Edge-Coloring Series-Parallel Multigraphs

Cristina G. Fernandes *
Departamento de Ciência da Computação
Instituto de Matemática e Estatística
Universidade de São Paulo - Brazil
E-mail: cris@ime.usp.br

and

Robin Thomas †
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
E-mail: thomas@math.gatech.edu

July 28, 2011

Abstract

We give a simpler proof of Seymour's Theorem on edge-coloring series-parallel multigraphs and derive a linear-time algorithm to check whether a given series-parallel multigraph can be colored with a given number of colors.

1 Introduction

All *graphs* in this paper are finite, may have parallel edges, but no loops. Let $k \geq 0$ be an integer. A graph G is *k-edge-colorable* if there exists a map $\kappa : E(G) \to \{1, \ldots, k\}$, called a *$k$-edge-coloring*, such that $\kappa(e) \neq \kappa(f)$ for any two distinct edges e, f of G that share at least one end. The *chromatic index* $\chi'(G)$ is the minimum $k \geq 0$ such that G is k-edge-colorable. Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G, but there is another lower bound. Let

$$\Gamma(G) = \max\left\{\frac{2|E(G[U])|}{|U|-1} : U \subseteq V(G), |U| \geq 3 \text{ and } |U| \text{ is odd}\right\}.$$

If U is as above, then every matching in $G[U]$, the subgraph induced by U, has size at most $\lfloor \frac{1}{2}|U| \rfloor$. Consequently, $\chi'(G) \geq \Gamma(G)$. If G is the Petersen graph, or the Petersen graph with one vertex deleted, then $\chi'(G) > \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$. However, Seymour conjectures that equality holds for planar graphs:

Conjecture 1.1 *If G is a planar graph, then $\chi'(G) = \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$.*

Conjecture 1.1 most likely does not have an easy proof, because it implies the Four-Color Theorem. Marcotte [5] proved that this conjecture holds for graphs which do not contain $K_{3,3}$ and do not contain $K_5 \setminus e$ as a minor (where $K_5 \setminus e$ is the graph obtained from K_5 by removing one of its edges). This result extended a previous result by Seymour [6], who proved that his conjecture holds for series-parallel graphs (a graph is *series-parallel* if it has no subgraph isomorphic to a subdivision of K_4):

Theorem 1.2 *If G is a series-parallel graph, and k is an integer with $k \geq \max\{\Delta(G), \Gamma(G)\}$ then G is k-edge-colorable.*

*Research supported in part by CNPq Proc. No. 301174/97-0, FAPESP Proc. No. 98/14329 and 96/04505-2 and PRONEX/CNPq 664107/1997-4 (Brazil).

†Research supported in part by NSA under Grant No. MDA904-98-1-0517 and by NSF under Grant No. DMS-9970514.

It should be noted that Theorem 1.2 is fairly easy for simple graphs; the difficulty lies in the presence of parallel edges. Seymour's proof is elegant and interesting, but the induction step requires the verification of a large number of inequalities. We give a simpler proof, based on a structural lemma about series-parallel graphs, which in turn is an easy consequence of the well-known fact that every simple series-parallel graph has a vertex of degree at most two. Our work was motivated by the list edge-coloring conjecture of [1] (see also [3, Problem 12.20]):

Conjecture 1.3 *Every graph is $\chi'(G)$-edge-choosable.*

At present there seems to be no credible approach for proving the conjecture in full generality. We were trying to gain some insight by studying it for series-parallel graphs. The conjecture has been verified for *simple* series-parallel graphs in [4], but it is open for series-parallel graphs with parallel edges. Our efforts only resulted in a simpler proof of Theorem 1.2 and in a linear-time algorithm for checking whether or not a series-parallel graph can be colored with a given number of colors. Our algorithm substantially simplifies an earlier algorithm of Zhou, Suzuki and Nishizeki [7].

