# Integration of Constraint Equations in Problems of a Disc and a Ball Rolling on a Horizontal Plane 

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#### Abstract

The problem of a disc and a ball rolling on a horizontal plane without slipping is considered. Differential constrained equations are shown to be integrated when the trajectory of the point of contact is taken in a form of the natural equation, i.e. when the dependence of the curvature of the trajectory is explicitly expressed in terms of the distance passed by the point.


## 1 The problem of a rolling disc

The rolling motion of a disc and a ball on a static plane was described many times, see e.g. [1, 2], and generalized in later publications [3]-8]. These are the classical examples of motion of mechanical systems with non-holonomic constraints.

Let a disc with radius $R$ be tangent to a plane $\pi$ with the system of coordinates $O x y$, let $P$ be the point of contact between the disc and the plane. The position of a disk is determined by five independent coordinates. For example, it can be fixed by coordinates $x_{P}$ and $y_{P}$, by the angle of rotation $\varphi$, by the angle of precession $\psi$, and by the angle of nutation $\vartheta$, see Figure 1 .


Figure 1: Generalized coordinates of a rolling disk.

Evolution of the coordinates always satisfies the two non-holonomic constraints which can be written in the following form:

$$
\begin{equation*}
d x_{P}=R \cos \psi d \varphi \quad, \quad d y_{P}=R \sin \psi d \varphi . \tag{1}
\end{equation*}
$$

The problem above is, therefore, to integrate the given dynamical system under applied arbitrary external forces. The problem is interesting because the methods of the classical mechanics it requires to use are rather involved. The solution can be essentially simplified if the rolling trajectory is known. Such a formulation is possible, for example, under modeling of the rolling with the help of a computer animation.

In fact, the kinematics of the rolling motion of the disk is determined by the three functions

$$
\begin{equation*}
\varphi=\varphi(t) \quad, \quad \vartheta=\vartheta(t) \quad, \quad \psi=\psi(t) \tag{2}
\end{equation*}
$$

Under the rolling motion without slipping the arc coordinate $s$ of the point of contact of the disc and the plane is related to the angle of rotation $\varphi$ as

$$
\begin{equation*}
\varphi=s / R \tag{3}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{\dot{\psi}}{\dot{\varphi}}=R k(s) . \tag{4}
\end{equation*}
$$

Here $k(s)=\frac{d \psi}{d s}$ is a curvature of the point of contact trajectory for the disc.
It is a well-known that given any function $k=k(s)$ one can find a curve $\vec{r}=\vec{r}(s)$ with the curvature equal $k(s)$. The curve is unique up to a congruence. Equation $k=k(s)$ is known as a natural equation of the curve. The parametric equations of the trajectory of the point of contact,

$$
\begin{equation*}
x_{P}=\int_{0}^{s} \cos \left(\int_{0}^{\tau} k(s) d s\right) d \tau, \quad y_{P}=\int_{0}^{s} \sin \left(\int_{0}^{\tau} k(s) d s\right) d \tau \tag{5}
\end{equation*}
$$

see [9], allow one to find the location of the disk on the plane at any moment of the motion.

Equations (2) enable one to describe kinematics of the disk with equations of the point of contact in form (5). The found solution satisfies non-holonomic constraint equations (1).

Let us give an example of how rolling of the disc can be described by three equations of motion (2) by using the natural equation for the trajectory of the point of contact with a horizontal plane.

Example 1: Let rolling of the disk be determined by following equations:

$$
\begin{equation*}
\varphi=\omega t, \quad \psi=\frac{\varepsilon t^{2}}{2}, \quad \vartheta=f(t) \tag{6}
\end{equation*}
$$

The curvature of the trajectory of the point of contact is

$$
\begin{equation*}
k(s)=\frac{\dot{\psi}}{\dot{s}}=\frac{\varepsilon t}{R \omega}=\frac{\varepsilon s}{(R \omega)^{2}} \tag{7}
\end{equation*}
$$

where $\varepsilon$ and $\omega$ are some parameters, $f(t)$ is a function. This trajectory is the clothoid whose asymptotic point has coordinates $x=y=(R \omega \sqrt{\pi / \varepsilon}) / 2$.

The equations of motion of the point of contact in this case are

$$
\begin{equation*}
x_{P}=\int_{0}^{s} \cos \frac{\varepsilon s^{2}}{2(R \omega)^{2}} d s \quad, \quad y_{P}=\int_{0}^{s} \sin \frac{\varepsilon s^{2}}{2(R \omega)^{2}} d s \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
x_{P}=R \omega \int_{0}^{t} \cos \frac{\varepsilon t^{2}}{2} d t \quad, \quad y_{P}=R \omega \int_{0}^{t} \sin \frac{\varepsilon t^{2}}{2} d t \tag{9}
\end{equation*}
$$

A direct substitution of these functions into the equations (2) shows that obtained solution satisfies the constraints.


Figure 2: Phases of motion of the disk along a clothoid $\left(\omega=\pi, \varepsilon=\frac{\pi}{16}, t^{*}=16\right)$

The phases of motion are shown on Fig. 2, They are obtained by using equation (6) with $f(t)=\frac{\pi}{t^{* 2}}\left(t-t^{*}\right)^{2}$.

## 2 The problem of a rolling ball

Let us now discuss a similar problem for a ball. We assume that the ball rolls on a plane and spins simultaneously. Like in case of the disc, the position of a ball (see Fig. 3) can be determined by three functions. To introduce these functions we decompose the vector of angular velocity of the ball $\vec{\Omega}$ into two parts, as shown on Fig. 3,

$$
\begin{equation*}
\vec{\Omega}=\vec{\omega}_{s}+\vec{\omega}_{r} \tag{10}
\end{equation*}
$$

The vector of angular velocity related to the spinning, $\vec{\omega}_{s}$, is a orthogonal to the plane. The vector of the angular velocity associated to the rolling, $\vec{\omega}_{r}$, is parallel to the plane.


