# A FURTHER GENERALIZATION OF THE COLOURFUL CARATHÉODORY THEOREM

FRÉDÉRIC MEUNIER AND ANTOINE DEZA

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ABSTRACT. Given $d+1$ sets, or colours, $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ of points in $\mathbb{R}^d$, a *colourful* set is a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| \leq 1$ for $i = 1, \ldots, d+1$. The convex hull of a colourful set $S$ is called a *colourful simplex*. Bárány's colourful Carathéodory theorem asserts that if the origin $\mathbf{0}$ is contained in the convex hull of $\mathbf{S}_i$ for $i = 1, \ldots, d+1$, then there exists a colourful simplex containing $\mathbf{0}$. The sufficient condition for the existence of a colourful simplex containing $\mathbf{0}$ was generalized to $\mathbf{0}$ being contained in the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ for $1 \leq i < j \leq d+1$ by Arocha et al. and by Holmsen et al. We further generalize the theorem by showing that a colourful simplex containing $\mathbf{0}$ exists if, for $1 \leq i < j \leq d+1$, there exists $k \notin \{i, j\}$ such that, for all $x_k \in \mathbf{S}_k$, the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ intersects the ray $\overrightarrow{x_k\mathbf{0}}$ in a point distinct from $x_k$. A slightly stronger version of this new colourful Carathéodory theorem is also given. This result provides a short and geometric proof of the previous generalization of the colourful Carathéodory theorem. We also give an algorithm to find a colourful simplex containing $\mathbf{0}$ under the generalized condition. In the plane an alternative and more general proof using graphs is given. In addition, we observe that, in general, the existence of one colourful simplex containing $\mathbf{0}$ implies the existence of at least $\min_i |\mathbf{S}_i|$ colourful simplices containing $\mathbf{0}$. In other words, any condition implying the existence of a colourful simplex containing $\mathbf{0}$ actually implies the existence of $\min_i |\mathbf{S}_i|$ such simplices.

## 1. COLOURFUL CARATHÉODORY THEOREMS

Given $d+1$ sets, or colours, $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ of points in $\mathbb{R}^d$, we call a set of points drawn from the $\mathbf{S}_i$'s *colourful* if it contains at most one point from each $\mathbf{S}_i$. A *colourful simplex* is the convex hull of a colourful set $S$, and a colourful set of $d$ points which misses $\mathbf{S}_i$ is called an $\widehat{i}$*-transversal*. The colourful Carathéodory Theorem 1.1 by Bárány provides a sufficient condition for the existence of a colourful simplex containing the origin $\mathbf{0}$.

**Theorem 1.1** ([Bár82])**.** *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$ such that $\mathbf{0} \in \mathrm{conv}(\mathbf{S}_i)$ for $i = 1 \ldots d+1$. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

Theorem 1.1 was generalized by Arocha et al. [ABB+09] and by Holmsen et al. [HPT08] who provided a more general sufficient condition for the existence of a colourful simplex containing the origin.

**Theorem 1.2** ([ABB+09, HPT08])**.** *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$ such that $\mathbf{0} \in \mathrm{conv}(\mathbf{S}_i \cup \mathbf{S}_j)$ for $1 \leq i < j \leq d+1$. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \mathrm{conv}(S)$.*

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We generalize Theorem 1.2 by further generalizing the sufficient condition for the existence of a colourful simplex containing the origin. Moreover, the proof, given in Section 4, provides a short and geometric alternative proof for Theorem 1.2. Let $\overrightarrow{x_k\mathbf{0}}$ denote the ray originating from $x_k$ towards $\mathbf{0}$.

**Theorem 1.3.** *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$. Assume that, for each $1 \leq i < j \leq d+1$, there exists $k \notin \{i, j\}$ such that, for all $x_k \in \mathbf{S}_k$, the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ intersects the ray $\overrightarrow{x_k\mathbf{0}}$ in a point distinct from $x_k$. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \operatorname{conv}(S)$.*

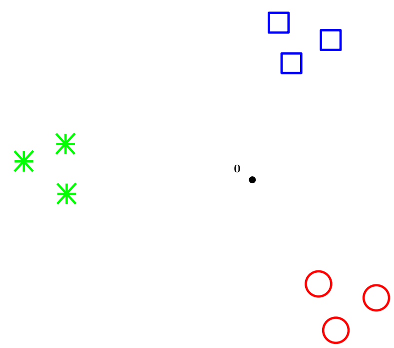


FIGURE 1. A set in dimension 2 satisfying the condition of Theorem 1.3 but not the one of Theorem 1.2.

