A general central limit theorem for capacity

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Abstract: In this paper,in the base of sublinear expectation space called 'G-expectation space' that introduced by Peng , adapting Peng's IID notion and applying Peng's new CLT under sublinear expectations, we investigate the general CLT for capacity and give an affirmative answer. For prove Theorm2.1,we deeply sduty the theorem of generalized G-expectation and the related results in Hu and Zhang's paper.

Keywords:G-expectation; G-distribution; generalized G-expectation; capacity; tightness; CLT

0 Introduction

The law of large numbers (LLN) and central limit theorem (CLT) is long and widely been known as two fundamental results in theory of probability and statistics. It is very useful tool in many other fields, such as mathematical finance, economics. Recently motivated by model uncertainties in statistics and economics, measures of risk and superhedging in finance, Peng[1-5] introduces a new notion of sublinear expectation space called 'G-expectation space'. Many results are established, for example, the corresponding LLN and CLT under a sublinear expectation. Peng[2] initiated the notion of IID random variables and the definition of *G*-normal distribution under sublinear expectations, he further proved law of large numbers (LLN) and central limit theorems (CLT) under sublinear expectations. Hu and Zhang[6] obtained central limit theorem for capacities in the framework of Peng[2].

A furthermore question is that: Can the CLT under a sublinear expectation be generalized for capacity in the new framework of Peng[5]? In this paper, adapting Peng's IID notion and applying Peng's new CLT under sublinear expectations, we investigate the general CLT for capacity and give an affirmative answer.

1 Preliminaries

We present some preliminaries in the theory of sublinear expectations space such as some basic notions and results of G-expectation space and the related space of random variables. More details of this section can be found in Peng[1-5].

Definition 1.1 Let ✱ be a given set and let H be a linear space of real valued functions defined on ✱. We assume that all constants are in H and that ✣❺H implies❽✣❽❺H. H is considered as the space of our "random variables". A nonlinear expectation $\hat{E}$ on H is a functional $\hat{E}$: H→ R satisfying the following properties: for all X, Y❺H, we have

(a) Monotonicity: If X>Y then $\hat{E}[X] > \hat{E}[Y]$.

(b) Constant preserving: $\hat{E}[c] = c$

The triple (✱, H, $\hat{E}$) is called a nonlinear expectation space (compare with a probability space (✱, Φ, N)).

We are mainly concerned with sublinear expectation where the expectation $\hat{E}$ satisfies also

(c) Sub-additivity: $\hat{E}$[X]«$\hat{E}$[Y]❨ $\hat{E}$[X«Y].

(d) Positive homogeneity: $\hat{E}$[●X] = ●$\hat{E}$[X], ∀●❩0.

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If only (c) and (d) are satisfied, $\hat{E}$ is called a sublinear functional.

Remark 1.1 we known H is an Stone lattice.

The following representation theorem for sublinear expectations or sublinear functional is well-known (see Peng[5] or others books):

Lemma 1.1 Let $\hat{E}$ be a sublinear functional defined on (Ω, H), i.e., (c) and (d) hold for $\hat{E}$. Then there exists a family $\{E_\theta: \theta \in \Theta\}$ of linear functionals on (Ω, H) such that

$$\hat{E}[X] = \max_{\theta\in\Theta} E_\theta[X]. \qquad (1)$$

If (a) and (b) also hold, then E_θ are linear expectations for $\theta\in\Theta$.

If we make furthermore the following assumption: (H) For each sequence $\{X_n\}_{n\geq 1} \subset H$ such that $X_n(\omega) \downarrow 0$ for ω, we have $\hat{E}[X_n] \downarrow 0$. Then for each $\theta \in \Theta$, there exists a unique (σ-additive) probability measure P_θ defined on $(\Omega, \sigma(H))$ such that

$$E_\theta[X] = \int_\Omega X(\omega)\, dP_\theta(\omega),\ X\in H. \qquad (2)$$

Remark 1.2 The above (2) is the well-known Daniell-Stone Theorem.

In this paper, we research about the following sublinear expectation:

$$\bar{E}[\cdot] = \sup_{Q\in\Pi} E_Q[\cdot],$$

where Π is a set of probability measures.

