Article ID: 1002-1175(2011)02-0131-10

Holomorphic 2-spheres in a complex Grassmann manifold $G(2,5)$ *

FEI Jie†, JIAO Xiao-Xiang

(*School of Mathematics, Graduate University, Chinese Academy of Sciences, Beijing* 100049, *China*)

(Received 21 June 2010; Revised 14 July 2010)

Fei J, Jiao X X. Holomorphic 2-spheres in a complex Grassmann manifold $G(2,5)$ [J]. Journal of the Graduate School of the Chinese Academy of Sciences, 2011, 28(2): 131 – 140.

Abstract We study the linearly full holomorphic 2-spheres in a complex Grassmann manifold $G(2, 5)$ by using harmonic sequence and moving frames. We construct some examples of homogeneous holomorphic 2-spheres in $G(2, 5)$ by applying the irreducible unitary representations of $SU(2)$. Then, we determine all linearly full degenerate holomorphic 2-spheres with constant Gaussian curvatures of 2/3 and 4/3, up to $U(5)$ equivalence. Moreover, we prove that all non-degenerate holomorphic 2-spheres with constant Gaussian curvature of 4/3 must be $U(5)$ equivalent under some conditions.

Key words holomorphic 2-sphere, Gaussian curvature, complex Grassmann manifold, harmonic sequence

CLC O186.1

Let f be a harmonic map from a Riemann surface M into a complex Grassmann manifold $G(k, n)$. Then by using the ∂-transform associated to the map f, one can obtain the following sequence of harmonic maps[1]

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \longrightarrow \cdots \tag{1}$$

where $f_{j+1} \equiv \partial f_j$ for $j = 0, 1, 2, \cdots$ and $f_j: M \to G(k_j, n)$ are harmonic maps. We call k_j the rank of f_j and define $k_{j+1} = 0$ if f_j is anti-holomorphic. The map f_j is called non-degenerate (resp. degenerate) when $k_j = k_{j+1}$ (resp. $k_j > k_{j+1}$). When f is holomorphic, the sequence (1) is orthogonal and therefore is finite. In this case, the sequence is called a pseudo-holomorphic sequence and f_j is called a pseudo-holomorphic map generated by f.

In 1989, Chi and Zheng[2] classified all holomorphic curves from 2-spheres into $G(2, 4)$ whose curvature is equal to 2 into two families, up to unitary equivalence, in which none of the curves is congruent. Xu and Jiao[3] studied the linearly full holomorphic 2-spheres into $G(2, 4)$ and gave several pinching theorems for the Gaussian curvature. Furthermore, they determined all holomorphic 2-spheres with constant Gaussian curvature of 1 in $G(2, 4)$ up to $U(4)$ equivalence.

In 2004, Jiao and Peng[4] classified all linearly full holomorphic 2-spheres in $G(2, 5)$ with the induced Gaussian curvatures $K = 4, 2, 4/3, 1$ and $4/5$ into some classes, up to unitary equivalence. Recently, Jiao

* Supported by the NSFC (11071248), and the Knowledge Innovation Program of the Chinese Academy of Sciences (KJCX3-SYW-SO3)

† E-mail: feijie311@163.com

proved that if f is a linearly full totally unramified pseudo-holomorphic curve with constant Gaussian curvature K in $G(2, 5)$ and f is non-± holomorphic, then K is either 1/2 or 4/5[5].

In this paper, we will use harmonic sequence and moving frames to discuss the holomorphic 2-spheres into a complex Grassmann manifold $G(2, 5)$. Preliminaries are given in section 1 and section 2. In section 2, we construct some examples of homogeneous holomorphic 2-spheres into $G(2, 5)$ by making use of the unitary representations of the 3-dimensional special unitary group $SU(2)$.

In section 3, we give a pinching theorem about the Gaussian curvature K of a degenerate holomorphic immersion from S^2 to $G(2, 5)$ (see Theorem 3.1). Furthermore, we determine all linearly full degenerate holomorphic 2-spheres with constant Gaussian curvatures of 2/3 and 4/3, up to $U(5)$ equivalence (see Theorem 3.2).

In section 4, we prove that all non-degenerate holomorphic 2-spheres with constant Gaussian curvature 4/3 must be $U(5)$ equivalent under some conditions (see Proposition 4.1).

1 Geometry of minimal surfaces in complex Grassmannians

We begin to give a description of the geometry of minimal surfaces in complex Grassmann manifolds[1].

Let $G(k, n)$ be the Grassmann manifold of all k-dimensional subspaces in a complex number space $\mathbb{C}^n$ of dimension n. In particular, $G(1, n)$ is just the complex projective space $\mathbb{C}P^{n-1}$.

