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Existence and pathwise uniqueness of solutions to SDE driven by a class of special semimartingale*

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Abstract We prove the existence and uniqueness of solutions to stochastic differential equations $\mathrm{d}X_t = F(X)_t \mathrm{d}Z_t$, where F is non-Lipschitz coefficient and Z is in a kind of special semimartingale.

Key words existence, pathwise uniqueness, special semimartingale, successive approximation

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The theory of classical stochastic differential equation (SDE) driven by Brownian motion has gone through a long period development and has already achieved fruitful results. For a general d-dimensional semimartingale Z with $Z_0 = 0$ a. s., here we consider the following stochastic differential equaiton on $\mathbb{R}^m$:

$$X_t = X_0 + \int_0^t F(X)_s \mathrm{d}Z_s, \tag{1}$$

where $X_0 : \Omega \to \mathbb{R}^m$ is $\mathscr{F}_0 / B(\mathbb{R}^m)$ measurable such that $\mathbb{E}(|X_0|^p) < \infty$ for some $p \geqslant 2$ and F is a mapping from the set of all m-dimensional cadlag adapted processes to the set of all d-dimensional locally bounded predictable processes such that for each stopping time τ, $F(X^{\tau-})$ coincides with $F(X)$ on $((0,\tau]]$, where $X^{\tau-} := XI_{[[0,\tau))} + X_{\tau-} I_{[[\tau,\infty))}$. There are few papers to investigate the existence and uniqueness of Eq. (1) without assuming a Lipschitz condition (see Ref. [1] for Lipschitz coefficient case and Ref. [2] for continuous Z).

We should bring the reader's attention to the paper of Taniguchi[3], in which he proved the existence and uniqueness of SDE driven by Brownian motion under quite weak non-Lipschitz assumptions by successive approximation. In fact, Taniguchi's method has been used by many authors[4-5]. Motivated by Jiang[2] and Taniguchi[3], in this paper we study Eq. (1) driven by a class of special semimartingale Z under the Taniguchi's conditions[4] and prove the existence and uniqueness by successive approximation.

1 Main results and proofs

Define $S^p := \{Z$ is a semimartingale: $[Z]^{\frac{p}{2}}$ is a locally integrable increasing process$\}$ for $p \geqslant 2$. By using

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Theorems 5.19 and 8.6 in Ref. [6], it is easy to see that $Z \in S^p$ if and only if it is a special semimartingale with the canonical decomposition $M + A$, where M is a locally integrable martingale such that $\left(\sum_{s \leqslant \cdot} (\Delta M_s)^2\right)^{\frac{p}{2}}$ is a locally integrable increasing process, and A is a predictable process with finite variation. In particular, a semimartingale with bounded jump or a predictable semimartingale is in S^p.

Before presenting our main result we need the following important predictable control theorem.

Theorem 1.1 Let Z be a d-dimensional semimartingale with $Z^i \in S^p$, where $p \geqslant 2$. There exists a zero initial valued predictable process $V^{(p)}$ with $V^{(p)} \upuparrows \infty$ and a constant $C_p > 0$ such that for every stopping time τ and every d-tuple H of predictable process which is integrable w.r.t. Z,

$$\mathbb{E}\left(\sup_{t \leqslant \tau}\left|\int_0^t H_s \mathrm{d}Z_s\right|^p\right) \leqslant C_p \mathbb{E}\left(\int_0^\tau |H_s|^p \mathrm{d}V_s^{(p)}\right). \tag{2}$$

If Z is quasi-left-continuous, then $V^{(p)}$ is continuous.

Proof As a result of Theorem 4.5.1 in Ref. [7], we can see that Z is a d-tuple of local L^p-integrators and there exists a strictly increasing predictable process $\Lambda^{(p)}$ (It is continuous if Z is quasi-left-continuous) such that

$$\mathbb{E}\left(\sup_{t \leqslant \tau}\left|\int_0^t H_s \mathrm{d}Z_s\right|^p\right) \leqslant C'_p \max_{q=1,2,p} \mathbb{E}\left(\int_0^\tau |H_s|^q \mathrm{d}\Lambda_s^{(p)}\right)^{\frac{p}{q}},$$

where $0 < C'_p \leqslant 9.5p$. Moreover, by Hölder inequality,

$$\begin{aligned}\mathbb{E}\left(\int_0^\tau |H_s| \mathrm{d}\Lambda_s^{(p)}\right)^p &= \mathbb{E}\left(\int_0^\tau |H_s| (1+\Lambda_s^{(p)})^{\frac{p-1}{p}} \frac{1}{(1+\Lambda_s^{(p)})^{\frac{p-1}{p}}} \mathrm{d}\Lambda_s^{(p)}\right)^p \\ &\leqslant \mathbb{E}\left[\left(\int_0^\tau |H_s|^p (1+\Lambda_s^{(p)})^{p-1} \mathrm{d}\Lambda_s^{(p)}\right)\left(\int_0^\tau \frac{1}{1+\Lambda_s^{(p)}} \mathrm{d}\Lambda_s^{(p)}\right)^{p-1}\right] \\ &\leqslant \left(\frac{\pi}{2}\right)^{p-1} \mathbb{E}\left(\int_0^\tau |H_s|^p (1+\Lambda_s^{(p)})^{p-1} \mathrm{d}\Lambda_s^{(p)}\right).\end{aligned}$$

