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Some remarks on a quasilinear elliptic equation with critical exponent*

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Abstract We investigate the following quasilinear elliptic equation:

$$\Delta_p u + u^q + \lambda u^{p^*-1} = 0, \quad u \in W_0^{1,p}(\Omega), \tag{1_λ}$$

where Ω is a bounded domain in R^N with smooth boundary, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $N \geqslant 3$, $2 \leq p < N$, $0 < q < 1$, and $p^* = \frac{Np}{N-p}$. By using variational methods, we obtain a lower bound of the extremal value $\lambda^*(\Omega,p,q)$ for equation (1_λ), which can be explicitly calculated.

Key words quasilinear elliptic equation, critical exponent, Ekeland's variational principle, extremal value

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In this article, we consider the following λ-parameter family of quasilinear elliptic problems:

$$\Delta_p u + u^q + \lambda u^{p^*-1} = 0, \quad u \in W_0^{1,p}(\Omega), \tag{1_λ}$$

where Ω is a bounded domain in R^N with smooth boundary, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $N \geqslant 3$, $2 \leq p < N$, $0 < q < 1$ and $p^* = \frac{Np}{N-p}$.

It is well known that there exists a constant $\lambda^* > 0$ such that problem (1_λ) admits at least two solutions if $\lambda \in (0, \lambda^*)$ and no solutions if $\lambda > \lambda^*$ [1-4]. We are now interested in the dependence of λ^* on Ω, N, p and q (i. e. how large is λ^*?). It is difficult to derive an exact result about λ^* for domains without symmetric properties and few general results are known for this type of estimates except in Gazzola and Malchiodi[5] and our recent papers[6-7]. Here, it must be said that the method of sub and supersolutions does not adapt for dealing with estimates of this kind, since for general Ω (without symmetric property, say) precise information about sub/supersolutions is no longer possible and explicit calculations for λ^* can not be actually carried out.

The energy functional corresponding to problem (1_λ) is the following:

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx - \frac{\lambda}{p^*}\int_\Omega |u|^{p^*}\,dx, \quad u \in W_0^{1,p}(\Omega).$$

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Define

$$T_{p,q} := \frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}}(S_N)^{\frac{p^*-q-1}{p-1-q}}\frac{1}{|\Omega|^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}},$$

where

$$S_N = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\int_\Omega |\nabla u|^p\,dx}{\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{p}{p^*}}}.$$

It is well known that S_N is independent of Ω[8].

We now state the main results.

Theorem A Assume $2\le p<N$, $0<q<1$, $p^*=\dfrac{Np}{N-p}$ and $T_{p,q}$ is defined as above. Then for all $\lambda\in(0,T_{p,q})$, problem (1_λ) admits at least one solution in $W_0^{1,p}(\Omega)$.

Theorem B Let λ^* be the extremal value for problem (1_λ), we have

$$\lambda^*(\Omega,p,q) > T_{p,q}.$$

1 Some preliminary results

Define $\Lambda_\lambda=\{u\in W_0^{1,p}(\Omega)\mid\langle I'_\lambda(u),u\rangle=0\}$ and we divide Λ_λ into three parts as follows:

$$\Lambda_\lambda^+ = \{u\in\Lambda_\lambda\mid (p-1-q)\int_\Omega|\nabla u|^p\,dx-\lambda(p^*-q-1)\int_\Omega|u|^{p^*}\,dx>0\};$$

$$\Lambda_\lambda^0 = \{u\in\Lambda_\lambda\mid (p-1-q)\int_\Omega|\nabla u|^p\,dx-\lambda(p^*-q-1)\int_\Omega|u|^{p^*}\,dx=0\};$$

$$\Lambda_\lambda^- = \{u\in\Lambda_\lambda\mid (p-1-q)\int_\Omega|\nabla u|^p\,dx-\lambda(p^*-q-1)\int_\Omega|u|^{p^*}\,dx<0\}.$$

Proposition 1.1 Let $\lambda<T_{p,q}$, then $\Lambda_\lambda^\pm\neq\phi$ and $\Lambda_\lambda^0=\{0\}$.