We will use the following ranges of indices in this section.

$$1 \leqslant A, B, C \cdots \leqslant n, \ 1 \leqslant i, j, l \cdots \leqslant k, \ k+1 \leqslant \alpha, \beta, \gamma \cdots \leqslant n.$$

A unitary frame of $\mathbb{C}^n$ consists of an ordered set of unitary basis $e = (e_1, \cdots, e_n)$ of $\mathbb{C}^n$. The space of unitary frames can be identified with the unitary group $U(n)$. Let ω_{AB} be the Maurer-Cartan forms of $U(n)$. They are skew-hermitian and satisfy the Maurer-Cartan structure equations of $U(n)$, i. e.

$$\omega_{AB} + \overline{\omega}_{BA} = 0, \tag{2}$$

$$d\omega_{AB} = -\omega_{AC} \wedge \omega_{CB}. \tag{3}$$

An element of $G(k, n)$ can be defined by the multivector $e_1 \wedge e_2 \cdots \wedge e_k \neq 0$, defined up to a factor. The vectors $\{e_i\}$ and their orthogonal vectors $\{e_\alpha\}$ are defined up to a transformation of $U(k)$ and $U(n-k)$ respectively, so that $G(k, n)$ has a G-structure with $G = U(k) \times U(n-k)$. The Kaehler metric on $G(k, n)$ is defined by

$$ds^2 = \sum_{\alpha, i} \omega_{\alpha i} \cdot \overline{\omega}_{\alpha i}. \tag{4}$$

When $k = 1$, this is just the Fubini-Study metric on $\mathbb{C}P^{n-1}$ of constant holomorphic curvature 4.

Let M be an oriented surface and $f: M \to G(k, n)$ an immersion. Then M acquires an induced Riemannian metric

$$ds_M^2 = f^* ds^2 = \varphi\overline{\varphi}, \tag{5}$$

where φ is a complex-valued one form, defined up to a factor of norm one. The structure equations of M with respect to the induced metric can be written as

$$d\varphi = -\rho \wedge \varphi, \tag{6}$$

$$d\rho = \frac{K}{2}\varphi \wedge \overline{\varphi}, \tag{7}$$

where ρ is the complex connection form and K is the Gaussian curvature. Choose a local unitary frame $e = (e_1, \cdots, e_n)$ along f such that $e_1, e_2, \cdots, e_k$ span $f(x)$ and still denote ω_{AB} the entries of the pull back of the Maurer- Cartan form of $U(n)$ via e. Restricted to M, we set

$$\omega_{\alpha i} = a_{\alpha i}\varphi + b_{\alpha i}\overline{\varphi}. \tag{8}$$

The map f is holomorphic if and only if $b_{\alpha i} = 0$. The isometric condition (5) implies

$$\sum_{\alpha, i} a_{\alpha i} b_{\alpha i} = 0, \tag{9}$$

$$\sum_{\alpha, i} (a_{\alpha i}\bar{a}_{\alpha i} + b_{\alpha i}\bar{b}_{\alpha i}) = 1. \tag{10}$$

The harmonicity condition for f is

$$Da_{\alpha i} := \mathrm{d}a_{\alpha i} + a_{\beta i}\omega_{\alpha\beta} - a_{\alpha j}\omega_{ji} - a_{\alpha i}\rho \equiv 0 \bmod \varphi, \tag{11}$$

which is equivalent to

$$Db_{\alpha i} := \mathrm{d}b_{\alpha i} + b_{\beta i}\omega_{\alpha\beta} - b_{\alpha j}\omega_{ji} + b_{\alpha i}\rho \equiv 0 \bmod \varphi. \tag{12}$$

The quadratic differential form

$$\prod_{\alpha, i}^{\mathbb{C}} = Da_{\alpha i}\varphi + Db_{\alpha i}\bar{\varphi} \tag{13}$$

is called the complex second fundamental form of the map f. The map f is called totally geodesic if $\prod_{\alpha,i}^{\mathbb{C}} = 0$ for all α and i.

The following Lemma[6] will be frequently used in this paper.

Lemma 1.1 Let $(M, \mathrm{d}s_M^2)$ be a surface and z a conformal coordinate on some open subset U of M. Let u be a smooth complex valued function and ω a purely imaginary 1-form on U. Assume

$$\mathrm{d}u \equiv u\omega \bmod \mathrm{d}z.$$

Then

$$\Delta \log |u| \varphi \wedge \bar{\varphi} = 2\mathrm{d}\omega,$$

where $\varphi = \lambda \mathrm{d}z$ as above and Δ the Laplacian of M.

2 Construction of some homogeneous holomorphic 2-spheres in $G(2, 5)$

In this section, we review some results on irreducible unitary representations of the 3-dimensional special unitary group $SU(2)$. We will use some notation in Ref. [7]. $SU(2)$ can be defined by

$$SU(2) = \left\{ g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

The Lie algebra $su(2)$ of $SU(2)$ is given by

$$su(2) = \left\{ X = \begin{pmatrix} \sqrt{-1}x & -\bar{y} \\ y & -\sqrt{-1}x \end{pmatrix}; x \in \mathbb{R}, y \in \mathbb{C} \right\}.$$

A basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $su(2)$ is given by

$$\varepsilon_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Put $T = \{\exp(t\varepsilon_1); t \in \mathbb{R}\}$ and we have $S^2 = SU(2)/T$.

