Ruin theory with a Markov chain interest model

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Abstract: In this paper we consider a discrete time risk process with a Markov chain interest model. We derive recursive equations satisfied respectively by the joint distribution of surplus immediately before ruin and the deficit at ruin, the distribution of the surplus immediately before ruin, and the distribution of the deficit at ruin. An integral equation for the (expected discounted) penalty function is obtained as a unified method to study ruin quantities with such an investment. Applications of integral equation are given to the Laplace transform of the time of ruin, the deficit at ruin, the amount of claim causing ruin, etc

Key words: Probability; Markov chain; Ruin function; Interest

0 Introduction

Consider the discrete time risk process

$$U_k = U_{k-1}(1+I_k) - Z_k, k = 1,2,\ldots \tag{0.1}$$

where $U_0 = u \geq 0$ is a constant, $\{Z_k; k \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $F(x) = P(Z_1 \leq x)$ and tail probability $\overline{F}(x) = 1 - F(x)$, and $\{I_k; k \geq 1\}$ is a sequence of random variables independent of $\{Z_k; k \geq 1\}$. In the insurance risk context, Z_k denotes the net loss in period k, namely from time $k-1$ to time k, and the distribution F is referred as the loss distribution. The net loss is calculated at the end of each period and equal to claim amount minus premium income in period k. And I_k denotes the rate of interest in period k. Thus, U_k defined by (0.1) is the surplus of an insurance company with initial capital of u at the end of period k.

It is easy to see that (0.1) implies

$$U_k = u\prod_{j=1}^{k}\left(1+I_j\right) - \sum_{j=1}^{k} Z_j \prod_{i=j+1}^{k}\left(1+I_i\right), k \geq 1 \tag{0.2}$$

where $\prod_{i=a}^{b} = 1$ and $\sum_{i=a}^{b} = 0$ if $a > b$.

In this paper we will study the discrete time risk model proposed in Cai and Dikson (2004). Assume that the interest rate $\{I_k; k \geq 1\}$ is a Markov chain with state space $\mathrm{I} = \{i_0, i_2, \ldots, i_N\}$. We also suppose that for all $n \geq 1$ and all state $i_s, i_t, i_{t_0}, \ldots, i_{t_{n-1}}$,

$$\begin{aligned} &P\left(I_{n+1} = i_t \middle| I_n = i_s, I_{n-1} = i_{t_{n-1}}, \ldots, I_0 = i_{t_0}\right) = P\left(I_{n+1} = i_t \middle| I_n = i_s\right) \\ &= p_{st} \geq 0, s,t = 0,1,\ldots N \end{aligned} \tag{0.3}$$

where $\sum_{t=0}^{N} p_{st} = 1$ for $s = 0,1,2,\ldots, N$.

In addition, we assume that $1 + I_k > 0$ for all $n \geq 1$, that is, $1 + i_s > 0$ for all $i_s \in \mathrm{I}$. The interest rates $\{I_k; k \geq 1\}$ can be negative. In this case, as said in Cai and Dikson (2004),

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$\{I_k; k \ge 1\}$ are the return rates of a risky investment.

Let $T = \inf\{k \ge 1 : U_k < 0\}$, the stopping time T, is the ruin time. Then the finite and infinite time ruin probabilities in risk model (0.2) with interest model (0.3), initial surplus u, and a given $I_0 = i_s$, are defined, respectively, by

$$\psi_{ni_s}(u) = P(T \le n | U_0 = u, I_0 = i_s), \psi_{i_s}(u) = P(T < \infty | U_0 = u, I_0 = i_s).$$

Since $\psi_{1i_s} \le \psi_{2i_s} \le \psi_{3i_s} \le \Lambda$, $\psi_{i_s}(u) = \lim_{n\to\infty} \psi_{ni_s}$.