2 Three lemmas

For our proof of Theorem 1.2 we need three lemmas. The first two are easy, and the third appeared in [4]. Let G be a graph, and let u, v be adjacent vertices of G. We use uv to denote the unique edge with ends u and v in the underlying simple graph of G. If G has m edges with ends u and v, then we say that uv has *multiplicity* m. If u and v are not adjacent, then we say that uv has multiplicity zero. Let G be a graph, let κ be a k-edge-coloring of a subgraph H of G, let $u \in V(G)$, and let $i \in \{1, 2, \ldots, k\}$. We say that u *sees* i and that i *is seen by* u if $\kappa(f) = i$ for some edge f of H incident with u.

Lemma 2.1 *Let G be a graph, let $u_0 \in V(G)$, let u_1, u_2 be distinct neighbors of u_0, let H be the graph obtained from G by deleting all edges with one end u_0 and the other end u_1 or u_2, and let κ be a k-edge-coloring of H. For $i = 1, 2$ let m_i be the multiplicity of u_0u_i in G, and for $i = 0, 1, 2$ let S_i be the set of colors seen by u_i. If $m_1 + |S_0 \cup S_1| \le k$, $m_2 + |S_0 \cup S_2| \le k$ and $m_1 + m_2 + |S_0 \cup (S_1 \cap S_2)| \le k$, then κ can be extended to a k-edge-coloring of G.*

Proof. Since $m_1 + |S_0 \cup S_1| \le k$, the edges with ends u_0 and u_1 can be colored using colors not in $S_0 \cup S_1$. We do that, using as many colors in S_2 as possible. If the u_0u_1 edges can be colored using colors in S_2 only, then there are at least $k - |S_0 \cup S_2| \ge m_2$ colors left to color the edges with ends u_0 and u_2, and so κ can be extended to a k-edge-coloring of G, as desired. Otherwise, the u_0u_1 edges of G will be colored using $|S_2 \backslash (S_0 \cup S_1)|$ colors from S_2, and $m_1 - |S_2 \backslash (S_0 \cup S_1)|$ other colors. Thus the number of colors available to color the u_0u_2 edges of G is at least $k - |S_0 \cup S_2| - (m_1 - |S_2 \backslash (S_0 \cup S_1)|) = k - m_1 - |S_0 \cup (S_1 \cap S_2)| \ge m_2$, and so the coloring can be completed to a k-edge-coloring of G, as desired. ■

Lemma 2.2 *Let k be an integer, and let G be a graph with $\Delta(G) \le k$. Then $\Gamma(G) \le k$ if and only if $2|E(G[U])| \le k(|U| - 1)$ for every set $U \subseteq V(G)$ such that $|U|$ is odd and at least three, and the underlying graph of $G[U]$ has no vertices of degree at most one.*

Proof. The "only if" part is clear. To prove the "if" part we must show that $2|E(G[U])| \le k(|U| - 1)$ for every set $U \subseteq V(G)$ such that $|U|$ is odd and at least three. We proceed by induction on $|U|$. We may assume that the underlying graph of $G[U]$ has a vertex u of degree at most one, for otherwise the conclusion follows from the hypothesis. If u has degree one in the underlying graph of $G[U]$, then let v be its unique neighbor; otherwise let $v \in U \backslash \{u\}$ be arbitrary. Let $U' = U \backslash \{u, v\}$. Then $2|E(G[U])| \le 2\Delta(G) + 2|E(G[U'])| \le 2k + k(|U'| - 1) \le k(|U| - 1)$ by the induction hypothesis if

$|U| > 3$ and trivially otherwise, as desired. ■

The third lemma appeared in [4]. For the sake of completeness we include its short proof.

Lemma 2.3 *Every non-null simple series-parallel graph G has one of the following:*

(a) a vertex of degree at most one,

(b) two distinct vertices of degree two with the same neighbors,

(c) two distinct vertices u, v and two not necessarily distinct vertices $w, z \in V(G) \backslash \{u, v\}$ such that the neighbors of v are u and w, and every neighbor of u is equal to v, w, or z, or

(d) five distinct vertices v_1, v_2, u_1, u_2, w such that the neighbors of w are u_1, u_2, v_1, v_2, and for $i = 1, 2$ the neighbors of v_i are w and u_i.