Proof For any $u\in W_0^{1,p}(\Omega)\setminus\{0\}$, define

$$\varphi(t) = t^{p-p^*}\int_\Omega|\nabla u|^p\,dx - t^{q+1-p^*}\int_\Omega|u|^{q+1}\,dx,\quad t\in(0,+\infty).$$

It is easily verified that

$$\max_{t\in(0,\infty)}\varphi(t) = \frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\int_\Omega|u|^{q+1}\,dx\right)^{\frac{p^*-p}{p-1-q}}}.$$

If $\lambda<T_{p,q}$, we apply the Hölder inequality and the Sobolev inequality to conclude

$$\begin{aligned}
&\max_{t\in(0,\infty)}\varphi(t)-\lambda\int_\Omega|u|^{p^*}\,dx\\
&=\frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\int_\Omega|u|^{q+1}\,dx\right)^{\frac{p^*-p}{p-1-q}}}-\lambda\int_\Omega|u|^{p^*}\,dx\\
&\geqslant\frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\|u\|_{p^*}^{q+1}|\Omega|^{\frac{p^*-q-1}{p^*}}\right)^{\frac{p^*-p}{p-1-q}}}-\lambda\int_\Omega|u|^{p^*}\,dx\\
&=\left[\frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\|u\|_{p^*}^{\frac{p(p^*-q-1)}{p-1-q}}|\Omega|^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}}-\lambda\right]\int_\Omega|u|^{p^*}\,dx
\end{aligned}$$

$$\geqslant (T_{p,q} - \lambda)\int_\Omega |u|^{p^*} dx > 0.$$

Consequently there exist two and only two positive numbers denoted by $t^- = t^-(u)$ and $t^+ = t^+(u)$ such that $\varphi(t^-) = \varphi(t^+) = \lambda\int_\Omega |u|^{p^*} dx$ and $\varphi'(t^-) > 0 > \varphi'(t^+)$, i.e. $t^+(u)u \in \Lambda_\lambda^-$ and $t^-(u)u \in \Lambda_\lambda^+$.

It remains to show that $\Lambda_\lambda^0 = \{0\}$. Let us argue by contradiction and assume $\exists u_0 \in \Lambda_\lambda^0$ and $u_0 \neq 0$. By the definition of Λ_λ^0, we have

$$(p-1-q)\int_\Omega |\nabla u_0|^p dx - \lambda(p^* - q - 1)\int_\Omega |u_0|^{p^*} dx = 0.$$

Then

$$0 = \int_\Omega |\nabla u_0|^p dx - \int_\Omega |u_0|^{q+1} dx - \lambda\int_\Omega |u_0|^{p^*} dx$$

$$= \frac{p^* - p}{p^* - q - 1}\int_\Omega |\nabla u_0|^p dx - \int_\Omega |u_0|^{q+1} dx.$$

Therefore, we have

$$[T_{p,q} - \lambda]\int_\Omega |u_0|^{p^*} dx$$

$$\leq \left[\frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}} \frac{\left(\int_\Omega |\nabla u_0|^p dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\|u_0\|_{p^*}^{\frac{p(p^*-q-1)}{p-1-q}} |\Omega|^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}} - \lambda\right]\int_\Omega |u_0|^{p^*} dx$$

$$= \frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}} \frac{\left(\int_\Omega |\nabla u_0|^p dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\|u_0\|_{p^*}^{q+1} |\Omega|^{\frac{p^*-q-1}{p^*}}\right)^{\frac{p^*-p}{p-1-q}}} - \lambda\int_\Omega |u_0|^{p^*} dx$$

$$\leq \frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}} \frac{\left(\int_\Omega |\nabla u_0|^p dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\int_\Omega |u_0|^{q+1} dx\right)^{\frac{p^*-p}{p-1-q}}} - \lambda\int_\Omega |u_0|^{p^*} dx$$

$$= \frac{p-q-1}{p^*-q-1}\left(\frac{p^*-p}{p^*-q-1}\right)^{\frac{p^*-p}{p-1-q}} \frac{\left(\int_\Omega |\nabla u_0|^p dx\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\frac{p^*-p}{p^*-q-1}\int_\Omega |\nabla u_0|^p dx\right)^{\frac{p^*-p}{p-1-q}}} - \frac{p-1-q}{p^*-q-1}\int_\Omega |\nabla u_0|^p dx$$

$$= 0.$$

From the inequality above, we get $u_0 = 0$. This is a contradiction. □

Proposition 1.2 Let $\lambda < T_{p,q}$, we have the following estimates:

$$\|\nabla U\|_p^{p^*-p} > B(\lambda) = \frac{1}{\lambda}\frac{p-1-q}{p^*-q-1}(S_N)^{\frac{p^*}{p}}, \quad \forall U \in \Lambda_\lambda^- \tag{1}$$

$$\|\nabla u\|_p^{p^*-p} < B(0) = \left(\frac{p^*-q-1}{p^*-p}\right)^{\frac{p^*-p}{p-q-1}} \frac{1}{(S_N)^{\frac{(q+1)(p^*-p)}{p(p-q-1)}}} |\Omega|^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}, \quad \forall u \in \Lambda_\lambda^+ \tag{2}$$