Let Ω be a given set and let Φ be an σ-algebra defined on Ω. Define $V(A) := \bar{E}[I_A] = \sup_{Q\in\Pi} E_Q[I_A]$, $v(A) := -\bar{E}[-I_A] = \inf_{Q\in\Pi} E_Q[I_A]$, $\forall A \in \Phi$, then V and v are two capacities.

Let $C_{l, Lip}(R^n)$ denote the space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x- y| \quad \forall x, y \in R^n, \quad \text{for some } C > 0,\ m \in N \text{ depending on } \varphi$$

and let $C_{b,Lip}(R^n)$ denote the space of bounded functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x- y| \quad \forall x, y \in R^n, \text{ for some } C > 0 \text{ depending on } \varphi.$$

The following is the notion of IID random variables under sublinear expectations introduced by Peng[1-5]:

Definition 1.2 Independence: Suppose that $Y_1, Y_2, \ldots, Y_n$ is a sequence of random variables such that $Y_i \in H$. Random variable Y_n is said to be independent of $X := (Y_1, \ldots, Y_{n-1})$ under $\bar{E}$, if for each function $\varphi \in C_{l, Lip}(R^n)$, we have

$$\bar{E}[\varphi(X, Y_n)] = \bar{E}[\bar{E}[\varphi(x, Y_n)]_{x=X}].$$

Definition 1.3 Identical distribution: Random variables X and Y are said to be identically distributed, denoted by $X \sim Y$, if for each function $\varphi \in C_{l, Lip}(R^n)$, we have

$$\bar{E}[\varphi(X)] = \bar{E}[\varphi(Y)].$$

Definition 1.4 IID random variables: A sequence of random variables $\{X_n\}_{n\geq 1}$ is said to be IID, if $X_n \sim X_1$ and X_{n+1} is independent of $Y := (X_1, \ldots, X_n)$ for each $n\geq 1$.

Definition 1.5 (G-normal distribution, see Definition 10 in Peng[2]). A random variable $\xi \in H$ under sublinear expectation $\tilde{E}_\sigma$ with $\sigma^2 = \tilde{E}_\sigma[\xi^2]$, $\underline{\sigma}^2 = -\tilde{E}_\sigma[-\xi^2]$ is called G-normal distribution, denoted by $N(0; [\sigma^2, \underline{\sigma}^2])$, if for any function $\varphi \in C_{l, Lip}(R^n)$, write $u(t, x) := \tilde{E}_\sigma[\varphi(x+\sqrt{t}\xi)]$, $(t, x) \in [0,\infty)\times R$, then u is the unique viscosity solution of PDE:

$$\partial_t u - G(\partial^2_{xx} u) = 0,\ u(0, x) = \varphi(x),$$

where $G(x) := (\sigma^2 x^+ - \underline{\sigma}^2 x^-)\frac{1}{2}$ and $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$.

Definition 1.6 (G-distribution, see Definition 4.5 in Peng[5]). A random variable $(\xi, \zeta) \in H$ under sublinear expectation $\tilde{E}$ with $\mu = \tilde{E}[\zeta]$, $\underline{\mu} = -\tilde{E}[-\zeta]$; $\sigma^2 = \tilde{E}[\xi^2]$, $\underline{\sigma}^2 = -\tilde{E}[-\xi^2]$ is called G-distribution, denoted by $N([\underline{\mu}, \mu]; [\underline{\sigma}^2, \sigma^2])$, if for any function $\varphi \in C_{l, Lip}(R^n)$, write

$u(t, x) := \tilde{E}[\varphi(x+ t\zeta+\sqrt{t}\xi)]$, $(t, x) \in [0,\infty)\times R$, then u is the unique viscosity solution of PDE:

$\partial_t u - G(\partial_x u, \partial^2_{xx} u) = 0,\ u(0, x) = \varphi(x)$,

where $G(x, y) := (\sigma^2 y^+ - \underline{\sigma}^2 y^-)/2 + (\mu x^+ - \underline{\mu} x^-)$ and $a^+ := \max\{a, 0\}$, $a^- := (-a)^+$.

With the notion of IID under sublinear expectations, Peng shows central limit theorem under sublinear expectations (see Theorem 5.1 in Peng[5]).