Proof. We proceed by induction on the number of vertices. Let G be a non-null simple series-parallel graph, and assume that the result holds for all graphs on fewer vertices. We may assume that G does not satisfy (a), (b), or (c). Thus G has no two adjacent vertices of degree two. By suppressing all vertices of degree two (that is, contracting one of the incident edges) we obtain a series-parallel graph without vertices of degree two or less. Therefore, by a well-known property of series-parallel graphs [2], this graph is not simple. Since G does not satisfy (b), this implies that G has a vertex of degree two that belongs to a cycle of length three. Let G' be obtained from G by deleting all vertices of degree two that belong to a cycle of length three. First notice that if G' has a vertex of degree less than two, then the result holds for G (cases (a), (b), or case (c) with $w = z$). Similarly, if G' has a vertex of degree two that does not have degree two in G, then the result holds (one of the cases (b)–(d) occurs). Thus we may assume that G' has minimum degree at least two, and every vertex of degree two in G' has degree two in G. By induction, (b), (c), or (d) holds for G', but it is easy to see that then one of (b), (c), or (d) holds for G. ■

3 Proof of Theorem 1.2

We proceed by induction on $|E(G)|$, and, subject to that, by induction on $|V(G)|$. The theorem clearly holds for graphs with no edges, so we assume that G has at least one edge, and that the theorem holds for graphs with fewer edges or the same number of edges but fewer vertices. Let S be the underlying simple graph of G. We apply Lemma 2.3 to S, and distinguish the corresponding cases.

If case (a) holds, let G' be the graph obtained from G by removing a vertex of degree at most one in S. The rest is straightforward: $k \geq \max\{\Delta(G'), \Gamma(G')\}$ and so, by induction, there is a k-edge-coloring of G'. From this k-edge-coloring, it is easy to obtain a k-edge-coloring for G.

If case (b) holds, let u and v be two distinct vertices of degree two in S with the same neighbors. Let the common neighbors be x and y. Let a, b, c, d be the multiplicities of ux, uy, vx, vy, respectively. See Figure 1(a). From the symmetry we may assume that $a \geq d$. Let G' be obtained from $G \backslash v$ by deleting d edges with ends u and x, and adding d edges with ends u and y. See Figure 1(b). Then clearly $\Delta(G') \leq k$, and it follows from Lemma 2.2 that $\Gamma(G') \leq k$. By the induction hypothesis the graph G' has a k-edge-coloring κ'. Let A be a set of colors of size d used by a subset of the edges of G' with ends u and y, chosen so that as few as possible of these colors are seen by x. By deleting those edges we obtain a coloring of $G \backslash v$, where d edges with ends u and x are uncolored. Next we color those d uncolored edges, first using colors in A not seen by x, and then using arbitrary colors not seen by x or u. This can be done: if at least one color in A is seen by x, then once we exhaust colors of A not seen by x, the choice of A implies that every color seen by u is seen by x, and so the coloring can be completed, because x has degree at most k. This results in a k-edge-coloring of $G\backslash v$ with the property that at least d of the colors seen by x (namely the colors in A) are not seen by y. Thus the number of colors seen by both x and y is at most $k-c-d$ (v sees no colors), and clearly the number of colors seen by x is at most $k-c$ and the number of colors seen by y is at most $k-d$. By Lemma 2.1 this coloring can be extended to a k-edge-coloring of G, as desired.

We now assume a special case of (c) of Lemma 2.3. Let u, v, w, z be as in that lemma, with $w = z$. Then clearly $\Delta(G\backslash v) \leq k$ and $\Gamma(G\backslash v) \leq k$, and so $G\backslash v$ has a k-edge-coloring. This k-edge-coloring can be extended to a k-edge-coloring of G by first coloring the edges with ends w and v (this can be done because the degree of w is at most k), and then coloring the edges with ends u and v (there are enough colors for this because $|E(G[U])| \leq k$ for $U = \{u, v, w\}$).