The discrete risk model (0.2) has been studied by many researchers. Yang (1999) obtained both exponential and non-exponential upper bounds for the ruin probabilities when the rates of interest are identical constants. Cai (2002a，2002b) derived the recursive and integral equations for the ruin probabilities in the case of i.i.d. interest rates and the case of dependent model of AR(1) structure for the rates of interest, respectively. The ruin probabilities in risk model (0.2) with i.i.d. rates of interest was also considered by Yang and Zhang (2003). Cai and Dickson (2004) considered the model (0.2) with rates of interest (0.3), and obtained the recursive and integral equations for the ruin probabilities. As for asymptotic properties for ruin probabilities in risk model (0.2) readers are referred to Nyrhinen (1999, 2001), Tang and Tsitsiashvili (2003) and references therein.

Except for the ruin probabilities, other important ruin quantities in ruin theory include the Laplace transform of the time of ruin $Ee^{-\alpha T}$, the surplus immediately before ruin U_{T-1}, the deficit at ruin $-U_T$, and the amount of claim causing ruin $U_{T-1} - U_T$, etc. A unified method to study these ruin quantities is to consider the (expected discounted) penalty function associated the time of ruin with initial reserve u and for a given $i_0 = i_s$ by defining

$$\Phi_{\alpha i_s}(u) = E\left[g(U_{T-1}, -U_T)e^{-\alpha T} 1_{\{T<\infty\}} \middle| U_0 = u, I_0 = i_s\right] \tag{0.4}$$

where $g(x, y)$, $x \ge 0$, $y \ge 0$, is a non-negative function such that $\Phi_{\alpha i_s}(u)$ exists; $\alpha \ge 0$, and $1_{\{C\}}$ is the indicator function of a set C .

A simple and sufficient condition on $g(x, y)$ for $\Phi_{\alpha i_s}(u) < \infty$ is that $g(x, y)$ is a bounded function. With suitable choice of $g(x, y)$, $\Phi_{\alpha i_s}(u)$ will yield different ruin functions, see Gerber and Shiu (1997) for $\Phi_\alpha(u)$ in the compound Poisson risk model, Cai and Dickson (2002), and Cai (2004) for it in the compound Poisson risk model with a constant interest rate and with stochastic ones, respectively.

In this paper we derive recursive equations satisfied respectively by the joint distribution of surplus immediately before ruin and the deficit at ruin, the distribution of the surplus immediately before ruin, and the distribution of the deficit at ruin. A unified treatment to the ruin quantities when $\{I_k; k \ge 1\}$ is a Markov chain of (0.3) is also given by studying the generalized penalty function $\Phi_{\alpha i_s}(u)$.

1 Recursive equations for distributions

For initial reserve u and a given $I_0 = i_s$ we define the joint distribution of surplus immediately before ruin time $T = k$ and the deficit at ruin time $T = k$ with

$$H_{ki_s}(u,x,y)=P\left(T=k,-U_T>y,U_{T-1}>x\middle|U_0=u,I_0=i_s\right),x\ge 0,y\ge 0$$

From definition, we get

$$H_{ki_s}(u,x,y)$$

$$=P\left(\begin{array}{l}\sum_{j=1}^{k}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}>u+y\prod_{i=1}^{k}(1+I_i)^{-1},\sum_{j=1}^{k-1}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}<u-x\prod_{i=1}^{k-1}(1+I_i)^{-1},\\ \sum_{j=1}^{k-2}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}<u,\sum_{j=1}^{k-3}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}<u,\Lambda\ ,Z_1(1+I_1)^{-1}<u\middle|I_0=i_s\end{array}\right).$$

Therefore,

$$\begin{aligned}&H_{1i_s}(u,x,y)\\&=P\left(Z_1(1+I_1)^{-1}>u+y(1+I_1)^{-1},0<u-x\middle|I_0=i_s\right)\\&=\begin{cases}P\left(Z_1(1+I_1)^{-1}>u+y(1+I_1)^{-1}\middle|I_0=i_s\right), & x<u,\\ 0, & x\ge u,\end{cases}\\&=\begin{cases}\sum_{t=0}^{N}p_{st}P\left(Z_1>u(1+i_t)+y\right), & x<u,\\ 0, & x\ge u,\end{cases}\\&=\begin{cases}\sum_{t=0}^{N}p_{st}\overline{F}\left(u(1+i_t)+y\right), & x<u,\\ 0, & x\ge u,\end{cases}\end{aligned}\tag{1.1}$$