Let V_n be the representation space of $SU(2)$, which is an $(n+1)$-dimensional complex vector space of all complex homogeneous polynomials of degree n in two variables z_0 and z_1. The standard irreducible representation ρ_n of $SU(2)$ on V_n is defined by

$$\rho_n(g)f(z_0, z_1) := f(az_0 + bz_1, -\bar{b}z_0 + \bar{a}z_1) \tag{14}$$

for $g \in SU(2)$ and $f \in V_n$. If we view elements of V_n as polynomial functions of $S^3 = \{(z_0, z_1) \in \mathbb{C}^2; |z_0|^2 + |z_1|^2 = 1\}$, we can define a $SU(2)$-invariant Hermitian inner product $(\ ,\ )$ on V_n as follows

$$(f, h) := \frac{(n+1)!}{2\pi^2}\int_{S^3} f \cdot \bar{h}\, \mathrm{d}v,$$

where $h \in V_n$ and $\mathrm{d}v$ is the volume element of S^3. It is easy to check that $\{u_k^{(n)}\}$ defined by

$$u_k^{(n)} := \frac{1}{\sqrt{k!(n-k)!}} z_0^{n-k} z_1^k, \quad 0 \leqslant k \leqslant n$$

is a unitary basis for V_n. Since $\rho_n(g) u_k^{(n)} \in V_n$ we can write

$$\rho_n(g) u_k^{(n)} = \sum_{l=0}^{n} \lambda_k^l(a,b) u_l^{(n)},$$

where $\{\lambda_k^l(a, b)\}$ are polynomials of degree n in $\{a, \bar{a}, b, \bar{b}\}$. By (14) we have

$$\lambda_k^l(a, b) = \sqrt{\frac{l!(n-l)!}{k!(n-k)!}} \sum_{p+q=n-l} \binom{n-k}{p} \binom{k}{q} a^p (\bar{a})^{k-q} b^{n-k-p} (-\bar{b})^q. \tag{15}$$

Let $\mathbb{C}^{n+1}$ be the complex number space of $n+1$ dimensions and $\{E_i\}_{i=0}^n$ be the standard basis of $\mathbb{C}^{n+1}$. With respect to the unitary basis $\{u_k^{(n)}\}$, we may identify V_n with $\mathbb{C}^{n+1}$ and represent each linear endomorphism $\rho_n(g)$ (for $g \in SU(2)$) by a matrix $(\lambda_k^l(a, b))$, then we still denote the matrix by $\rho_n(g)$. It is easy to see $\rho_n(g) \in U(n+1)$, and thus we have a Lie group homomorphism:

$$\rho_n : SU(2) \to U(n+1)$$
$$g \mapsto \rho_n(g) = (\lambda_k^l(a, b)).$$

The representation ρ_n of $SU(2)$ induces an action of $su(2)$ on V_n which is described as follows

$$\begin{aligned} \rho_{n*}(X)(u_k^{(n)}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho(\exp tX)(u_k^{(n)}) \\ &= -\sqrt{k(n-k+1)}\, \bar{y} u_{k-1}^{(n)} + (n-2k)\sqrt{-1} x u_k^{(n)} + \\ &\quad \sqrt{(n-k)(k+1)}\, y u_{k+1}^{(n)} \end{aligned} \tag{16}$$

for $0 \leqslant k \leqslant n$ and $X \in su(2)$. Using the matrix notation, we get a Lie algebra homomorphism $\rho_{n*}: su(2) \to u(n+1)$, $X \mapsto \rho_{n*}(X)$, which is the differential of ρ_n. From (16), $\rho_{n*}(X)$ is given by

$$\rho_{n*}(X) = \begin{pmatrix} n\sqrt{-1}x & -\sqrt{n}\,\bar{y} & & & \\ \sqrt{n}y & (n-2)\sqrt{-1}\,x & -\sqrt{2(n-1)}\,\bar{y} & & \\ & \sqrt{2(n-1)}\,y & \vdots & \vdots & \\ & & \cdots & \cdots & -\sqrt{n}\,\bar{y} \\ & & & \sqrt{n}y & -n\sqrt{-1}x \end{pmatrix}. \tag{17}$$