Finally we assume that case (d) of Lemma 2.3 holds and we will show that our analysis includes the remainder of case (c) as a special case. Let v_1, v_2, u_1, u_2 and w be as in the statement of Lemma 2.3, and let a, b, c, d, e and f be the multiplicities of u_1v_1, u_1w, v_1w, v_2w, u_2w and u_2v_2, respectively, as in Figure 2(a). In order to include case (c) we will not be assuming that a, b, c, d, e and f are nonzero; we only assume that $c + d > 0$. (This is why the primary induction is on $|E(G)|$.) If $a+b+c+d+e+f \leq k$, then a k-edge-coloring of $G\backslash\{v_1w, v_2w\}$ can be extended to a k-edge-coloring of G, and so we may assume that $k < a+b+c+d+e+f$. Since w has degree at most k we have $b+c+d+e \leq k$, and by considering the sets $U = \{u_1, v_1, w\}$ and $U = \{u_2, v_2, w\}$ we deduce that $a+b+c \leq k$ and $d+e+f \leq k$. Let $z_1 = \max\{0, a+b+c+e-k\}$, $z_2 = \max\{0, b+d+e+f-k\}$ and $s = k-(b+c+d+e)$. Thus $z_1 \leq e$, $z_2 \leq b$, $s \geq 0$ and

$$a+f-z_1-z_2-s = \begin{cases} k-(b+e) & \text{if } z_1 > 0 \text{ and } z_2 > 0 \\ a+c & \text{if } z_1 = 0 \text{ and } z_2 > 0 \\ d+f & \text{if } z_1 > 0 \text{ and } z_2 = 0 \\ a+f-s & \text{if } z_1 = z_2 = 0. \end{cases} \quad (1)$$

We claim that there exist nonnegative integers s_1 and s_2 such that $s = s_1 + s_2$, $s_1 \leq a - z_1$ and $s_2 \leq f - z_2$. To prove this claim it suffices to check that $a - z_1 \geq 0$, $f - z_2 \geq 0$ and $a - z_1 + f - z_2 \geq s$. We have $a - z_1 \geq \min\{a, k-(b+c+e)\} \geq \min\{a, d\} \geq 0$, and by symmetry $f - z_2 \geq 0$. The third inequality follows from (1). This proves the existence of s_1 and s_2.

Let G' be obtained from G by removing the vertices v_1, v_2, w, adding two new vertices, x and y, and adding $a - z_1 - s_1$ edges with ends x and u_1, $f - z_2 - s_2$ edges with ends x and u_2, $b - z_2$ edges with ends y and u_1, $e - z_1$ edges with ends y and u_2, and $z_1 + z_2$ edges with ends u_1 and u_2. See Figure 2(b). Thus $|E(G')| < |E(G)|$.

It follows from (1) that x has degree at most k. Since all other vertices of G' clearly have degree at most k, we see that $k \geq \Delta(G')$. We claim that $k \geq \Gamma(G')$. By Lemma 2.2 we must

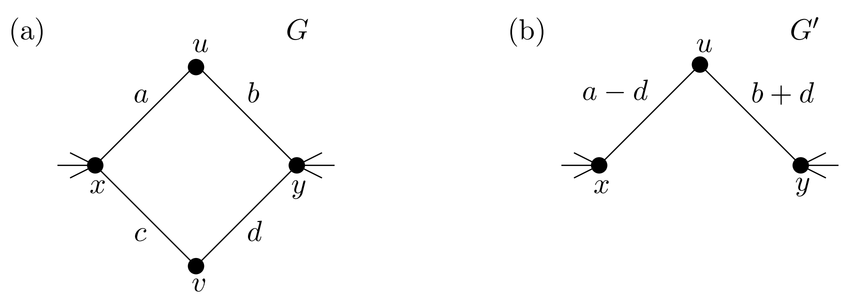

Figure 1: Configurations referring to Case (b).