$$\begin{aligned}&H_{2i_s}(u,x,y)\\&=P\left(\begin{array}{l}Z_1(1+I_1)^{-1}+Z_2(1+I_1)^{-1}(1+I_2)^{-1}>u+y(1+I_1)^{-1}(1+I_2)^{-1},\\ Z_1(1+I_1)^{-1}<u-x(1+I_1)^{-1}\middle|I_0=i_s\end{array}\right)\\&=P\left(Z_2(1+I_2)^{-1}>u(1+I_1)-Z_1+y(1+I_2)^{-1},Z_1<u(1+I_1)-x\middle|I_0=i_s\right)\\&=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)-x}P\left(Z_2(1+I_2)^{-1}>u(1+i_t)-z+y(1+I_2)^{-1}\middle|I_1=i_t\right)dF(z)\end{aligned}\tag{1.2}$$

and

$$H_{3i_s}(u,x,y)$$

$$=P\left(\begin{array}{l}\sum_{j=1}^{3}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}>u+y\prod_{i=1}^{3}(1+I_i)^{-1},\sum_{j=1}^{2}Z_j\prod_{i=1}^{j}(1+I_i)^{-1}<u-x\prod_{i=1}^{2}(1+I_i)^{-1},\\ Z_1(1+I_1)^{-1}<u\middle|I_0=i_s\end{array}\right)$$

$$=P\left(\begin{array}{l}\sum_{j=2}^{3}Z_j\prod_{i=2}^{j}(1+I_i)^{-1}>u(1+I_1)-Z_1+y\prod_{i=2}^{3}(1+I_i)^{-1},Z_2(1+I_2)^{-1}<u(1+I_1)-Z_1-x(1+I_2)^{-1},\\ Z_1<u(1+I_1)\middle|I_0=i_s\end{array}\right)$$

$$=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}P\left(\begin{array}{l}\sum_{j=2}^{3}Z_j\prod_{i=2}^{j}(1+I_i)^{-1}>u(1+i_t)-z+y\prod_{i=2}^{3}(1+I_i)^{-1},\\ Z_2(1+I_2)^{-1}<u(1+i_t)-z-x\middle|I_1=i_t\end{array}\right)dF(z)$$

$$=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{2i_t}\left(u(1+i_t)-z,x,y\right)dF(z)$$

Similarly, we obtain that for k⩾3,

$$H_{ki_s}(u,x,y)=\sum_{t=0}^{N} p_{st}\int_{-\infty}^{u(1+i_t)} H_{(k-1)i_t}(u(1+i_t)-z,x,y)dF(z) \quad (1.3)$$

For initial reserve u and a given $I_0=i_s$ we denote the distribution of the surplus immediately before ruin time $T=k$ by

$$B_{ki_s}(x,u)=P(T=k,U_{T-1}>x|U_0=u,I_0=i_s),x\geq 0,$$

and the distribution of the deficit at ruin time $T=k$ by

$$G_{ki_s}(x,u)=P(T=k,-U_T>y|U_0=u,I_0=i_s),y\geq 0.$$

Notice that

$$B_{ki_s}(x,u)=H_{ki_s}(x,0),G_{ki_s}(x,y)=H_{ki_s}(0,y).$$

Then we get

$$B_{1i_s}(u,x)=\begin{cases}\sum_{t=0}^{N} p_{st}\overline{F}(u(1+i_t)), & x<u,\\ 0, & x\geq u,\end{cases}$$

$$B_{2i_s}(u,x)=\sum_{t=0}^{N} p_{st}\int_{-\infty}^{u(1+i_t)-x} P(Z_2(1+I_2)^{-1}>u(1+i_t)-z|I_t=i_t)dF(z),$$

$$B_{ki_s}(u,x)=\sum_{t=0}^{N} p_{st}\int_{-\infty}^{u(1+i_t)} B_{(k-1)i_t}(u(1+i_t)-z,x)dF(z),k\geq 3;$$

and

$$G_{1i_s}(u,y)=\sum_{t=0}^{N} p_{st}\overline{F}(u(1+i_t)+y),$$

$$G_{2i_s}(u,y)=\sum_{t=0}^{N} p_{st}\int_{-\infty}^{u(1+i_t)} P(Z_2(1+I_2)^{-1}>u(1+i_t)-z+y(1+I_2)^{-1}|I_1=i_t)dF(z),$$

$$G_{ki_s}(u,x,y)=\sum_{t=0}^{N} p_{st}\int_{-\infty}^{u(1+i_t)} G_{(k-1)i_t}(u(1+i_t)-z,y)dF(z),k\geq 3.$$