Let $\omega = (\omega_{kl})$ be the pull back of Maurer-Cartan forms of $U(n+1)$ via ρ_n and

$$g^{-1}\mathrm{d}g = \begin{pmatrix} \bar{a}\mathrm{d}a + \bar{b}\mathrm{d}b & -\bar{a}\mathrm{d}\bar{b} + \bar{b}\mathrm{d}\bar{a} \\ a\mathrm{d}b - b\mathrm{d}a & -\bar{a}\mathrm{d}a - \bar{b}\mathrm{d}b \end{pmatrix} := \begin{pmatrix} \psi & -\bar{\varphi} \\ \varphi & -\psi \end{pmatrix}$$

be the Maurer-Cartan form of $SU(2)$. By a straightforward computation, we have

$$\omega_{lk} = (\mathrm{d}e_k, e_l) = \begin{cases} -\sqrt{k(n-k+1)}\,\bar{\varphi}, & l = k-1, \\ (n-2k)\psi, & l = k, \\ \sqrt{(k+1)(n-k)}\varphi, & l = k+1, \end{cases}$$

where $e_k = \rho_n(g) u_k^{(n)}$, i. e.

$$\omega = \begin{pmatrix} n\psi & -\sqrt{n}\,\bar{\varphi} & & & \\ \sqrt{n}\,\varphi & (n-2)\psi & -\sqrt{2(n-1)}\,\bar{\varphi} & & \\ & \sqrt{2(n-1)}\varphi & \vdots & & \\ & & & \cdots & -\sqrt{n}\,\bar{\varphi} \\ & & & \sqrt{n}\varphi & -n\psi \end{pmatrix}. \tag{18}$$

It was proved in Ref. [7] that $\varphi_k^{(n)}$ given as follows

$$\phi_k^{(n)}: S^2 = SU(2)/T \to \mathbb{C}P^n$$
$$gT \mapsto [f_k^{(n)}] = [\lambda_k^0, \lambda_k^1, \cdots, \lambda_k^n]$$

are $SU(2)$ equivariant minimal immersions of S^2 in $\mathbb{C}P^n$, which is well known as Veronese sequence in $\mathbb{C}P^n$. In particular, $\phi_0^{(n)}$ is called Veronese surface of $\mathbb{C}P^n$. Moreover, the Gaussian curvature K and the Kaehler angle α of $\phi_k^{(n)}$ are given by

$$K = \frac{4}{n + 2k(n-k)}, \quad \cos\alpha = \frac{n-2k}{n+2k(n-k)}.$$

Now we construct some examples of homogeneous holomorphic 2-spheres in $G(2, 5)$.

Example 2.1 Let $h: S^2 \to G(2, 5)$ be spanned by the first two elements of Veronese sequence in $\mathbb{C}P^4$, i.e. h maps gT to

$$\begin{bmatrix} a^4 & 2a^3b & \sqrt{6}a^2b^2 & 2ab^3 & b^4 \\ -2a^3\bar{b} & a^2(|a|^2 - 3|b|^2) & \sqrt{6}ab(|a|^2 - |b|^2) & b^2(3|a|^2 - |b|^2) & 2\bar{a}b^3 \end{bmatrix}.$$

It is well known that h is a holomorphic 2-sphere in $G(2, 5)$ with Gaussian curvature $K = 2/3$.

Example 2.2 Let $\rho: SU(2) \to U(5)$ be a Lie group homomorphism defined by

$$\rho(g) = \begin{pmatrix} \rho_3(g) & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. $\rho = \rho_3 \oplus \rho_0$. Let $\rho(g)_A$ denote the $A-$th column of $\rho(g)$. It is easy to check that the map

$$h: S^2 = SU(2)/T \to G(2, 5)$$
$$gT \mapsto [\rho(g)_1 \wedge \rho(g)_5] = \begin{bmatrix} a^3 & \sqrt{3}a^2b & \sqrt{3}ab^2 & b^3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is a well defined holomorphic 2-sphere in $G(2, 5)$. We set

$$e_1 = \rho(g)_1,\ e_2 = \rho(g)_5,\ e_3 = \rho(g)_2,\ e_4 = \rho(g)_3,\ e_5 = \rho(g)_4.$$

Then $e = (e_1, \cdots, e_5)$ is a unitary frame along h. It is easy to see from (18) that the pull back of Maurer-Cartan forms of $U(5)$ via e is given by

$$\omega = \begin{pmatrix} 3\psi & 0 & -\bar{\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \phi & 0 & \psi & -\frac{2}{\sqrt{3}}\bar{\phi} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}}\phi & -\psi & -\bar{\phi} \\ 0 & 0 & 0 & \phi & -3\psi \end{pmatrix}, \tag{19}$$

where $\phi = \sqrt{3}\varphi$. By simple computation, we have the Gaussian curvature $K = 4/3$.