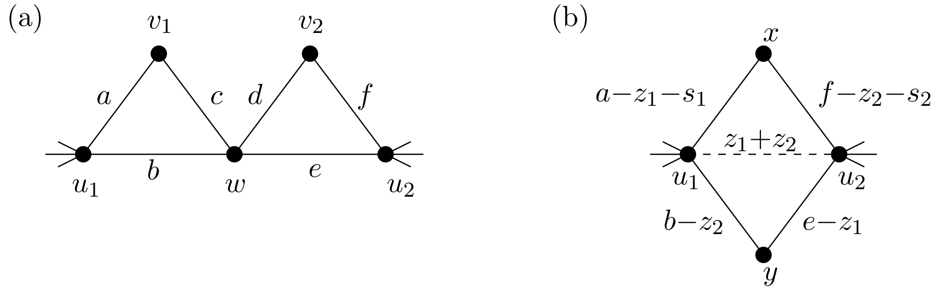

Figure 2: Configurations referring to Case (d).

show that $2|E(G'[X'])| \leq k(|X'|-1)$ for every set $X' \subseteq V(G')$ such that $|X'|$ is odd, $|X'| \geq 3$ and the underlying graph of $G'[X']$ has no vertices of degree at most one. If $|X' \cap \{u_1, u_2\}| \leq 1$, then $G[X'] = G'[X']$, and the result follows. Thus we may assume that $u_1, u_2 \in X'$. We need to distinguish several cases. If $x, y \in X'$, then let $X = X' \setminus \{x, y\}$. We have $2|E(G'[X'])| = 2|E(G[X])| + 2(a - z_1 - s_1 + f - z_2 - s_2 + z_1 + z_2 + b - z_2 + e - z_1) \leq k(|X'|-1)$, using the induction hypothesis and the relations $s_1 + s_2 = k - (b + c + d + e)$, $z_1 \geq a + b + c + e - k$ and $z_2 \geq b + d + e + f - k$. If $x \in X'$ and $y \notin X'$ we put $X = X' \setminus \{x\} \cup \{w, v_1, v_2\}$, and if $x \notin X'$ and $y \in X'$ we put $X = X' \setminus \{y\} \cup \{w\}$. In either of these two cases the counting is straightforward. Finally, we assume that $x, y \notin X'$. If $z_1 = z_2 = 0$, then $G[X'] = G'[X']$, and so the conclusion holds. If $z_1 > 0$ and $z_2 > 0$, then let $X = X' \setminus \{u_1, u_2\}$. We have $2|E(G'[X'])| \leq 2|E(G[X])| + 2(k-(a+b)+k-(e+f)+z_1+z_2) \leq k(|X|-1)+2(b+c+d+e) \leq k(|X'|-1)$, where the second inequality follows from the induction hypothesis (or is trivial if $|X| = 1$) and the definition of z_1 and z_2. Finally, from the symmetry between z_1 and z_2 it suffices to consider the case $z_1 = 0$ and $z_2 > 0$. In that case we put $X = X' \cup \{w, v_2\}$. Then $2|E(G'[X'])| = 2|E(G[X])| + 2(z_1 + z_2 - (b+d+e+f)) \leq k(|X'|-1)$, using the induction hypothesis and the definition of z_1 and z_2. This completes the proof that $k \geq \Gamma(G')$.

By induction there exists a k-edge-coloring κ' of G'. Let $Z_1 \cup Z_2$ be the colors used on the $z_1 + z_2$ edges of $E(G') \setminus E(G)$ with ends u_1 and u_2, so that $|Z_1| = z_1$ and $|Z_2| = z_2$. Let G'' be the graph obtained from G by deleting all edges with one end w and the other end v_1 or v_2. We first construct a suitable k-edge-coloring κ'' of G''. To do so we start with the restriction of κ' to $E(G'') \cap E(G')$, and then use Z_1 and the colors of the xu_1 edges of G' to color a subset of the u_1v_1 edges of G, we use Z_2 and the colors of the yu_1 edges of G' to color all of the wu_1 edges of G, and symmetrically we use Z_1 and the colors of the u_2y edges of G' to color all the wu_2 edges of G, and we use Z_2 and all the colors of the xu_2 edges of G' to color a subset of the v_2u_2 edges of G. We color the s_1 uncolored u_1v_1 edges and the s_2 uncolored u_2v_2 edges arbitrarily. That can be done, because u_i is the only neighbor of v_i in G''. This completes the definition of κ''. Now the number of colors seen by v_1 or w is at most $a - z_1 - s_1 + z_1 + z_2 + b - z_2 + e - z_1 + s_1 = a + b + e - z_1 \leq k - c$, and similarly the number of colors seen by v_2 or w is at most $k - d$. The number of colors seen by w, or by both v_1 and v_2 is at most $b - z_2 + e - z_1 + z_1 + z_2 + s \leq k - (c + d)$. By Lemma 2.1 the k-edge-coloring κ'' can be extended to a k-edge-coloring of G, as desired.