Moreover, $B(\lambda) > B(0)$ for all $\lambda \in \Lambda(0, T_{p,q})$.

Proof Let $U \in \Lambda_\lambda^-$, by the definition of Λ_λ^-, we have

$$(p-1-q)\|\nabla U\|_p^p < \lambda(p^*-q-1)\|U\|_{p^*}^{p^*}. \tag{3}$$

From the Sobolev inequality, we derive

$$(p-1-q)\|\nabla U\|_p^p < \lambda(p^*-q-1)\|\nabla U\|_p^{p^*}\frac{1}{(S_N)^{\frac{p^*}{p}}}. \tag{4}$$

Thus

$$\| \nabla U \|_p^{p^*-p} > B(\lambda) = \frac{1}{\lambda}\frac{p-1-q}{p^*-q-1}(S_N)^{\frac{p^*}{p}}. \tag{5}$$

Let $u \in \Lambda_\lambda^+$, by the definition of Λ_λ and Λ_λ^+, we have

$$\lambda \| u \|_{p^*}^{p^*} = \| \nabla u \|_p^p - \int_\Omega | u |^{q+1} dx; \tag{6}$$

$$(p-q-1) \| \nabla u \|_p^p - \lambda(p^*-q-1)\int_\Omega | u |^{p^*} dx > 0. \tag{7}$$

Then

$$\begin{aligned}&(p^*-q-1)\int_\Omega | u |^{q+1} dx - (p^*-p) \| \nabla u \|_p^p \\ &= (p-q-1) \| \nabla u \|_p^p - (p^*-q-1)(\| \nabla u \|_p^p - \int_\Omega | u |^{q+1} dx) \\ &= (p-q-1) \| \nabla u \|_p^p - \lambda(p^*-q-1)\int_\Omega | u |^{p^*} dx \\ &> 0.\end{aligned} \tag{8}$$

By the Hölder inequality and the Sobolev inequality, we obtain

$$(p^*-p) \| \nabla u \|_p^p < (p^*-q-1) \frac{\| \nabla u \|_p^{q+1}}{(S_N)^{\frac{q+1}{p}}} | \Omega |^{\frac{p^*-q-1}{p^*}}. \tag{9}$$

From (9), we get

$$\| \nabla u \|_p^{p^*-p} < B(0) = \left(\frac{p^*-q-1}{p^*-p}\right)^{\frac{p^*-p}{p-q-1}} \frac{1}{(S_N)^{\frac{(q+1)(p^*-p)}{p(p-q-1)}}} | \Omega |^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}. \tag{10}$$

It is easily verified that $T_{p,q} = \frac{\lambda B(\lambda)}{B(0)}$. Therefore, $\frac{B(\lambda)}{B(0)} = \frac{T_{p,q}}{\lambda} > 1$, $\forall \lambda \in (0, T_{p,q})$, that is $B(\lambda) > B(0)$ for all $\lambda \in (0, T_{p,q})$.

Proposition 1.3 Let $0 < \lambda < T_{p,q}$, then $\Lambda_\lambda^+ \cup \Lambda_\lambda^0$ and Λ_λ^- are both closed in $W_0^{1,p}(\Omega)$.

Proof For any $u_n \to u_0$ strongly in $W_0^{1,p}(\Omega)$ with $\{u_n\} \subset \Lambda_\lambda^+ \cup \Lambda_\lambda^0$, it follows that $u_0 \in \Lambda_\lambda$ and for all $n \in N^+$, we have

$$(p-1-q)\int_\Omega | \nabla u_n |^p dx - \lambda(p^*-q-1)\int_\Omega | u_n |^{p^*} dx \geqslant 0.$$

Passing to the limit as $n \to \infty$, we conclude that

$$(p-1-q)\int_\Omega | \nabla u_0 |^p dx - \lambda(p^*-q-1)\int_\Omega | u_0 |^{p^*} dx \geqslant 0.$$

Thus, u_0 belongs to $\Lambda_\lambda^+ \cup \Lambda_\lambda^0$.