Figure 3: The rolling and spinning angular velocity vectors of the ball.

Evolution of the velocity vectors at the center of the ball is determined by the same angle $\psi$. By taking this into account one can find the complete set of functions which describe the motion of the ball:

$$
\begin{equation*}
\varphi=\int_{0}^{t} \omega_{r}(t) d t, \quad \chi=\int_{0}^{t} \omega_{s}(t) d t, \quad \psi=\psi(t) \tag{11}
\end{equation*}
$$

Angles $\varphi, \chi, \psi$ are quasi-coordinates, they do not enable one to establish position of the ball in different moments of time. The position can be fixed by integrating constraint equations which are identical to (1). A solution to these equations formally coincides with equalities (5).

It allows one to find the position of the ball at any moment of motion. For the motion without spinning the position of the ball is determined by only two functions $\varphi(t)$ and $\psi(t)$. Consider again an example.

Example 2: Let the ball of the radius $R$ roll according to the following equations:

$$
\begin{equation*}
\varphi=\omega_{1} t, \quad \psi=\omega_{2} t \tag{12}
\end{equation*}
$$

where $\omega_{1}$ and $\omega_{2}$ are the corresponding angular velocities which are assumed to be some constants. The curvature of the trajectory of the point of contact with the plane is a constant,

$$
\begin{equation*}
k=\frac{\dot{\psi}}{\dot{s}}=\frac{\omega_{2}}{R \omega_{1}} \tag{13}
\end{equation*}
$$

This means that the point of contact of the ball with the plane and its center moves along a circle with the radius $R \omega_{1} / \omega_{2}$. The equations of the circle are

$$
\begin{equation*}
x_{P}=\frac{R \omega_{1}}{\omega_{2}} \sin \omega_{2} t, \quad y_{P}=\frac{R \omega_{1}}{\omega_{2}}\left(1-\cos \omega_{2} t\right) \tag{14}
\end{equation*}
$$

The motion of other points of the ball can be determined by the Euler formula which defines a velocity $\vec{v}$ for a point of a ball

$$
\begin{equation*}
\vec{v}=\left[\vec{\omega}_{r}, \vec{r}-\vec{r}_{p}\right], \tag{15}
\end{equation*}
$$

where $\vec{\omega}_{r}=\left\{-\sin \omega_{2} t, \cos \omega_{2} t, 0\right\}$.
By using (14), (15) one finds

$$
\frac{d \mathbf{r}}{d t}=\omega_{1}\left(\begin{array}{c}
z \cos \omega_{2} t  \tag{16}\\
z \sin \omega_{2} t \\
-x \cos \omega_{2} t-y \sin \omega_{2} t+\frac{R \omega_{1}}{\omega_{2}} \sin \omega_{2} t
\end{array}\right)
$$

In coordinates with the origin at the center of the circle trajectory (of the point of contact) this equation takes the form

$$
\frac{d \mathbf{r}}{d t}=\omega_{1}\left(\begin{array}{c}
z \cos \omega_{2} t  \tag{17}\\
z \sin \omega_{2} t \\
-x \cos \omega_{2} t-y \sin \omega_{2} t
\end{array}\right)
$$

One immediately checks that one of the integrals of this equation is expressed as

$$
\begin{equation*}
x \frac{\omega_{1}}{\omega_{2}} \sin \omega_{2} t-y \frac{\omega_{1}}{\omega_{2}} \cos \omega_{2} t+z=C \tag{18}
\end{equation*}
$$

Fig. 4 shows the behavior of $x, y, z$ coordinates of a point of the ball being initially a "south" pole of the ball.

t
Figure 4: The behavior of the "south" pole of the ball under rolling along the circle $\left(\omega_{1}=\pi, \omega_{2}=\frac{\pi}{3}, R=1\right)$.

## 3 Conclusions

Our analysis shows that in problems of a disk and a ball rolling on a plane it is a sofficient to choose three functions to determine the laws of motion (5)), (11). (The choice of other parameters is also possible.) For the disc these functions are $x_{P}=x_{P}(t), y_{P}=y_{P}(t)$, and $\vartheta=\vartheta(t)$. For the ball they are $x_{P}=x_{P}(t), y_{P}=y_{P}(t)$, and $\chi=\chi(t)$. Other coordinates which fix orientations of the given bodies are determined by coordinates of the point of contact

$$
\begin{equation*}
\psi=\arccos \frac{\dot{x}_{P}}{\sqrt{\dot{x}_{P}^{2}+\dot{y}_{P}^{2}}} \quad, \quad \varphi=\frac{1}{R} \int_{0}^{t} \sqrt{\dot{x}_{P}^{2}+\dot{y}_{P}^{2}} d t \tag{19}
\end{equation*}
$$

This form of kinematic equations of motion is essentially convenient when the trajectories of the rolling are given.

The following statement is an important consequence of the considered kinematic description: a vertical rolling of a disc or a (non spinning) ball on a plane is completely determined by (or equivalent to) the motion of a point (of the same mass) on a plane under the force equal to the main vector of external forces applied to rolling bodies.

The validity of this statement is based on Eq. (19) and on the equivalence of equations of motion of a point on a plane to equations of motion of the
center of mass of a disc or a ball under the action of the same system of forces.

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