Up to an additional argument to handle degenerate configurations, Theorem 1.3 can be derived from the slightly stronger Theorem 1.4 where $H^+(T_i)$ denotes, for any $\widehat{\imath}$-transversal $T_i$, the open half-space defined by $\operatorname{aff}(T_i)$ and containing $\mathbf{0}$. The proof of Theorem 1.4 and the argument to handle degeneracy for Theorem 1.3 are given in Section 4.

**Theorem 1.4.** *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ be finite sets of points in $\mathbb{R}^d$ such that the points in $\bigcup_i \mathbf{S}_i \cup \{\mathbf{0}\}$ are distinct and in general position. Assume that, for any $i \neq j$, $(\mathbf{S}_i \cup \mathbf{S}_j) \cap H^+(T_j) \neq \emptyset$ for any $\widehat{\jmath}$-transversal $T_j$. Then there exists a set $S \subset \bigcup_i \mathbf{S}_i$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$ and $\mathbf{0} \in \operatorname{conv}(S)$.*

Note that, as the conditions of Theorems 1.1 and 1.2, but unlike the one of Theorem 1.4, the condition of Theorem 1.3 is computationally easy to check. In the plane and assuming general position, Theorem 1.3 can be generalized to Theorem 1.5, see Section 6 for a proof.

**Theorem 1.5.** *Let $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ be finite sets of points in $\mathbb{R}^2$ such that the points in $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \cup \{\mathbf{0}\}$ are distinct and in general position. Assume that, for pairwise distinct $i, j, k \in \{1, 2, 3\}$, the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ intersects the line $\operatorname{aff}(x_k, \mathbf{0})$ for all $x_k \in \mathbf{S}_k$. Then there exists a set $S \subset \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3$ such that $|S \cap \mathbf{S}_i| = 1$ for $i = 1, 2, 3$ and $\mathbf{0} \in \operatorname{conv}(S)$.*

Figures 1 and 2 illustrate sets satisfying the condition of Theorem 1.3 but not the ones of Theorems 1.1 and 1.2. The $d$-dimensional version of the set given in Figure 1 corresponds to the points in $\mathbf{S}_i$ being clustered around the $i^{th}$ vertex of a simplex containing $\mathbf{0}$. Note that while all the $(d+1)^{d+1}$ colourful simplices of this configuration contain $\mathbf{0}$, for $d \geq 3$ it does not satisfy the condition of Theorem 1.3 but does satisfy the one of Theorem 1.4.

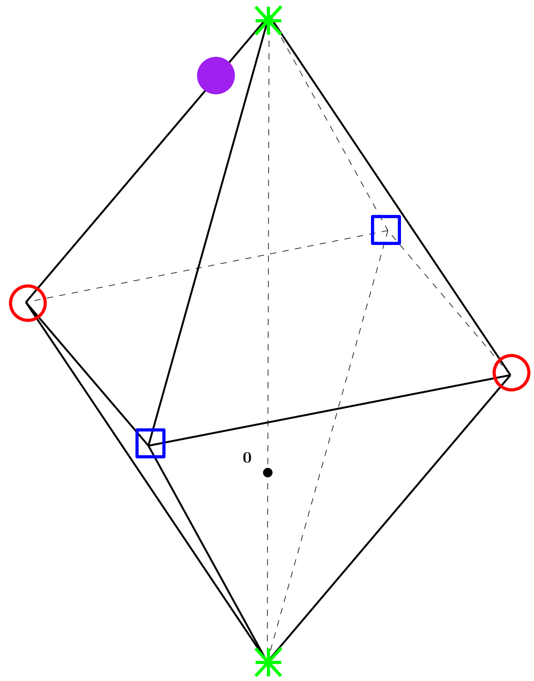


FIGURE 2. A set in dimension 3 satisfying the condition of Theorem 1.3 but not the one of Theorem 1.2.