Lemma 1.2 (Central limit theorem under sublinear expectations). Let $\{(X_i,Y_i)\}_{i\geq 1}$ be a sequence of IID random variables. We further assume that $\bar{E}[X_1] = \bar{E}[-X_1] = 0$. Then the sequence $\{S_n\}_{n\geq 1}$ defined by $S_n := (\sum_{i=1}^n X_i)/n^{1/2} + (\sum_{i=1}^n Y_i)/n$ converges in law to $\xi+\zeta$, i.e.,

$\lim_{n\to\infty} \bar{E}[\varphi(S_n)] = \tilde{E}[\varphi(\xi+\zeta)]$,

for any continuous function φ satisfying linear growth condition, where (ξ,ζ) is a G-distribution.

2 Main result

The following lemma is very useful in this paper:

Lemma 2.1 Suppose that (ξ, ζ) is G- distributed by $N([\underline{\mu}, \mu]; [\underline{\sigma}^2, \sigma^2])$. Let P be a probability measure and φ be a bounded continuous function. If $\{W_t\}_{t\geq 0}$ is a P-Brownian motion, then

$\tilde{E}[\varphi(\xi+\zeta)] = \sup_{(\theta,\mu)\in\Theta\times M} E_P[\varphi(\int_{[0,1]}(\theta_s dW_s + \mu_s ds))]$,

where $\Theta := \{\theta_t\}_{t\geq 0}$: θ_t is Φ_t-adapted process such that $\underline{\sigma} \leq \theta_t \leq \sigma$,

$M := \{\mu_t\}_{t\geq 0}$: μ_t is Φ_t-adapted process such that $\underline{\mu} \leq \mu_t \leq \mu$,

$\Phi_t := \sigma\{W_s, 0 \leq s \leq t\} \vee N$, N is the collection of P-null subsets.

Proof. By Theorem 3.3 in Su[7], we have: for each $\varphi \in C_{b,Lip}(R^n)$,

$\tilde{E}[\varphi(\xi+\zeta)] = \sup_{(\theta,\mu)\in\Theta\times M} E_P[\varphi(\int_{[0,1]}(\theta_s dW_s + \mu_s ds))]$. (3)

Furthermore by Theorem 3.3 in Su[7], we known the family $\{P_{\theta,\mu}\}_{(\theta,\mu)\in\Theta\times M}$ of law of the processes $\int_{[0,t]}(\theta_s dW_s + \mu_s ds)$ is tight. So if φ is a bounded continuous function, we also get

$\tilde{E}[\varphi(\xi+\zeta)] = \sup_{(\theta,\mu)\in\Theta\times M} E_P[\varphi(\int_{[0,1]}(\theta_s dW_s + \mu_s ds))]$.

Remark 2.1 The above lemma generalizes the lemma 2.2 in Hu and Zhang[6].

Now we give our main result:

Theorem 2.1 (Central limit theorem for capacities). Let $\{(X_i,Y_i)\}_{i\geq 1}$ be a sequence of IID random variables. We further assume that $\bar{E}[X_1] = \bar{E}[-X_1] = 0$. Denote $S_n := (\sum_{i=1}^n X_i)/n^{1/2} + (\sum_{i=1}^n Y_i)/n$. Then

if z is a point at which V is continuous, we have

$\lim_{n\to\infty} V(S_n \leq z) = V_G(z)$,

if z is a point at which v is continuous, we have

$\lim_{n\to\infty} v(S_n \leq z) = v_G(z)$,

where $V_G(z) = \sup_{(\theta,\mu)\in\Theta\times M} E_P[I_{\{\int_{[0,1]}(\theta_s dW_s+\mu_s ds)\leq z\}}]$

and $v_G(z) = \inf_{(\theta,\mu)\in\Theta\times M} E_P[I_{\{\int_{[0,1]}(\theta_s dW_s+\mu_s ds)\leq z\}}]$.

Proof. Suppose that z is a point at which V_G is continuous. Let ε be any positive number, and take δ small enough that $V_G(z+\delta) - V_G(z-\delta) \leq \varepsilon$.

Construct two bounded continuous functions f, g such that

$f(x) = 1$ for $x \leq z-\delta$, $f(x) = 0$ for $x \geq z$, $0 < f(x) \leq 1$ for $z-\delta < x < z$;

$g(y) = 1$ for $y \leq z$, $g(y) = 0$ for $y \geq z+\delta$, $0 < g(y) \leq 1$ for $z < y < z+\delta$.