Remark 2.1 In fact by the recursive equations (1.1), (1.2) and (1.3) we can derive the integral equation for the following joint distribution of the surplus immediately before ruin and the deficit at ruin

$$H_{i_s}(u,x,y)=P(T<\infty,-U_T>y,-U_{T-1}>x|U_0=u,I_0=i_s),x\geq 0,y\geq 0.$$

(1) When $x<u$,

$$
\begin{aligned}
&H_{i_s}(u,x,y)\\
&=P(T<\infty,-U_T>y,U_{T-1}>x)\\
&=\sum_{k=1}^{\infty}H_{ki_s}(u,x,y)\\
&=\sum_{t=0}^{N}p_{st}\overline{F}(u(1+i_t)+y)+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)-x}P\left(Z_2(1+I_2)^{-1}>u(1+i_t)-z+y(1+I_2)^{-1}\middle|I_1=i_t\right)dF(z)\\
&+\sum_{k=3}^{\infty}\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{(k-1)i_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\overline{F}(u(1+i_t)+y)+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{1i_t}(u(1+i_t)-z,x,y)dF(z)\\
&+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}\sum_{k=3}^{\infty}H_{(k-1)i_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\overline{F}(u(1+i_t)+y)+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}\sum_{k=1}^{\infty}H_{ki_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\overline{F}(u(1+i_t)+y)+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{i_t}(u(1+i_t)-z,x,y)dF(z)
\end{aligned}
$$

(2) When $x\ge u$,

$$
\begin{aligned}
&H_{i_s}(u,x,y)\\
&=P(T<\infty,-U_T>y,-U_{T-1}>x)\\
&=\sum_{k=1}^{\infty}H_{ki_s}(u,x,y)\\
&=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)-x}P\left(Z_2(1+I_2)^{-1}>u(1+i_t)-z+y(1+I_2)^{-1}\middle|I_1=i_t\right)dF(z)\\
&+\sum_{k=3}^{\infty}\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{(k-1)i_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{1i_t}(u(1+i_t)-z,x,y)dF(z)+\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}\sum_{k=3}^{\infty}H_{(k-1)i_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}\sum_{k=1}^{\infty}H_{ki_t}(u(1+i_t)-z,x,y)dF(z)\\
&=\sum_{t=0}^{N}p_{st}\int_{-\infty}^{u(1+i_t)}H_{i_t}(u(1+i_t)-z,x,y)dF(z)
\end{aligned}
$$

Similarly, using the recursive equations for $B_{ki_s}(u,x)$ and $G_{ki_s}(u,y)$ we can obtain the corresponding integral equations for the distribution of the surplus immediately before ruin

$$B_{i_s}(u,x)=P\left(T<\infty,U_{T-1}>x\middle|U_0=u,I_0=i_s\right),x\ge 0$$

and the distribution of the deficit at ruin

$$G_{i_s}(u,x)=P\left(T<\infty,U_T>y\middle|U_0=u,I_0=i_s\right),y\ge 0.$$

Remark 1.2 As we known that the joint distribution of surplus immediately before ruin and the deficit at ruin with initial surplus u and a given $I_0 = i_s$ is usually given by

$$J_{i_s}(u,x,y) = P(T<\infty, -U_T \le y, U_{T-1} \le x | U_0 = u, I_0 = i_s), x \ge 0, y \ge 0.$$

By the relationship

$$J_{i_s}(u,x,y) = H_{i_s}(u,0,0) - H_{i_s}(u,x,0) - H_{i_s}(u,0,y) + H_{i_s}(u,x,y),$$

we get the integral equation for the joint distribution $J_{i_s}(u,x,y)$,

$$J_{i_s}(u,x,y) = \sum_{t=0}^{N} p_{st}\left[\left(\overline{F}(u(1+i_t)) - \overline{F}(u(1+i_t)+y)\right)1_{\{x \ge u\}} + \int_{-\infty}^{u(1+i_t)} J_{i_s}(u(1+i_t)-z, x, y)dF(z)\right]$$

2 Integral equation for penalty function

In this section we derive an integral equation for $\Phi_{\alpha i_s}(u)$, and the main result is the following theorem.

