sigma(T)$, then $\overline{\lambda_0} \in \mathrm{iso}\,\sigma(T^*)$. Since $T^* \in$ weak $-H(p)$, $\overline{\lambda_0}$ is a pole of the resolvent of T^*. So λ_0 is a pole of the resolvent of T, that is, $\lambda_0 \notin \sigma_D(T)$. Thus $\sigma_D(T) = \sigma_1(T)$. By Corollary 1.3, we know that $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$. □

We know that generalized property (ω') may fail for T if $T \in$ weak $-H(p)$.

Theorem 2.3 Suppose that $T \in$ weak $-H(p)$ and $F \in B(H)$ is a finite rank operator commuting with T. If $\sigma_2(T) \subseteq \sigma_a(T)$, then $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$.

Proof Suppose $T \in$ weak $-H(p)$ and $\sigma_2(T) \subseteq \sigma_a(T)$, we claim that $\sigma_D(T) = \sigma_1(T)$. In fact, let $\lambda_0 \notin \sigma_1(T)$, then there exists $\varepsilon > 0$ such that $T-\lambda I \in SF_+^-(H)$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0<|\lambda-\lambda_0|<\varepsilon$. Since $\mathrm{asc}(T-\lambda I)<\infty$, then $N(T-\lambda I) = N(T-\lambda I) \cap \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n] = \{0\}$, which implies that $T-\lambda I$ is bounded below. Thus $T-\lambda I$ is invertible since $\sigma_2(T) \subseteq \sigma_a(T)$. Now we have that $\lambda_0 \in \mathrm{iso}\,\sigma(T) \cup \rho(T)$. Without loss of generality, we suppose that $\lambda_0 \in \mathrm{iso}\,\sigma(T)$. Since $T \in$ weak $-H(p)$, λ_0 is a pole of the resolvent of T. That is $\lambda_0 \notin \sigma_D(T)$. Thus $\sigma_D(T) = \sigma_1(T)$. By Corollary 1.3, we know that $f(T)+F$ satisfies generalized property (ω') for any $f \in H(T)$. □

The next results deal with the generalized property (ω') for quasinilpotent operators. We first recall three well-known results:

1) For any $T \in B(H)$ and a finite rank operator F, then $n(T)<\infty$ if and only if $n(T+F)<\infty$;

2) For any $T \in B(H)$ and a finite rank operator F, then $\sigma_{BW}(T) = \sigma_{BW}(T+F)$ (Proposition 3.3 in Ref. [7]);

3) Let $T \in B(H)$ be such that $n(T)<\infty$. Suppose that there exists an injective quasinilpotent operator $Q \in B(H)$ commuting with T. Then $n(T)=0$ (Lemma 2.11 in Ref. [15]).

If Q is an injective quasinilpotent operator, then Q satisfies generalized property (ω'). In fact, if Q is an injective quasinilpotent operator, then Q^n is an injective quasinilpotent operator for all positive integer n. It follows that $R(Q^n)$ is not closed for all positive integer n, then we have $\sigma(Q) = \sigma_a(Q) = \sigma_{BW}(Q) = \{0\}$, $E^a(Q) = \emptyset$. Thus Q satisfies generalized property (ω').

Theorem 2.4 Suppose that $Q \in B(H)$ is an injective quasinilpotent operator and $F \in B(H)$ is a finite rank operator commuting with Q. Then $Q+F$ satisfies generalized property (ω').

Proof Suppose Q is an injective quasinilpotent operator, then $\sigma_{BW}(Q+F) = \sigma_{BW}(Q) = \{0\}$ and $n(Q)=0$. Thus $n(Q+F)<\infty$. Since Q is an injective quasinilpotent operator commuting with $Q+F$, this implies that $n(Q+F)=0$. Then we claim that $\sigma(Q+F)=\{0\}$. In fact, for any $\lambda \neq 0$, $Q-\lambda I$ is invertible, in particular a Weyl operator, so that $Q+F-\lambda I$ is a Weyl operator, which implies that $n(Q+F-\lambda I)<\infty$. Therefore $n(Q+F-\lambda I)=0$, then $Q+F-\lambda I$ is invertible. So we have that $\sigma(Q+F) = \sigma_{BW}(Q+F) = \{0\}$ and $E^a(Q+F)=\emptyset$, and consequently generalized property (ω') holds for $Q+F$. □

Generally, a non-injective quasinilpotent operator Q may fail to satisfy generalized property (ω'). For example, $Q \in B(l^2)$ is defined by

$$Q(x_1, x_2, x_3, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots\right), \tag{26}$$

then $\sigma(Q) = \sigma_{BW}(Q) = \sigma_a(Q) = \{0\}$ and $E^a(Q) = \{0\}$, these show that Q does not satisfy generalized property (ω').

Theorem 2.5 Suppose that $Q \in B(H)$ is a non-injective quasinilpotent operator and $F \in B(H)$ is a finite rank operator commuting with Q. If Q satisfies generalized property (ω'), then $Q + F$ satisfies generalized property (ω').

Proof Suppose Q is a non-injective quasinilpotent operator, then Q is a-isoloid and $\sigma_2(Q) \subseteq \sigma_a(Q)$. By Theorem 1.3, we know that $Q + F$ satisfies generalized property (ω').

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算子有限秩摄动的广义(ω')性质

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摘 要 广义(ω')性质是 Weyl 型定理的一种新变化.利用由一致 Fredholm 指标算子定义的新谱集，研究了算子 T 摄动有限秩算子后的广义(ω')性质，其中 T 是 a-isoloid 的，并将主要结果应用于几类算子.

关键词 广义(ω')性质，一致 Fredholm 指标算子，谱，Weyl 定理