Example 2.3 Let $\rho: SU(2) \to U(5)$ be a Lie group homomorphism defined as follows

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

i.e. $\rho = \rho_1 \oplus \rho_2$. It is also easy to check that the map

$$h: S^2 = SU(2)/T \to G(2, 5)$$
$$gT \mapsto [\rho(g)_1 \wedge \rho(g)_3] = \begin{bmatrix} a & b & 0 & 0 & 0 \\ 0 & 0 & a^2 & \sqrt{2}ab & b^2 \end{bmatrix}$$

is a well defined holomorphic 2-sphere in $G(2, 5)$. We set

$$e_1 = \rho(g)_1,\ e_2 = \rho(g)_3,\ e_3 = \rho(g)_2,\ e_4 = \rho(g)_4,\ e_5 = \rho(g)_5.$$

Then $e = (e_1, \cdots, e_5)$ is a unitary frame along h. It is easy to see from (18) that the pull back of Maurer-

Cartan forms of $U(5)$ via e is given by

$$\omega = \begin{pmatrix} \psi & 0 & -\sqrt{\frac{1}{3}}\bar{\phi} & 0 & 0 \\ 0 & 2\psi & 0 & -\sqrt{\frac{2}{3}}\bar{\phi} & 0 \\ \sqrt{\frac{1}{3}}\phi & 0 & -\psi & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}}\phi & 0 & 0 & -\sqrt{\frac{2}{3}}\bar{\phi} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}}\phi & -2\psi \end{pmatrix}, \tag{20}$$

where $\phi = \sqrt{3}\varphi$. By simple computation, we have the Gaussian curvature $K = 4/3$. We know that $h_1 = [e_3 \wedge e_4]: S^2 \to G(2, 5)$, $h_2 = [e_5]: S^2 \to G(1, 5) = \mathbb{C}P^4$ are harmonic maps. Then we obtain an example of nonde-generate pseudo-holomorphic sequence from S^2 to $G(2, 5)$[5]

$$h = h_0 \xrightarrow{\partial} h_1 \xrightarrow{\partial} h_2 \xrightarrow{\partial} 0$$

with $K_0 = 4/3$, $K_1 = 4/5$ and $K_2 = 2$, where K_i, $i = 0, 1, 2$ are Gaussian curvatures of S^2 with respect to the induced metric $h_i^* \mathrm{d}s^2$.

3 Degenerate holomorphic 2-spheres in $G(2, 5)$

Let f be a linearly full holomorphic immersion from S^2 to $G(2, 5)$. If f is degenerate (i. e. rank $\partial f = 1$), then via the ∂-transform, it will generate a pseudo-holomorphic sequence

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} f_3 \xrightarrow{\partial} 0,$$

where f_1, f_2, and f_3 are harmonic maps from S^2 to $\mathbb{C}P^4$. We choose a local unitary frame $e = (e_1, e_2, \cdots, e_5)$ along f so that

$$f = \mathrm{span}\{e_1, e_2\},\ \ker\partial = \mathrm{span}\{e_2\},\ f_1 = \mathrm{span}\{e_3\},\ f_2 = \mathrm{span}\{e_4\},\ f_3 = \mathrm{span}\{e_5\}.$$

The local unitary frames e_A are defined up to the change $e_A \to e_A^* = \exp(\sqrt{-1}\tau_A)e_A$, τ_A real, i. e. to a transformation of the group $U(1) \times \cdots \times U(1)$. Then the pull back of Maurer-Cartan forms under such frames are

$$\begin{pmatrix} \omega_{11} & \omega_{12} & -\bar{\phi} & 0 & 0 \\ \omega_{21} & \omega_{22} & 0 & 0 & 0 \\ \phi & 0 & \omega_{33} & \omega_{34} & 0 \\ 0 & 0 & \omega_{43} & \omega_{44} & \omega_{45} \\ 0 & 0 & 0 & \omega_{54} & \omega_{55} \end{pmatrix}, \tag{21}$$

where ϕ is a local unitary coframe of bidegree (1,0) with respect to the induced metric.

The harmonicity condition for f implies

$$\rho = \omega_{33} - \omega_{11}, \tag{22}$$

$$\omega_{12} = p\varphi,\ \omega_{43} = q\varphi,\ \omega_{54} = r\varphi, \tag{23}$$

where p, q and r are local defined smooth functions, while $|p|^2$, $|q|^2$, and $|r|^2$ are globally defined on S^2. We assume f is totally unramified, i. e. $|q|^2 \neq 0$, $|r|^2 \neq 0$. Taking the exterior derivatives of (22) and using (2), (3), (23), we get

$$\mathrm{d}\rho = (2 - |p|^2 - |q|^2)\phi \wedge \bar{\phi} = \frac{K}{2}\phi \wedge \bar{\phi},$$

which gives

$$K = 4 - 2(|p|^2 + |q|^2). \tag{24}$$

By differentiating (23) and using the Maurer-Cartan structure equations again, we get

$$dp \equiv p(\rho + \omega_{22} - \omega_{11}) \bmod \phi, \tag{25}$$

$$dq \equiv q(\rho + \omega_{33} - \omega_{44}) \bmod \phi, \tag{26}$$

$$dr \equiv r(\rho + \omega_{44} - \omega_{55}) \bmod \phi. \tag{27}$$

Making use of Lemma 1.1 to (26) and (27), we get

$$\Delta \log |q| = K + 2(1 - 2|q|^2 + |r|^2), \tag{28}$$

$$\Delta \log |r| = K + 2(|q|^2 - 2|r|^2). \tag{29}$$

By a lemma in Ref. [8], p is of analytic type, which implies that either p vanishes identically or it vanishes at finitely many points. Combining (28) and (29), together with (24), we have