Theorem 2.1 For any $u \ge 0$ and $i_s \in \mathrm{I}$,

$$\Phi_{\alpha i_s}(u) = \sum_{t=0}^{N} p_{st} e^{-\alpha}\left[\int_{-\infty}^{u(1+i_t)} \Phi_{\alpha i_t}(u(1+i_t)-z)dF(z) + \int_{u(1+i_t)}^{+\infty} g(u(1+i_t), z-u(1+i_t))dF(z)\right].$$

Proof Conditioning on $I_1 = i_t$, $Z_1 = z$, if $z \le u(1+i_t)$, then ruin does not occur, $U_1 = u(1+i_t) - z$, $\Phi_{\alpha i_t}(u(1+i_t)-z)$ is the expected discounted value at time 1, then $e^{-\alpha}\Phi_{\alpha i_t}(u(1+i_t)-z)$ gives the expected discounted value at time 0. If $z > u(1+i_t)$, then ruin occurs with $T=1$, $U_{T-1} = u(1+i_t)$ and $-U_T = z - u(1+i_t)$. Thus, from the independence of $\{Z_k; k \ge 1\}$ and $\{I_k; k \ge 1\}$, it follows that

$$\begin{aligned}
&\Phi_{\alpha i_s}(u) \\
&= E\left[g(U_{T-1}, -U_T)e^{-\alpha T}1_{\{T<\infty\}} | U_0 = u, I_0 = i_s\right] \\
&= \sum_{t=0}^{N} p_{st} \int_{-\infty}^{+\infty} E\left[g(U_{T-1}, U_T)e^{-\alpha T}1_{\{T<\infty\}} | U_0 = u, I_1 = i_t, Z_1 = z\right]dF(z) \\
&= \sum_{t=0}^{N} p_{st}\left[\int_{-\infty}^{u(1+i_t)} e^{-\alpha}\Phi_{\alpha i_t}(u(1+i_t)-z)dF(z) + \int_{u(1+i_t)}^{+\infty} g(u(1+i_t), z-u(1+i_t))e^{-\alpha}dF(z)\right]
\end{aligned}$$

The proof of Theorem 2.1 is completed

3 Applications

In this section we give some examples to illustrate the applications of integration derived in Section 2.

Example 3.1 Let $g(x,y) = 1$ and $\alpha > 0$, then

$$\Phi_{\alpha i_s}(u) = E\left[e^{-\alpha T}1_{\{T<\infty\}} | U_0 = u, I_0 = i_s\right] = E\left[e^{-\alpha T} | U_0 = u, I_0 = i_s\right] := \tilde{q}_{\alpha i_s}(u)$$

is the Laplace transform of the time of ruin with an initial surplus u and a given $I_0 = i_s$. Thus, by Theorem 2.1, we get

$$\tilde{q}_{\alpha i_s}(u) = \sum_{t=0}^{N} p_{st} e^{-\alpha}\left[\int_{-\infty}^{u(1+i_t)} \tilde{q}_{\alpha i_t}(u(1+i_t)-z)dF(z) + \overline{F}(u(1+i_t))\right].$$

Example 3.2 Let $g(x,y)=1$ and $\alpha=0$, then

$$\Phi_{\alpha i_s}(u)=E\left[1_{\{T<\infty\}}\middle|U_0=u,I_0=i_s\right]=\psi_{i_s}(u)$$

is the infinite time ruin probability with an initial surplus u and a given $I_0=i_s$. Thus, by Theorem 2.1, we have

$$\psi_{i_s}(u)=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}\psi_{i_t}(u(1+i_t)-z)dF(z)+\overline{F}(u(1+i_t))\right].$$

which is Equation (2.3) of Cai and Dickson (2004).