For any $U_n \to U_0$ strongly in $W_0^{1,p}(\Omega)$ with $\{U_n\} \subset \Lambda_\lambda^-$, it follows that $U_0 \in \Lambda_\lambda$. By Proposition 1.2, $\| \nabla U_0 \|_p^{p^*-p} > B(\lambda) > B(0) > \| \nabla u \|_p^{p^*-p}$, $\forall u \in \Lambda_\lambda^+$, provided $0 < \lambda < T_{p,q}$. Therefore, U_0 does not belong to Λ_λ^+. By Proposition 1.1, U_0 does not belong to Λ_λ^0. In turn, it follows that U_0 belongs to Λ_λ^-. □

Proposition 1.4 Given $u \in \Lambda_\lambda^+$, there exists $\varepsilon_0 > 0$ and a differentiable functional $f = f(\omega) > 0$, $\omega \in W_0^{1,p}(\Omega)$, $\| \omega \| < \varepsilon_0$ satisfying the following:

$$f(0) = 1, f(\omega)(u+\omega) \in \Lambda_\lambda^+, \ \forall \omega \in W_0^{1,p}(\Omega), \| \omega \| < \varepsilon_0.$$

And

$$f'(0)\varphi = \frac{-p\int_\Omega | \nabla u |^{p-2} \nabla u \nabla \varphi \, dx + (1+q)\int_\Omega | u |^q \operatorname{sgn}(u)\varphi \, dx + \lambda p^* \int_\Omega | u |^{p^*-2} u\varphi \, dx}{(1-q) \| \nabla u \|_p^p - \lambda(p^*-1-q)\int_\Omega | u |^{p^*} dx}.$$

Proof Let $u \in \Lambda_\lambda^+$. Define G: $W_0^{1,p}(\Omega) \times R^+ \to R$ as follows:

$$G(\omega,t) = t^p\int_\Omega |\nabla(u+\omega)|^p\,dx - t^{1+q}\int_\Omega |u+\omega|^{q+1}\,dx - \lambda t^{p^*}\int_\Omega |u+\omega|^{p^*}\,dx.$$

It is obvious that $G(0,1) = 0$ and

$$G_t(0,1) = (p-1-q)\int_\Omega |\nabla u|^p\,dx - \lambda(p^*-q-1)\int_\Omega |u|^{p^*}\,dx > 0.$$

Then we can apply the implicit function theorem at the point $(0,1)$ and obtain $\varepsilon_0 > 0$ and a differentiable functional $f = f(\omega)$, $\omega \in W_0^{1,p}(\Omega)$, $\|\omega\| < \varepsilon_0$ satisfying that

$$f(0) = 1, f(\omega)(u+\omega) \in \Lambda_\lambda^+,\ \forall\ \omega \in W_0^{1,p}(\Omega),\ \|\omega\| < \varepsilon_0.$$ □

2 Proof of the Theorems

Proof of Theorem A For every $u \in \Lambda_\lambda$, we can easily obtain

$$I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |\nabla u|^p\,dx - \left(\frac{1}{q+1} - \frac{1}{p^*}\right)\int_\Omega |u|^{1+q}dx,$$

Clearly, $I_\lambda(u)$ is coercive on Λ_λ. Thus, $I_\lambda(\Lambda_\lambda)$ has a lower bound and $\inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda$ are finite.

From Ekeland's variational principle (see Theorem 4.8.1 in Ref. [9]), there exists a sequence $\{u_n\} \subset \Lambda_\lambda^+ \cup \Lambda_\lambda^0$ with the following properties:

1) $I_\lambda(u_n) \le \inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda + \frac{1}{n}$;

2) $I_\lambda(\omega) \geq I_\lambda(u_n) - \frac{1}{n}\|u_n - \omega\|$, $\forall\ \omega \in \Lambda_\lambda^+ \cup \Lambda_\lambda^0$.