By the same approach, we have

$$\mathbb{E}\left(\int_0^\tau |H_s|^2 \mathrm{d}\Lambda_s^{(p)}\right)^{\frac{p}{2}} \leqslant \left(\frac{\pi}{2}\right)^{\frac{p}{2}-1} \mathbb{E}\left(\int_0^\tau |H_s|^p (1+\Lambda_s^{(p)})^{\frac{p}{2}-1} \mathrm{d}\Lambda_s^{(p)}\right).$$

Eq. (2) holds as long as we define $V_t^{(p)} := \int_0^t (1+\Lambda_s^{(p)})^{p-1} \mathrm{d}\Lambda_s^{(p)} + \int_0^t (1+\Lambda_s^{(p)})^{\frac{p}{2}-1} \mathrm{d}\Lambda_s^{(p)} + \Lambda_t^{(p)} - \Lambda_0^{(p)} + t$. □

In the following we fix Z with $Z^i \in S^p$ and $V := V^{(p)}$ as above. Set $V_t^{-1} := \inf\{s \geqslant 0 : V_s > t\}$. Then V^{-1} is a continuous process with $V^{-1} \uparrow \infty$ and $\forall t > 0$, V_t^{-1} is a predictable time so it is a.s. foretellable. Thus we can get the following estimation.

Lemma 1.2 For any $t > 0$,

$$\mathbb{E}\left(\sup_{r < V_t^{-1}}\left|\int_0^r H_s \mathrm{d}Z_s\right|^p\right) \leqslant C_p \mathbb{E}\left(\int_0^t |H_{V_s^{-1}}|^p \mathrm{d}s\right). \tag{3}$$

Proof Thanks to Eq. (2) and Lebesgue's lemma, for each stopping time τ,

$$\mathbb{E}\left(\sup_{r \leqslant \tau}\left|\int_0^r H_s \mathrm{d}Z_s\right|^p\right) \leqslant C_p \mathbb{E}\left(\int_0^\tau |H_s|^p \mathrm{d}V_s\right) = C_p \mathbb{E}\left(\int_0^{V_\tau} |H_{V_s^{-1}}|^p \mathrm{d}s\right).$$

Let τ_n be a sequence of stopping times such that $\tau_n < V_t^{-1}$ a.s. and $\tau_n \uparrow V_t^{-1}$ a.s. on $\{V_t^{-1} > 0\}$, then $V_s^{-1} \leqslant \tau_n$ (i.e. $s \leqslant V_{\tau_n}$) does imply $s < t$. It follows that

$$\mathbb{E}\left(\sup_{r < V_t^{-1}}\left|\int_0^r H_s \mathrm{d}Z_s\right|^p\right) = \mathbb{E}\left(\lim_{n \to \infty} \sup_{r \leqslant \tau_n}\left|\int_0^r H_s \mathrm{d}Z_s\right|^p\right) \leqslant C_p \mathbb{E}\left(\int_0^t |H_{V_s^{-1}}|^p \mathrm{d}s\right).$$ □

Finally, we can prove the following result.

Theorem 1.3 Suppose that the following Taniguchi's conditions hold:

(T1) There exists a continuous nondecreasing nonnegative function H on $\mathbb{R}_+$ such that for any stopping time τ,

$$\mathbb{E}(|F(X)_\tau|^p) \leqslant H(\mathbb{E}(\sup_{r<\tau}|X_r|^p)),$$

and for any constant $K>0$ and any initial value u_0, the differential equation $u_t = u_0 + K\int_0^t H(u_s)\,\mathrm{d}s$ has a global solution.

(T2) There exists a continuous nondecreasing nonnegative function G on $\mathbb{R}_+$ such that $G(0)=0$, for any stopping time τ,

$$\mathbb{E}(|F(X)_\tau - F(Y)_\tau|^p) \leqslant G(\mathbb{E}(\sup_{r<\tau}|X_r - Y_r|^p)),$$

and for any constant $K>0$, if a nonnegative function g satisfies that $g_t \leqslant K\int_0^t G(g_s)\,\mathrm{d}s$ for all $t\in\mathbb{R}_+$, then $g\equiv 0$.

Then Eq. (1) has a pathwise unique solution.