4 A linear-time algorithm

In this section we present a linear-time algorithm to decide whether $\chi'(G) \leq k$, where the series-parallel graph G and the integer k are part of the input instance. The idea of the algorithm is very simple – we repeatedly find vertices of the underlying simple graph satisfying one of (a)–(d) of Lemma 2.3, construct the graph G' as in the proof of Theorem 1.2, apply the algorithm recursively to G' to check whether $\chi'(G') \le k$, and from that knowledge we deduce whether $\chi'(G) \le k$. The construction of G' is straightforward, and the decision whether $\chi'(G) \le k$ is easy: suppose, for instance, that we find vertices v_1, v_2, u_1, u_2, w as in Lemma 2.3(d), and let a, b, c, d, e, f be as in the proof of Theorem 1.2. If $a+b+c+d+e+f \ge k$, then construct G' as in the proof; we have $\chi'(G) \le k$ if and only if $\chi'(G') \le k$ and $a+b+c \le k$ and $d+e+f \le k$. If $a+b+c+d+e+f \le k$, then $\chi'(G) \le k$ if and only if $\chi'(G \backslash w) \le k$. Thus it remains to describe how to find the vertices as in Lemma 2.3. That can be done by a slight modification of a linear-time recognition algorithm for series-parallel graphs. We need a few definitions in order to describe the algorithm.

Let H be a graph, and let λ be a function assigning to each edge $e \in E(H)$ a set $\lambda(e)$ disjoint from $V(H)$ in such a way that $\lambda(e) \cap \lambda(e') = \emptyset$ for distinct edges $e, e' \in E(H)$. Let H_λ be the graph obtained from H by adding, for each edge $e \in E(H)$ and each $x \in \lambda(e)$, a vertex x of degree two, adjacent to the two ends of e. Then H_λ is unique up to isomorphism, and so we can speak of the graph H_λ. Now let $\mu : E(H_\lambda) \to \mathbb{Z}_0^+$ be a function, and let H_λ^μ be the graph obtained from H_λ by replacing each edge $e \in E(H_\lambda)$ by $\mu(e)$ parallel edges with the same ends. In those circumstances we say that (H, λ, μ) is an *encoding*, and that it is an *encoding* of H_λ^μ.

For a graph H and $v \in V(H)$ we let $\deg_H(v)$ denote the number of edges incident to v in H and $\mathrm{val}_H(v)$ denote the number of distinct neighbors of v in H. Thus $\mathrm{val}_H(v) \le \deg_H(v)$ with equality if and only if v is incident with no parallel edges. We say that a function $C : V(H) \to \mathbb{Z}_0^+$ is a *counter* for a graph H if $\deg_H(v) - \mathrm{val}_H(v) \le C(v)$ for every vertex $v \in V(H)$. We say that a vertex $v \in V(H)$ is *active* if either $\deg_H(v) \le 2$ or $\deg_H(v) \le 3C(v)$.

The following lemma guarantees that if there are no active vertices, then the graph is null.

Lemma 4.1 *Let H be a non-null series-parallel graph, and let C be a counter for H. Then there exists an active vertex.*

Proof. As noted in the proof of Lemma 2.3, the underlying simple graph of H has a vertex of degree at most two. Thus H has a vertex v with $\mathrm{val}_H(v) \le 2$. If $\deg_H(v) > 3C(v)$, then

$$\deg(v) - 2 \le \deg_H(v) - \mathrm{val}_H(v) \le C(v) < \deg_H(v)/3,$$

which implies $\deg_H(v) \le 2$. Thus v is active, as desired. ■

4.1 The algorithm

The input for the algorithm is a series-parallel graph G and a non-negative integer k, where the graph G is presented by means of its underlying undirected graph and a function $E(G) \to \mathbb{Z}^+$ that describes the multiplicity of each edge.