Since $2 \le p < p^*$, we derive

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{q+1}\right)\|\nabla u\|_p^p + \lambda\left(\frac{1}{q+1} - \frac{1}{p^*}\right)\int_\Omega |u|^{p^*}\,dx \\ &< -\frac{1}{p(1+q)}\left[(p-q-1)\|\nabla u\|_p^p - \lambda(p^*-q-1)\int_\Omega |u|^{p^*}\,dx\right] < 0,\ \forall\ u \in \Lambda_\lambda^+. \end{aligned}$$

Therefore, $\inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda = \inf_{\Lambda_\lambda^+} I_\lambda < 0$. Thus, $I_\lambda(u_n) < 0$ for n large enough and we can assume $u_n \in \Lambda_\lambda^+$. Since $I_\lambda(|u|) = I_\lambda(u)$, we can assume that $u_n > 0$. The coercivity of I_λ implies that $\{u_n\}$ is bounded. Going if necessary to a subsequence, we can assume $u_n \rightharpoonup u_\lambda$ weakly in $W_0^{1,p}(\Omega)$ and pointwise a. e. in Ω. Let $g_n = u_n - u_\lambda$, then $g_n \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$. By the compactness of the embedding $W_0^{1,p}(\Omega) \to L^{1+q}(\Omega)$, we have

$$\int_\Omega |u_n|^{1+q}dx \to \int_\Omega |u_\lambda|^{1+q}dx;$$

$$\int_\Omega |g_n|^{1+q}dx \to 0.$$

We divide the arguments below into three steps.

Step 1 $u_\lambda \not\equiv 0$.

On the contrary, we assume that $u_\lambda \equiv 0$. Then $g_n \in \Lambda_\lambda^+$ and $I_\lambda(g_n) \to \inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda$. That is

$$0 < (p-q-1)\|\nabla g_n\|_p^p - \lambda(p^*-q-1)\int_\Omega |g_n|^{p^*}dx + o(1), \tag{11}$$

and

$$\frac{1}{p}\|\nabla g_n\|_p^p - \frac{\lambda}{p^*}\int_\Omega |g_n|^{p^*}dx = \inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda + o(1), \tag{12}$$

which leads to the following contradiction:

$$\begin{aligned} 0 &< \frac{p(p-q-1) - p^*(p^*-q-1)}{p}\|\nabla g_n\|_p^p + p^*(p^*-q-1)\inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda + o(1) \\ &\le p^*(p^*-q-1)\inf_{\Lambda_\lambda^+\cup\Lambda_\lambda^0} I_\lambda + o(1) < 0. \end{aligned}$$

Therefore, $u_\lambda \not\equiv 0$.

Step 2

$$\liminf_{n\to\infty}(p^* - p)\|\nabla u_n\|_p^p < (p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x. \tag{13}$$

Since for $\{u_n\} \subset \Lambda_\lambda^+$, we have:

$$(p^* - q - 1)\|\nabla u_n\|_p^p = (p^* - q - 1)\int_\Omega |u_n|^{1+q}\mathrm{d}x + \lambda(p^* - q - 1)\int_\Omega |u_n|^{p^*}\mathrm{d}x. \tag{14}$$

Then

$$(p^* - p)\|\nabla u_n\|_p^p - (p^* - q - 1)\int_\Omega |u_n|^{1+q}\mathrm{d}x$$
$$= -\left[(p - 1 - q)\|\nabla u_n\|_p^p - \lambda(p^* - q - 1)\int_\Omega |u_n|^{p^*}\mathrm{d}x\right] < 0.$$

Therefore,

$$\liminf_{n\to\infty}(p^* - p)\|\nabla u_n\|_p^p \le (p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x.$$

It remains to show the inequality above strictly holds. Let us argue by contradiction and assume

$$\liminf_{n\to\infty}(p^* - p)\|\nabla u_n\|_p^p = (p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x .$$

Then

$$(p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x \ge \limsup_{n\to\infty}(p^* - p)\|\nabla u_n\|_p^p$$
$$\ge \liminf_{n\to\infty}(p^* - p)\|\nabla u_n\|_p^p = (p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x.$$

That is

$$(p^* - p)\|\nabla u_n\|_p^p \to (p^* - q - 1)\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x,$$

as $n \to \infty$, which gives:

$$\lambda\int_\Omega |u_n|^{p^*}\mathrm{d}x = \|\nabla u_n\|_p^p - \int_\Omega |u_n|^{1+q}\mathrm{d}x \to \frac{p - q - 1}{p^* - p}\int_\Omega |u_\lambda|^{1+q}\mathrm{d}x,$$

as $n \to \infty$.