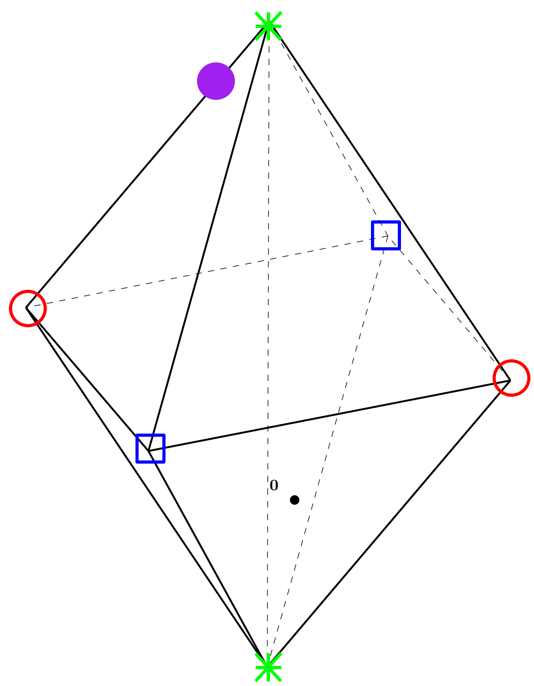


FIGURE 3. A set in dimension 3 satisfying, up to a slight perturbation, the condition of Theorem 1.4 but not the one of Theorem 1.3.

*Remark* 1.6. By a standard perturbation argument, Theorem 1.4 still holds if the general position assumption is replaced by the following milder condition: for each transversal $T_i$, there exists a hyperplane containing $T_i$ but not $\mathbf{0}$. Indeed, for each transversal $T_i$, denote by $H^0(T_i)$ a hyperplane containing $T_i$ and not $\mathbf{0}$. Up to a perturbation of the points in $\bigcup_i \mathbf{S}_i$ and the $H^0(T_i)$ accordingly, the configuration is in general position. With a sufficiently small perturbation, Theorem 1.4 can be applied because a point of $\bigcup_i \mathbf{S}_i \cup \{\mathbf{0}\}$ not in $H^0(T_i)$ remains on the same side of $H^0(T_i)$. The colourful simplex containing $\mathbf{0}$ given by Theorem 1.4 did contain $\mathbf{0}$ before being perturbed since $\mathbf{0}$ is in its interior. Similarly, Theorem 1.5 holds if $\mathbf{0}$ is not aligned with two points of distinct colours.

For example, Remark 1.6 implies the existence of a colourful simplex containing $\mathbf{0}$ for the set given in Figure 3 while the condition of Theorem 1.3 is not satisfied for $i = \square$ and $j = \bullet$. Figure 4 illustrates a set satisfying the condition of Theorem 1.5 but not the one of Theorem 1.4.

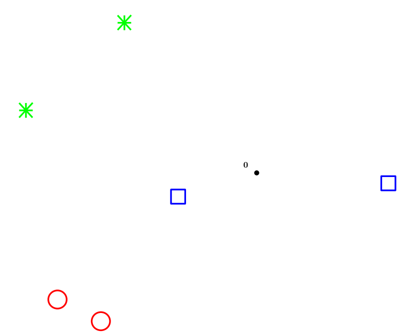

FIGURE 4. A set in dimension 2 satisfying the condition of Theorem 1.5 but not the one of Theorem 1.4.

## 2. GIVEN ONE, FIND ANOTHER ONE

In this section we present results that can be directly derived from the *Octahedron Lemma*, see the end of Section 4.1 for more details about this lemma. Our objective is twofold:

$(i)$ to stress in Corollary 2.2 that, in general, the existence of one colourful simplex containing $\mathbf{0}$ implies the existence of at least $\min_i |\mathbf{S}_i|$ colourful simplices containing $\mathbf{0}$. In other words, any condition implying the existence of a colourful simplex containing $\mathbf{0}$ actually implies the existence of $\min_i |\mathbf{S}_i|$ such simplices, and

$(ii)$ to obtain in Section 5 an algorithm finding a colourful simplex for sets satisfying the condition of Theorem 1.4 based on the algorithmic proof of Proposition 2.1.

**Proposition 2.1.** *Given $d+1$ sets, or colours, $\mathbf{S}_1^*, \mathbf{S}_2^*, \ldots, \mathbf{S}_{d+1}^*$ of points in $\mathbb{R}^d$ with $|\mathbf{S}_i^*| = 2$ for $i = 1, \ldots, d+1$, if there is a colourful simplex containing $\mathbf{0}$, then there is another colourful simplex containing $\mathbf{0}$.*

*Proof.* Without loss of generality we assume that the points in $\bigcup_i \mathbf{S}_i^* \cup \{\mathbf{0}\}$ are distinct and in general position. Consider the graph $G$ whose nodes are of three types: $(i)$ $T_1$ of cardinality