$$\Delta \log |q|^2 |r| = 6K - 8 + 6|p|^2. \tag{30}$$

If p is identically zero, the above equation reduces to

$$\Delta \log |q|^2 |r| = 6K - 8. \tag{31}$$

If p is not identically zero, then away from its zeros we get from (25) that

$$\Delta \log |p| = K + 2(1 - 2|p|^2). \tag{32}$$

Combining (28), (29), and (32), together with (24), we have

$$\Delta \log |p|^3 |q|^4 |r|^2 = 5(3K - 2). \tag{33}$$

Thus, we have the following pinching theorem about Gaussian curvature K.

Theorem 3.1 Let f be a degenerate holomorphic immersion from S^2 to $G(2, 5)$ and K the Gaussian curvature.

1) If $K \geqslant 4/3$, then $K = 4/3$;

2) If $2/3 \leqslant K \leqslant 4/3$, then $K = 2/3$ or $4/3$.

Proof 1) By using (30), $K \geqslant 4/3$ implies that $\Delta \log |q|^2 |r| \geqslant 0$. So $\log |q|^2 |r|$ is a subharmonic function on S^2. Since S^2 is compact, $\log |q|^2 |r|$ then must contain a maximum in S^2. Hence it is constant by the maximum principle for subharmonic functions. Thus $K = 4/3$ and $|p| = 0$.

2) If p is identically zero, then $2/3 \leqslant K \leqslant 4/3$ together with (31) says that $\Delta \log |q|^2 |r| \leqslant 0$. Making use of the minimum principle and analogous argument of 1), we have $K = 4/3$. If p is not identically zero, similarly, we get $K = 2/3$ by (33) and the maximum principle. □

Furthermore, we show that any two degenerate holomorphic 2-spheres with constant Gaussian curvature $K = 4/3$ or $2/3$ in $G(2, 5)$ must be $U(5)$ equivalent.

Theorem 3.2 1) Any degenerate linearly full holomorphic 2-spheres with constant Gaussian curvature of $2/3$ in $G(2, 5)$ is $U(5)$ equivalent to h: $S^2 = \mathbb{C}P^1 \to G(2, 5)$ defined by

$$\begin{bmatrix} a^4 & 2a^3 b & \sqrt{6}a^2 b^2 & 2ab^3 & b^4 \\ -2a^3\bar{b} & a^2(|a|^2 - 3|b|^2) & \sqrt{6}ab(|a|^2 - |b|^2) & b^2(3|a|^2 - |b|^2) & 2\bar{a}b^3 \end{bmatrix}; \tag{34}$$

2) Any degenerate linearly full holomorphic 2-spheres with constant Gaussian curvature of $4/3$ in $G(2, 5)$ is $U(5)$ equivalent to h: $S^2 = \mathbb{C}P^1 \to G(2, 5)$ defined by

$$h([a, b]) = \begin{bmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

Proof 1) Assume $K = 2/3$. It is easy to see that $[e_2]$ describes a linearly full holomorphic curve in $\mathbb{C}P^4$ with $[e_2 e_1 e_3 e_4 e_5]$ as its Frenet frames. Here f is first osculating curve of $[e_2]$. Then it follows directly from Theorem 6.2 of Ref. [9] that $[e_2]$ is Veronese surface up to a holomorphic isometry of $\mathbb{C}P^4$. Thus f is equivalent to h which is spanned by the first two elements of Veronese sequence in $\mathbb{C}P^4$.

2) If $K = 4/3$, then $|p|^2 = 0$ and $|q|^2 = 4/3$ by (24). Then using (28), we get $|r|^2 = 1$. Since e is uniquely determined up to $U(1) \times \cdots \times U(1)$ transformations, rotations $e_A \to e_A^* = \exp(\sqrt{-1}\tau_A) e_A$, τ_A real, induce the change

$$\omega_{AA} \to \omega_{AA}^* = \sqrt{-1}\mathrm{d}\tau_A + \omega_{AA} \tag{36}$$

on the Maurer-Cartan forms. Since ω_{22} and $\omega_{11} - 3\omega_{33}$ are closed and purely imaginary by the Maurer-Cartan structure equations, we can take suitable local functions τ_1 and τ_2 such that

$$\omega_{22} = 0,\ \omega_{11} - 3\omega_{33} = 0. \tag{37}$$

Furthermore, we can specify e_4 and e_5 so that $q = 2/\sqrt{3}$ and $r = 1$. Then it follows from (26) and (27) that

$$\rho = \omega_{44} - \omega_{33} = \omega_{55}\ \omega_{44}. \tag{38}$$

Combining (22), (37), (38), the pull back of Maurer-Cartan forms are

$$\begin{pmatrix} 3\psi & 0 & -\bar\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \phi & 0 & \psi & -\frac{2}{\sqrt{3}}\bar\phi & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}}\phi & -\psi & -\bar\phi \\ 0 & 0 & 0 & \phi & -3\psi \end{pmatrix}, \tag{39}$$

where $\psi = \omega_{33}$. In this case f is $U(5)$ equivalent to h defined in Example 2.2 by comparing with (19).