Therefore, we apply the Sobolev inequality and the Hölder inequality to conclude

$$[T_{p,q} - \lambda]\int_\Omega |u_n|^{p^*}\mathrm{d}x$$
$$\le\left[\frac{p - q - 1}{p^* - q - 1}\left(\frac{p^* - p}{p^* - q - 1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{p^*-q-1}{p-1-q}}}{\|u_n\|_{p^*}^{\frac{p(p^*-q-1)}{p-1-q}}|\Omega|^{\frac{(p^*-p)(p^*-q-1)}{p^*(p-1-q)}}} - \lambda\right]\int_\Omega |u_n|^{p^*}\,\mathrm{d}x$$
$$=\frac{p - q - 1}{p^* - q - 1}\left(\frac{p^* - p}{p^* - q - 1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\|u_n\|_{p^*}^{q+1}|\Omega|^{\frac{p^*-q-1}{p^*}}\right)^{\frac{p^*-p}{p-1-q}}} - \lambda\int_\Omega |u_n|^{p^*}\,\mathrm{d}x$$
$$\le\frac{p - q - 1}{p^* - q - 1}\left(\frac{p^* - p}{p^* - q - 1}\right)^{\frac{p^*-p}{p-1-q}}\frac{\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{p^*-q-1}{p-1-q}}}{\left(\int_\Omega |u_n|^{q+1}\,\mathrm{d}x\right)^{\frac{p^*-p}{p-1-q}}} - \lambda\int_\Omega |u_n|^{p^*}\,\mathrm{d}x$$
$$=\frac{p - q - 1}{p^* - p}\left(\frac{p^* - p}{p^* - q - 1}\right)^{\frac{p^*-q-1}{p-1-q}}\frac{\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{p^*-q-1}{p-q-1}}}{\left(\int_\Omega |u_n|^{1+q}\mathrm{d}x\right)^{\frac{p^*-p}{p-1-q}}} - \lambda\int_\Omega |u_n|^{p^*}\mathrm{d}x \to 0.$$

It implies that $u_n \to 0$ strongly in L^{p^*} and hence $u_\lambda \equiv 0$. This is a contradiction.

Step 3　u_λ is a solution of eq. (1_λ).

By Step 2, there exists a constant $C > 0$ independent of n such that a subsequence of $\{u_n\}$ (still called $\{u_n\}$) satisfying the following inequality:

$$(p^* - p)\|\nabla u_n\|_p^p - (p^* - q - 1)\int_\Omega |u_n|^{1+q}dx < -C. \tag{15}$$

By Proposition 1.4, there exist a suitable functional $f(u_n)$ corresponding to each u_n such that

$$f(\omega)(u_n + \omega) \in \Lambda_\lambda^+,\ \forall \omega \in W_0^{1,p}(\Omega),\ \|\omega\| < \varepsilon_n.$$

Hence, for each $\phi \in W_0^{1,p}(\Omega)$ and $t \in \left(0, \dfrac{\varepsilon_n}{\|\phi\|}\right)$,

$$\begin{aligned}
&\frac{1}{n}[|f_n(t\phi) - 1|\|u_n\| + tf_n(t\phi)\|\phi\|] \geqslant \frac{1}{n}\|f_n(t\phi)(u_n + t\phi) - u_n\| \\
&\geqslant I_\lambda[u_n] - I_\lambda[f_n(t\phi)(u_n + t\phi)] \\
&= \frac{1}{p}\|\nabla u_n\|_p^p - \frac{1}{1+q}\int_\Omega |u_n|^{1+q}dx - \frac{\lambda}{p^*}\int_\Omega |u_n|^{p^*}dx - \frac{1}{p}[f_n(t\phi)]^p\|\nabla(u_n + t\phi)\|_p^p + \\
&\qquad \frac{1}{1+q}[f_n(t\phi)]^{1+q}\int_\Omega |u_n + t\phi|^{1+q}dx + \frac{\lambda}{p^*}[f_n(t\phi)]^{p^*}\int_\Omega |u_n + t\phi|^{p^*}dx \\
&= -\left[\frac{[f_n(t\phi)]^p - 1}{p}\right]\|\nabla(u_n + t\phi)\|_p^p - \frac{1}{p}[\|\nabla(u_n + t\phi)\|_p^p - \|\nabla u_n\|_p^p] + \\
&\qquad \left[\frac{[f_n(t\phi)]^{1+q} - 1}{1+q}\right]\int_\Omega |u_n + t\phi|^{1+q}dx + \frac{1}{1+q}\left[\int_\Omega |u_n + t\phi|^{1+q}dx - \int_\Omega |u_n|^{1+q}dx\right] + \\
&\qquad \lambda\left[\frac{[f_n(t\phi)]^{p^*} - 1}{p^*}\right]\int_\Omega |u_n + t\phi|^{p^*}dx + \frac{\lambda}{p^*}\left[\int_\Omega |u_n + t\phi|^{p^*}dx - \int_\Omega |u_n|^{p^*}dx\right].
\end{aligned}$$