$d+2$ with $\mathbf{0} \in \mathrm{conv}(T_1)$, $|T_1 \cap \mathbf{S}_i^*| = 1$ for $i = 1,\dots,d$, and $|T_1 \cap \mathbf{S}_{d+1}^*| = 2$; $(ii)$ $T_2$ of cardinality $d+1$ with $\mathbf{0} \in \mathrm{conv}(T_2)$, $|T_2 \cap \mathbf{S}_i^*| = 1$ for $i = 1,\dots,d$ except for exactly one $i$, and $|T_2 \cap \mathbf{S}_{d+1}^*| = 2$; and $(iii)$ $T_3$ of cardinality $d+1$ with $\mathbf{0} \in \mathrm{conv}(T_3)$ and $|T_3 \cap \mathbf{S}_i^*| = 1$ for $i = 1,\dots,d+1$. The adjacency between the nodes of $G$ is defined as follows. There is no edge between nodes of type $T_2$ and $T_3$. A pair $(T_1, T_2)$, respectively $(T_1, T_3)$, is adjacent if and only if $T_2 \subseteq T_1$, respectively $T_3 \subseteq T_1$.

We now check the degree of $T_1, T_2$ and $T_3$ nodes. First consider a $T_1$ node. We recall that, under the general position assumption, there are exactly two subsets $X$ and $Y$ of cardinality $d+1$ containing $\mathbf{0}$ in their convex hull. Using the simplex method terminology, there is a unique leaving variable in a pivot step of the simplex method assuming non-degeneracy. Both of $X$ and $Y$ intersect $\mathbf{S}_i^*$ for $i = 1,\dots,d$ in at least one point except maybe for one $i$. Thus, $X$ and $Y$ are $T_2$ or $T_3$ nodes, hence the degree of a $T_1$ node is 2. Consider now a $T_2$ node, there is a $i_0 \neq d+1$ such that $|T_2 \cap \mathbf{S}_{i_0}^*| = 0$. The $T_2$ node is contained in exactly two $T_1$ nodes, each of them obtained by adding one of the points in $\mathbf{S}_{i_0}^*$. Hence the degree of a $T_2$ node is 2. Finally, consider a $T_3$ node, it is contained exactly one $T_1$ node obtained by adding the missing point of $\mathbf{S}_{d+1}^*$. Hence, the degree of a $T_3$ node is 1. The graph $G$ is thus a collection of node disjoint paths. Therefore, the existence of a colourful simplex containing $\mathbf{0}$ provides a $T_3$ node, and following the path in $G$ until reaching the other endpoint provides another node of degree 1, i.e. a $T_3$ node corresponding to a distinct colourful simplex containing $\mathbf{0}$. □

**Corollary 2.2.** *Given $d+1$ sets, or colours, $\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_{d+1}$ of points in $\mathbb{R}^d$, if there is a colourful simplex containing $\mathbf{0}$, then there are at least $\min_i |\mathbf{S}_i|$ colourful simplices containing $\mathbf{0}$.*

*Proof.* Let $I = \min_i |\mathbf{S}_i|$ and $\mathbf{S}_i = \{v_i^1, v_i^2, \dots\}$, and assume without loss of generality that the given colourful simplex containing $\mathbf{0}$ in its convex hull is $\mathrm{conv}(v_1^1, v_2^1, \dots, v_{d+1}^1)$. Applying Proposition 2.1 $(I-1)$ times with $\mathbf{S}_i^* = \{v_i^1, v_i^k\}$ we obtain an additional distinct colourful simplex containing $\mathbf{0}$ for each $k = 2,\dots,I$. □

Proposition 2.1 raises the following problem, which we call SECOND COVERING COLOURFUL SIMPLEX. Given $d+1$ sets, or colours, $\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_{d+1}$ of points in $\mathbb{R}^d$ with $|\mathbf{S}_i| \geq 2$ for $i = 1,\dots,d+1$, and a colourful set $S \subset \bigcup_i \mathbf{S}_i$ containing $\mathbf{0}$ in its convex hull, find another such set.