□

4 Non degenerate holomorphic 2-spheres in $G(2, 5)$

Let f be a non degenerate linearly full holomorphic immersion from S^2 to $G(2, 5)$. It will generate a pseudo-holomorphic sequence

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} 0,$$

where f_1 is harmonic map from S^2 to $G(2, 5)$ and f_2 is an anti-holomorphic map into $\mathbb{C}P^4$. We choose a local unitary frame $e = (e_1, e_2, \cdots, e_5)$ along f so that

$$f = \mathrm{span}\{e_1, e_2\},\ f_1 = \mathrm{span}\{e_3, e_4\},\ f_2 = \{e_5\}.$$

We can further specify the frames by demanding that ∂ take $[e_1]$ to $[e_3]$ and $[e_4]$ to $[e_5]$. This means that

$$\omega_{41} = 0,\ \omega_{51} = 0,\ \omega_{52} = 0,\ \omega_{53} = 0. \tag{40}$$

Then the local unitary frames e_A are defined up to a transformation of the group $U(1) \times \cdots \times U(1)$. The pull back of Maurer-Cartan forms under such frames have the form

$$\begin{pmatrix} \omega_{11} & \omega_{12} & -\bar a_1\bar\phi & 0 & 0 \\ \omega_{21} & \omega_{22} & -\bar a_2\bar\phi & -\bar a_3\bar\phi & 0 \\ a_1\phi & a_2\phi & \omega_{33} & \omega_{34} & 0 \\ 0 & a_3\phi & \omega_{43} & \omega_{44} & \omega_{45} \\ 0 & 0 & 0 & \omega_{54} & \omega_{55} \end{pmatrix}, \tag{41}$$

with

$$a_1 a_3 \neq 0,\ |a_1|^2 + |a_2|^2 + |a_3|^2 = 1, \tag{42}$$

where ϕ is a local unitary coframe of bidegree (1,0) with respect to the induced metric.

By taking the exterior derivatives of (40) and the harmonicity condition of f, we have

$$\omega_{21} = p\phi,\ \omega_{43} = q\phi,\ \omega_{54} = r\phi, \tag{43}$$

and

$$da_1 \equiv a_1(\rho + \omega_{11} - \omega_{33}) \bmod \phi, \tag{44}$$

$$da_3 \equiv a_3(\rho + \omega_{22} - \omega_{44}) \bmod \phi, \tag{45}$$

$$da_2 \equiv a_2(\rho + \omega_{22} - \omega_{33}) + a_1\omega_{12} - a_3\omega_{34} \bmod \phi, \tag{46}$$

where p, q, and r are local defined smooth functions, while $|p|^2$, $|q|^2$ and $|r|^2$ are globally defined on S^2. By differential the last equation of (43), we have

$$dr \equiv r(\rho + \omega_{44} - \omega_{55}) \bmod \phi. \tag{47}$$

By using Lemma 1.1 again to (44), (45) and (47), we obtain

$$\Delta\log|a_1| = K - 2(2|a_1|^2 + |a_2|^2 - |q|^2 + |p|^2), \tag{48}$$

$$\Delta\log|a_3| = K - 2(|a_2|^2 + 2|a_3|^2 + |q|^2 - |p|^2), \tag{49}$$

$$\Delta\log|r| = K + 2(|a_3|^2 + |q|^2 - 2|r|^2), \tag{50}$$

outside of their zero points. Combining (48), (49) and (50), together with (42), we have

$$\Delta\log|a_1^3a_3^4r^2| = 9K - 12 - 2|a_2|^2 + 2(|p|^2 + |q|^2). \tag{51}$$

Under some assumption, we have the following Proposition.

Proposition 4.1 Let f be a non-degenerate holomorphic immersion from S^2 to $G(2,5)$ and K be the Gaussian curvature. If $|a_2|^2 = 0$ and $K \geqslant 4/3$. Then $K = 4/3$ and f is $U(5)$ equivalent to h defined in Example 2.3.