Example 3.3 Let $g(x_1,x_2)=1_{\{x_1\le x\}}$ and $\alpha=0$, then

$$\Phi_{\alpha i_s}(u)=P(U_{T-1}\le x,T<\infty|U_0=u,I_0=i_s)=A_{i_s}(u,x)$$

is the distribution of surplus immediately before ruin. Thus, by Theorem 2.1, we get

$$A_{i_s}(u,x)=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}A_{i_t}(u(1+i_t)-z,x)dF(z)+\overline{F}(u(1+i_t))1_{\{x\ge u\}}\right]. \quad (3.1)$$

Example 3.4 Let $g(x_1,x_2)=1_{\{x_2\le y\}}$ and $\alpha=0$, then

$$\Phi_{\alpha i_s}(u)=P(-U_T\le y,T<\infty|U_0=u,I_0=i_s)=K_{i_s}(u,y)$$

is the distribution of deficit at ruin. Thus, by Theorem 2.1, we get

$$K_{i_s}(u,y)=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}K_{i_t}(u(1+i_t)-z,y)dF(z)+\overline{F}(u(1+i_t))-\overline{F}(u(1+i_t)+y)\right]. \quad (3.2)$$

Remark 3.1 Notice that $A_{i_s}(u,x)$ and $K_{i_s}(u,y)$ are two marginal distributions corresponding to $J_{i_s}(u,x,y)$, that is,

$$A_{i_s}(u,x)=\lim_{y\to\infty}J_{i_s}(u,x,y),K_{i_s}(u,x)=\lim_{x\to\infty}J_{i_s}(u,x,y),$$

Then we can get (3.1) and (3.2) from (1.4).

Example 3.5 Let $g(x_1,x_2)=e^{-rx_2}$, $r\ge 0$ and $\alpha=0$, then

$$\Phi_{\alpha i_s}(u)=E\left[e^{-r(-U_T)}1_{\{T<\infty\}}\middle|U_0=u,I_0=i_s\right]=\tilde{K}_{i_s}(u,r)$$

is the Laplace transform of the deficit at ruin. Thus, by Theorem 2.1, we get

$$\tilde{K}_{i_s}(u,r)=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}\tilde{K}_{i_t}(u(1+i_t)-z,r)dF(z)+\int_{u(1+i_t)}^{+\infty}e^{-r(z-u(1+i_t))}dF(z)\right]$$

$$=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}\tilde{K}_{i_t}(u(1+i_t)-z,r)dF(z)+e^{ru(1+i_t)}\int_{u(1+i_t)}^{+\infty}e^{-rz}dF(z)\right].$$

Example 3.6 Let $g(x_1,x_2)=e^{-r(x_1+x_2)}$, $r\ge 0$ and $\alpha=0$, then

$$\Phi_{\alpha i_s}(u)=E\left[e^{-r(U_{T-1}-U_T)}1_{\{T<\infty\}}\middle|U_0=u,I_0=i_s\right]=\tilde{D}_{i_s}(u,r)$$

is the Laplace transform of the amount of claim causing ruin. Thus, by Theorem 2.1, we get

$$\tilde{D}_{i_s}(u,r)=\sum_{t=0}^{N}p_{st}\left[\int_{-\infty}^{u(1+i_t)}\tilde{D}_{i_t}(u(1+i_t)-z,r)dF(z)+\int_{u(1+i_t)}^{+\infty}e^{-rz}dF(z)\right]$$

4 Conclusion

In this paper we derive recursive equations satisfied respectively by the joint distribution of surplus immediately before ruin and the deficit at ruin, the distribution of the surplus immediately before ruin, and the distribution of the deficit at ruin. A unified treatment to the ruin quantities of is also given by studying the generalized penalty function.

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马氏链利率风险模型的破产理论

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摘要：本文考虑带有马氏链利率的离散风险模型，分别给出了该模型破产前盈余、破产后赤字的联合分布以及边缘分布所满足的递归方程。本文得到了该模型的惩罚函数的期望贴现值所满足的积分方程，该结果应用到破产时、破产时赤字及导致破产的索赔等破产度量的研究中。

关键词：概率；马氏过程；破产函数；利率

中图分类号：G22