The algorithm starts by checking whether $\deg_G(v) \le k$ for all $v \in V(G)$. If not, it outputs "no, $\chi'(G) \not\le k$" and terminates. Otherwise let H be the underlying undirected graph of G, let $\lambda(e) := \emptyset$ for every edge $e \in E(H)$, let $\mu(e)$ be the multiplicity of e in G, and let $C(v) := 0$ for every $v \in V(H)$. Then (H, λ, μ) is an encoding of G and C is a counter for H. The algorithm computes the list of all active vertices of H. It does not matter how L is implemented as long as elements can be deleted and added in constant time.

After this, the algorithm is iterative. Each iteration starts with an encoding (H, λ, μ) of the current series-parallel graph G, a counter C for H and a list L which includes all active vertices of H.

Each iteration consists of the following. If $L = \emptyset$, then we output "yes, $\chi'(G) \leq k$" and terminate, else we let v be a vertex in L. If $v \notin V(H)$ or v is not active, then we remove v from L and move to the next iteration. If $v \in V(H)$ and v is active, then there are three possible cases.

If $\deg_H(v) > 2$, then $\deg_H(v) \leq 3C(v)$, because v is active. We rearrange the adjacency list of v, removing all but one edge from each class of parallel edges incident with v, adjusting λ and μ so that (H, λ, μ) is still an encoding of G. We set $C(v) := 0$, include in L all vertices whose degree decreased and move to the next iteration.

If $\deg_H(v) = \mathrm{val}_H(v) = 2$ and $\lambda(vx) = \lambda(vy) = 0$, where x and y are the two distinct neighbors of v, then we remove v from H and add a new edge $f = xy$ to H. We set $\mu(f) := 0, \lambda(f) := \{v\}$, increase both $C(x)$ and $C(y)$ by one, add x and y to L and move to the next iteration.

If $\deg_H(v) \leq 2$ but the previous case does not apply, then we have located vertices of G satisfying one of (a) to (d) of Lemma 2.3. We check if the local conditions are satisfied or not (for example, in case (d), if $a+b+c+d+e+f \geq k$, we check whether $a+b+c \leq k$ and $d+e+f \leq k$); if they are not, we output "no, $\chi'(G) \not\leq k$" and terminate. Otherwise, we modify the encoding (H, λ, μ) to get an encoding of the graph G' described in the proof of Theorem 1.2. This involves deleting vertices from H and adding edges to H. Every time an edge of H incident with a vertex $z \in V(H)$ is deleted or added we increase $C(z)$ by one and add z to L. We move to the next iteration.

The correctness of the algorithm follows from Lemma 4.1 and from the proof of Theorem 1.2.

To analyze the running-time, let n denote the number of vertices of the input graph G. The initial steps of the algorithm can be done in $O(n)$ time. Each iteration takes time proportional to the decrease in the quantity

$$2K \cdot |V(H)| + K \cdot \sum_{e \in E(H)} \lambda(e) + |L| + 4 \cdot \sum_{v \in V(H)} C(v),$$

where K is a sufficiently large constant. Thus the running-time of the algorithm is $O(n)$.

References

[1] B. Bollobás, A. J. Harris, *List colorings of graphs*, Graphs and Combinatorics 1 (1985), 115–127.

[2] J. Duffin, *Topology of series-parallel networks*, Journal of Mathematical Analysis and Applications 10 (1965) 303–318.

[3] T. R. Jensen, B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

[4] M. Juvan, B. Mohar, and R. Thomas, *List Edge-Colorings of Series-Parallel Graphs*, Electronic Journal of Combinatorics 6 (1999), no. 1, Research Paper 42.

[5] O. Marcotte, *Optimal Edge-Colourings for a Class of Planar Multigraphs*, Combinatorica **21** (3) (2001) 361–394.

[6] P.D. Seymour, *Colouring Series-Parallel Graphs*, Combinatorica **10** (4) (1990) 379–392.

[7] X. Zhou, H. Suzuki, and T. Nishizeki, *A Linear-Time Algorithm for Edge-Coloring Series-Parallel Multigraphs*, Journal of Algorithms **20** (1996), 174–201.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-9970514. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.