Proof By the same argument of the proof of Theorem 3.1(1) and the equation (51), we obtain $K = 4/3$ and $|p|^2 = |q|^2 = 0$. It is easy to see that $\phi_1 = [e_1]$ and $\phi_2 = [e_2]$ describe two holomorphic maps from S^2 to $\mathbb{C}P^4$ and $f = [e_1 \wedge e_2]$. The induced metric by ϕ_1 and ϕ_2 are

$$\phi_1^* ds^2 = |a_1|^2\phi\bar{\phi},\ \phi_2^* ds^2 = |a_3|^2\phi\bar{\phi}$$

with Gaussian curvature $K(\phi_1)$ and $K(\phi_2)$. Then by a Lemma in Ref. [10], $K(\phi_1)$ and $K(\phi_2)$ are also constant, which implies that $|a_1|^2$ and $|a_3|^2$ are constant. From (48) ~ (51), we get

$$|a_1|^2 = 1/3,\ |a_3|^2 = 2/3,\ |r|^2 = 2/3.$$

By the Maurer-Cartan equations, we know that ω_{44} and $\omega_{11} + \omega_{33}$ are closed and purely imaginary. Since e is uniquely determined up to rotations $e_A \to e_A^* = \exp(\sqrt{-1}\tau_A)e_A$, τ_A real, which induces the change (36), we can take suitable local functions τ_3 and τ_4 such that

$$\omega_{44} = 0,\ \omega_{11} + \omega_{33} = 0.$$

Furthermore, we can specify e_1, e_2 and e_5 so that $a_1 = \sqrt{1/3}$, $a_3 = \sqrt{2/3}$ and $r = \sqrt{2/3}$. By (44), (45) and (47), we have

$$\rho = \omega_{33} - \omega_{11} = \omega_{44} - \omega_{22} = \omega_{55} - \omega_{44}.$$

So the pull back of the Maurer-Cartan forms via such frame are

$$\omega = \begin{pmatrix} \psi & 0 & -\sqrt{\frac{1}{3}}\bar{\phi} & 0 & 0 \\ 0 & 2\psi & 0 & -\sqrt{\frac{2}{3}}\bar{\phi} & 0 \\ \sqrt{\frac{1}{3}}\phi & 0 & -\psi & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}}\phi & 0 & 0 & -\sqrt{\frac{2}{3}}\bar{\phi} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}}\phi & -2\psi \end{pmatrix}, \tag{52}$$

where $\psi = \omega_{11}$. Thus comparing with (20), we know that f is $U(5)$ equivalent to h defined in Example 2.3.

Remark 4.2 We make a conjecture that without the assumption $|a_2|^2 = 0$, the Proposition 4.1 is still valid.

The authors would like to express appreciation to Professor Jiagui Peng for his helpful guidance.

References

[1] Chern S S, Wolfson J G. Harmonic maps of the two-sphere into a complex Grassmann manifold II[J]. Ann of Math, 1987, 125: 301-335.

[2] Chi Q S, Zheng Y B. Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifold[J]. Trans Amer Math Soc, 1989, 313: 393-406.

[3] Xu X W, Jiao X X. Holomorphic 2-spheres in complex Grassmann manifold $G(2, 4)$[C]// Proc Indian Acad Sci (Math Sci). 2008, 118: 381-388.

[4] Jiao X X, Peng J G. Classification of holomorphic two-spheres with constant curvature in complex Grassmann manifold $G(2, 5)$[J]. Differential Geometry and its Applications, 2004, 20: 267-277.

[5] Jiao X X. On pseudo-holomorphic curves from two-spheres into a com-plex Grassmannian $G(2, 5)$[J]. Acta Mathematic Sinica, English Series, 2010, 26: 759-762.

[6] Eschenberg J H, Gaudalupe I V, Tribuzy R. The fundamental equations of minimal surfaces in $\mathbb{C}P^2$[J]. Math Ann, 1985, 270: 571-598.

[7] Bando S, Ohnita Y. Minimal 2-spheres with constant curvature in $\mathbb{C}P^n$[J]. J Math Soc Japan, 1987, 3: 477-487.

[8] Chern S S. On the minimal immersions of the two sphere in a space of constant curvature[C]// Problems in Analysis. Princeton Univ Press, Princeton, N J, 1970: 27-42.

[9] Shen Y B. The Riemannian Geometry of superminimal surfaces in complex space forms[J]. Acta Mathematica Sinica, 1996, 12: 298-313.

[10] Li Z Q. Holomorphic 2-spheres in Grassmann manifolds[J]. Science in China(Series A), 1998, 41(4): 357-364.

复格拉斯曼流形 $G(2,5)$ 中的全纯 2 球

费　杰，焦晓祥

（中国科学院研究生院数学科学学院，北京 100049）

摘　要　利用调和序列和活动标架研究了复格拉斯曼流形 $G(2,5)$ 中线性满的全纯 2 球. 利用 $SU(2)$ 的不可约酉表示构造了 $G(2,5)$ 中的一些齐性全纯 2 球. 在 $U(5)$ 等价意义下确定常高斯曲率为 2/3 和4/3 的所有线性满的退化全纯 2 球. 最后证明在某一特定条件下常高斯曲率为 4/3 的非退化的全纯 2 球一定是$U(5)$等价的.

关键词　全纯 2 球，高斯曲率，复格拉斯曼流形，调和序列