This problem is similar to the problem of, given one Hamiltonian circuit in a cubic graph, finding another one [Pap94, Tho78], or to the room partitioning problem [ES10]. These problems are such that the existence of an object of a certain type implies the existence of another object of same type. The basic approach for such problems consists in representing the objects by nodes of odd degrees in a graph. Since there is always an even number of odd degree nodes in a graph, the existence of one odd degree node implies the existence of another one. It means these problems belong to the *Polynomial Parity Argument* (PPA) class defined by Papadimitriou [Pap94] as search problems for which every instance is guaranteed to have a solution and for which the existence is proven through the following argument: In any graph with one odd degree node, there must be another odd degree node. Examples of PPA problems include BROUWER, BORSUK-ULAM, SECOND HAMILTONIAN CIRCUIT, NASH, see [Pap94].

The PPA class has a subclass of PPA-complete problems for which the existence of a polynomial algorithm would imply the existence of a polynomial algorithm for any problem in PPA. The first example of a PPA-complete problem was given by Grigni [Gri01]. We do not know whether SECOND COVERING COLOURFUL SIMPLEX is PPA-complete.

## 3. MINIMUM NUMBER OF COLOURFUL SIMPLICES CONTAINING $\mathbf{0}$

The minimum number $\mu(d)$ of colourful simplices containing $\mathbf{0}$ for sets satisfying the condition of Theorem 1.1 was investigated in [DHST06]. This paper shows that $2d \le \mu(d) \le d^2+1$, that $\mu(d)$ is even for odd $d$, and that $\mu(2)=5$, and conjectures that $\mu(d)=d^2+1$ for all $d \ge 1$. Subsequently, [BM07] verified the conjecture for $d=3$ and provided a lower bound of $\mu(d) \ge \max(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil)$ for $d \ge 3$, while [ST08] independently provided a lower bound of $\mu(d) \ge \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, before [DSX11] showed that $\mu(d) \ge \left\lceil \frac{(d+1)^2}{2} \right\rceil$. Note that since the points $\bigcup_i \mathbf{S}_i \cup \{\mathbf{0}\}$ are distinct and in general position, $\mathbf{0} \in \mathrm{conv}(\mathbf{S}_i)$ implies $|\mathbf{S}_i| \ge d+1$. Therefore, one can consider the analogous quantities $\mu^{\diamond}(d)$, respectively $\mu^{\circ}(d)$, defined as the the minimum number of colourful simplices containing $\mathbf{0}$ for sets satisfying the condition of Theorem 1.2, respectively Theorem 1.4, and $|\mathbf{S}_i| = d+1$ for $i=1,\dots,d+1$. It was noted in [CDSX11] that $\mu^{\diamond}(d)=d+1$. Since $\mu^{\circ}(d) \le \mu^{\diamond}(d)$, Theorem 1.4 and Corollary 2.2 imply the following equality.

**Proposition 3.1.** *For any $d \ge 2$, we have $\mu^{\circ}(d) = d+1$.*

## 4. PROOF OF THEOREM 1.4 AND ARGUMENT TO HANDLE DEGENERACY FOR THEOREM 1.3

4.1. **Proof of Theorem 1.4.** We recall that a $k$-simplex $\sigma$ is the convex hull of $(k+1)$ affinely independent points. An *abstract simplicial complex* is a family $\mathscr{F}$ of subsets of a finite ground set such that whenever $F \in \mathscr{F}$ and $G \subseteq F$, then $G \in \mathscr{F}$. These subsets are called *abstract* simplices. The *dimension* of an abstract simplex is its cardinality minus one. The *dimension* of a simplicial complex is the dimension of largest simplices. A *pure* abstract simplicial complex is a simplicial complex whose maximal simplices have all the same dimension. A *combinatorial $d$-pseudomanifold $\mathscr{M}$* is a pure abstract $d$-dimensional simplicial complex such that any abstract $(d-1)$-simplex is contained in exactly 2 abstract $d$-simplices.

Consider a ray $\mathbf{r}$ originating from $\mathbf{0}$ and intersecting at least one colourful $(d-1)$-simplex. Under the general position assumption for points in $\bigcup_i \mathbf{S}_i \cup \{\mathbf{0}\}$, one can choose $\mathbf{r}$ such that it intersects the interior of the colourful $(d-1)$-simplex, and that no two colourful simplices have the same intersection with $\mathbf{r}$. Let $\sigma$ be the *first* colourful $(d-1)$-simplex intersected by $\mathbf{r}$. Note that, given $\mathbf{r}$, $\sigma$ is uniquely defined. Without loss of generality, we can assume that the vertices of $\sigma$ form the $\widehat{d+1}$-*transversal* $\{v_1,\dots,v_d\}$.