Dividing by $t > 0$ and passing to the limit as $t \to 0$, we derive

$$\begin{aligned}
&\frac{1}{n}[|f'_n(0)\phi|\|u_n\| + \|\phi\|] \\
&\geqslant -[f'_n(0)\phi]\left[\int_\Omega |\nabla u_n|^p dx - \int_\Omega |u_n|^{q+1}dx - \lambda\int_\Omega |u_n|^{p^*}dx\right] - \\
&\qquad \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi\, dx + \int_\Omega (u_n)^q\phi\, dx + \lambda\int_\Omega (u_n)^{p^*-1}\phi\, dx \\
&= -\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi\, dx + \int_\Omega (u_n)^q\phi\, dx + \lambda\int_\Omega (u_n)^{p^*-1}\phi\, dx.
\end{aligned} \tag{16}$$

By Proposition 1.4, we have

$$\begin{aligned}
f'_n(0)\phi &= \frac{-p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\, dx + (1+q)\int_\Omega |u_n|^q \operatorname{sgn}(u_n)\phi\, dx + \lambda p^*\int_\Omega |u_n|^{p^*-2}u_n\phi\, dx}{(1-q)\|\nabla u_n\|_p^p - \lambda(p^* - 1 - q)\int_\Omega |u_n|^{p^*}dx} \\
&= \frac{p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\, dx - (1+q)\int_\Omega |u_n|^q \operatorname{sgn}(u_n)\phi\, dx - \lambda p^*\int_\Omega |u_n|^{p^*-2}u_n\phi\, dx}{(p^* - p)\|\nabla u_n\|_p^p - (p^* - q - 1)\int_\Omega |u_n|^{1+q}dx}.
\end{aligned}$$

Thus, by the boundedness of u_n and (15), we have

$$|f'_n(0)\phi| \leq C_1,$$

where C_1 is a positive constant independent of n.

Therefore, from (16), we obtain

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\, dx - \int_\Omega (u_n)^q\phi\, dx - \lambda\int_\Omega (u_n)^{p^*-1}\phi\, dx \geqslant -\frac{C_2}{n},$$

and passing to the limit as $n \to \infty$, we conclude that

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda\nabla\phi\,dx-\int_\Omega (u_\lambda)^q\phi\,dx-\lambda\int_\Omega (u_\lambda)^{p^*-1}\phi\,dx\geq 0,\ \forall\phi\in W_0^{1,p}(\Omega).$$

That is, u_λ is a weak solution of eq. (1_λ) with $\lambda<T_{p,q}$. The proof of Theorem A is completed. □

Proof of Theorem B From the definition of $\lambda^*(\Omega,p,q)$ and Theorem A, it is obvious that $\lambda^*(\Omega,p,q)>T_{p,q}$.

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一个含临界指数的拟线性椭圆型方程的注记

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摘 要 研究了如下的拟线性椭圆型方程：

$$\Delta_p u+u^q+\lambda u^{p^*-1}=0,\quad u\in W_0^{1,p}(\Omega),\qquad (1_\lambda)$$

其中，Ω 是 R^N 中具有光滑边界的有界区域，$\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$，$N\geq 3$，$2\leq p<N$，$0<q<1$，$p^*=\frac{Np}{N-p}$. 设 $\lambda^*(\Omega,p,q)$ 是拟线性椭圆型方程(1_λ)可解的参数集的上确界. 运用变分方法，在不要求具有对称性质的一般区域 Ω 上得到了 $\lambda^*(\Omega,p,q)$ 的一个可以精确计算的下界.

关键词 拟线性椭圆型方程，临界指数，Ekeland 变分原理，参数计算