Then $V_G(z-\delta) \leq \sup_{\theta\in\Theta} E_P[f(\int_{[0,1]}(\theta_s dW_s + \mu_s ds))] \leq V_G(z)$

$\leq \sup_{\theta\in\Theta} E_P[g(\int_{[0,1]}(\theta_s dW_s + \mu_s ds))] \leq V_G(z+\delta)$, (4)

and for each n,

$\bar{E}[f(S_n)] \leq V(S_n \leq z) \leq \bar{E}[g(S_n)]$. (5)

Obviously, f and g are bounded continuous functions. By Lemmas 2.1 and 1.2, we have

$\lim_{n\to\infty}\bar{E}[f(S_n)] = \sup_{\theta\in\Theta} E_P[f(\int_{[0,1]}(\theta_s dW_s+\mu_s ds))]$,

$\lim_{n\to\infty}\bar{E}[g(S_n)] = \sup_{\theta\in\Theta} E_P[g(\int_{[0,1]}(\theta_s dW_s+\mu_s ds))]$.

So that

$\sup_{\theta\in\Theta} E_P[f(\int_{[0,1]}(\theta_s dW_s+\mu_s ds))] \le \lim_{n\to\infty}\inf V(S_n \le z) \le \lim_{n\to\infty}\sup V(S_n \le z)$
$\le \sup_{\theta\in\Theta} E_P[g(\int_{[0,1]}(\theta_s dW_s+\mu_s ds))]$. (6)

Hence

$V_G(z) - \varepsilon \le \lim_{n\to\infty}\inf V(S_n \le z) \le \lim_{n\to\infty}\sup V(S_n \le z) \le V_G(z) + \varepsilon$.

Since this is true for every ε, $\lim_{n\to\infty} V(S_n \le z) = V_G(z)$.

In a similar manner as in the above, we can obtain $\lim_{n\to\infty} v(S_n \le z) = v_G(z)$.

Remark 2.1. (1) Obviously, V is an increasing function, then V is continuous in R except in, at most, countable points. Similarly, v is continuous in R except in, at most, countable points.

(2) The above proof of Theorem 2.1 follows Hu and Zhang[6],but the result generalizes theirs. For Hu and Zhang prove their result under the represent theorem of G-expectation in Denis, Hu and Peng[8].

(3) In Theorem 3.1, if $\bar{E}[Y_1] = -\bar{E}[-Y_1] = \mu$, $\bar{E}[X_1^2] = -\bar{E}[-X_1^2] = \sigma^2 > 0$, then for each $z \in R$,

$\lim_{n\to\infty} V(S_n \le z) = \lim_{n\to\infty} v(S_n \le z) = \Phi(z)$. Where $\Phi(\cdot)$ is the distribution function of normal distribution with mean μ and variance σ^2

Corollary2.1 (Law of large numbers for capacities) Let $\{Y_i\}_{i\ge 1}$ be a sequence of IID random variables. We further assume that $\mu = \tilde{E}_\mu[\zeta]$, $\underline{\mu} = -\tilde{E}_\mu[-\zeta]$. Denote $S_n := (\sum_{i=1}^n Y_i)/n$.

Then

(1)if z is a point at which V is continuous, we have

$\lim_{n\to\infty} V(S_n \le z) = V_\mu(z)$,

(2)if z is a point at which v is continuous, we have

$\lim_{n\to\infty} v(S_n \le z) = v_\mu(z)$,

where $V_\mu(z) = \sup_{\mu\in M} E_P[I_{\{\int_{[0,1]}\mu_s ds \le z\}}]$ and $v_\mu(z) = \inf_{\mu\in M} E_P[I_{\{\int_{[0,1]}\mu_s ds \le z\}}]$.

3 Conclusion

In this paper we get the generalize results than Hu and Zhang[6]. We also find the condition about capacity V is continuous is essential; in the following we will study other better condition.

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关于容度的一般中心极限定理

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摘要：本文在次线性期望空间-G-期望空间的大数定律与中心极限定理基础之上，利用推广的 G-期望的表示定理与现有的关于容度的大数定律与中心极限定理等相关结论得到了关于容度的一个一般的中心极限定理，进一步还得到相应的关于容度的大数定律。

关键词：G-期望；G-分布；推广的 G-期望；容度；胎紧；中心极限定理

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