Proof Let $X_t^0 := X_0$ and for $n\in\mathbb{N}$, we define the following successive approximation sequence:

$$X_t^n := X_0 + \int_0^t F(X^{n-1})_s\,\mathrm{d}Z_s. \tag{4}$$

Due to Eq. (3) and (T1), we have

$$\begin{aligned}\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^n|^p) &\leqslant 2^{p-1}\mathbb{E}(|X_0|^p) + 2^{p-1}\mathbb{E}\left(\sup_{r<V_t^{-1}}\left|\int_0^r F(X^{n-1})_s\,\mathrm{d}Z_s\right|^p\right)\\ &\leqslant 2^{p-1}\mathbb{E}(|X_0|^p) + 2^{p-1}C_p\int_0^t \mathbb{E}|F(X^{n-1})_{V_s^{-1}}|^p\,\mathrm{d}s\\ &\leqslant 2^{p-1}\mathbb{E}(|X_0|^p) + 2^{p-1}C_p\int_0^t H(\mathbb{E}(\sup_{r<V_s^{-1}}|X_r^{n-1}|^p))\,\mathrm{d}s.\end{aligned}$$

By (T1) there is a $\{u_t\}$ satisfying $u_t = 2^{p-1}\mathbb{E}(|X_0|^p) + 2^{p-1}C_p\int_0^t H(u_s)\,\mathrm{d}s$. By induction, we obtain $\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^0|^p) = \mathbb{E}(|X_0|^p) \leqslant u_0 \leqslant u_t$ and if $\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^{n-1}|^p) \leqslant u_t$, then

$$\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^n|^p) \leqslant u_0 + 2^{p-1}C_p\int_0^t H(u_s)\,\mathrm{d}s = u_t. \tag{5}$$

On the other hand, by the same way as above, (T2) yields that

$$\begin{aligned}\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^m - X_r^n|^p) &\leqslant \mathbb{E}\left(\sup_{r<V_t^{-1}}\left|\int_0^r (F(X^{m-1})_s - F(X^{n-1})_s)\,\mathrm{d}Z_s\right|^p\right)\\ &\leqslant C_p\int_0^t \mathbb{E}(|F(X^{m-1})_{V_s^{-1}} - F(X^{n-1})_{V_s^{-1}}|^p)\,\mathrm{d}s\\ &\leqslant C_p\int_0^t G(\mathbb{E}(\sup_{r<V_s^{-1}}|X_r^{m-1} - X_r^{n-1}|^p))\,\mathrm{d}s.\end{aligned}$$

Let $g_t = \overline{\lim}_{m,n\to\infty}\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^m - X_r^n|^p)$, in virtue of Eq. (5) and Fatou's lemma, it can be easily seen $g_t \leqslant C_p\int_0^t G(g_s)\,\mathrm{d}s$. By (T2), we immediately get $g\equiv 0$, which implies that

$$\lim_{m,n\to\infty}\mathbb{E}(\sup_{r<V_t^{-1}}|X_r^m - X_r^n|^p) = 0.$$

Since $V^{-1}\uparrow\infty$, by a diagonal procedure argument it is clear that there exists a subsequence X^{n_k} such that $X_t = \lim_{k\to\infty}X_t^{n_k}$ exists for every t and almost surely the convergence is uniform on $[0, V_t^{-1})$ for each fixed t. Now passing the limit in Eq. (4), we conclude that X satisfies Eq. (1). In other words, we have shown the existence of the solutions to Eq. (1).

Let both X and X' be two solutions to Eq. (1), then by the same way we can obtain

$$\mathbb{E}\left(\sup_{r<V_t^{-1}}|X_r - X'_r|^p\right) \leqslant C_p\int_0^t G\left(\mathbb{E}\sup_{r<V_s^{-1}}|X_r - X'_r|^p\right)\mathrm{d}s.$$

We can apply (T2) again, and deduce $\mathbb{E}\left(\sup_{r<V_t^{-1}}|X_r - X'_r|^p\right) = 0$. Let $t\to\infty$, hence $\mathbb{E}\left(\sup_r |X_r - X'_r|^p\right) = 0$, then we have shown the uniqueness of the solution to Eq. (1). □

Example 1.4 (See Ref. [4]) Let G be a continuous nondecreasing nonnegative function on $\mathbb{R}_+$ such that $\int_{0^+}\frac{1}{G(u)}\mathrm{d}u = +\infty$, $G(u)$ or $\frac{G(u)^2}{u}$ is a concave function and $H(u) = C_1 + C_2G(u)$, where $C_1, C_2 > 0$, then we know that conditions (T1) and (T2) in Theorem 1.3 hold.

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一类特殊半鞅驱动的随机微分方程解的存在唯一性

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摘　要　研究了随机微分方程 $\mathrm{d}X_t = F(X)_t\mathrm{d}Z_t$ 解的存在唯一性,其中 F 为非 Lipschitz 系数,Z 属于一类特殊半鞅.

关键词　存在性,轨道唯一性,特殊半鞅,逐次逼近