Setting $j=d+1$, and $T_{d+1}=\{v_1,\dots,v_d\}$ in Theorem 1.4 gives $(\mathbf{S}_i \cup \mathbf{S}_{d+1}) \cap H^+(T_{d+1}) \neq \emptyset$. In other words, there is, for each $i$, a point either in $\mathbf{S}_{d+1} \cap H^+(T_{d+1})$ or in $(\mathbf{S}_i \setminus \{v_i\}) \cap H^+(T_{d+1})$. Assume first that for one $i$ the corresponding point belongs to $\mathbf{S}_{d+1}$, and name it $v'_{d+1}$. Then $\mathbf{r}$ intersects the boundary of $\mathrm{conv}(v_1,\dots,v_d,v'_{d+1})$ in only one point as otherwise $\mathbf{r}$ would intersect another colourful $(d-1)$-simplex before intersecting $\sigma$. Indeed, $\mathbf{r}$ leaves $H^+(T_{d+1})$ after intersecting $\sigma$. Thus, $\mathbf{r}$ intersects $\mathrm{conv}(v_1,\dots,v_d,v'_{d+1})$ in exactly one point; that is, $\mathbf{0} \in \mathrm{conv}(v_1,\dots,v_d,v'_{d+1})$. Therefore, we can assume that for each $i$ there is a point $v'_i \neq v_i$ in $\mathbf{S}_i \cap$

$H^+(T_{d+1})$, and consider the $\widehat{d+1}$-transversal $T' = \{v'_1, \ldots, v'_d\}$ and the associated colourful $(d-1)$-simplex $\sigma' = \mathrm{conv}(v'_1, \ldots, v'_d)$. Let $\mathscr{M}$ be the abstract simplicial complex defined by:

$$\mathscr{M} = \{F \cup F' : F \subseteq V(\sigma),\, F' \subseteq V(\sigma') \text{ and } c(F) \cap c(F') = \emptyset\}$$

where $V(\sigma)$ denotes the vertex set of $\sigma$ and $c(x) = i$ for $x \in \mathbf{S}_i$. The simplicial complex $\mathscr{M}$ is a combinatorial $(d-1)$-pseudomanifold. Note that $V(\sigma)$ and $V(\sigma')$ are abstract simplices of $\mathscr{M}$. Let $M$ be the collection of the convex hulls of the abstract simplices of $\mathscr{M}$. Note that the vertices of all maximal simplices of $M$ form $\widehat{d+1}$-transversals and that $\mathscr{M}$ is not necessarily a simplicial complex in the geometric meaning as some pairs of geometric $(d-1)$-simplices might have intersecting interiors.

We recall that for any generic ray originating from $\mathbf{0}$, the parity of the number of times its intersects $M$ is the same. We remark that this number can not be even as, otherwise, we would have a colourful $(d-1)$-simplex closer to $\mathbf{0}$ than $\sigma$ on $\mathbf{r}$ since, $M$ being contained in the closure of $H^+(T_{d+1})$, when $\mathbf{r}$ intersects $\sigma$, it is the last intersection. Thus, the number of times $\mathbf{r}$ intersects $M$ is odd, and actually equal to 1. Take now any point $v \in \mathbf{S}_{d+1}$ and consider the ray originating from $\mathbf{0}$ towards the direction opposite to $v$. This ray intersects $M$ in a colourful $(d-1)$-simplex $\tau$; that is, $\mathbf{0} \in \mathrm{conv}(\tau \cup \{v\})$. □

Assuming $\bigcup_i \mathbf{S}_i$ lies on the sphere $\mathbb{S}^{d-1}$, note that $\widehat{i}$-transversals generate full dimensional colourful cones pointed at $\mathbf{0}$. We say that a transversal *covers* a point if the point is contained in the associated cone. Colourful simplices containing $\mathbf{0}$ are generated whenever the antipode of a point of colour $i$ is covered by an $\widehat{i}$-transversal. In particular, one can consider combinatorial *octahedra*, or cross polytopes, generated by pairs of disjoint $\widehat{i}$-transversals, and rely on the fact that every octahedron $\Omega$ either covers all of $\mathbb{S}^{d-1}$ with colourful cones, or, every point $x \in \mathbb{S}^{d-1}$ that is covered by colourful cones from $\Omega$ is covered by at least two distinct such cones. In the case where the points of $\Omega$ form an octahedron in the geometric sense, these correspond to the cases where $\mathbf{0}$ is inside and outside $\Omega$ respectively. For a proof, see for example the *Octahedron Lemma* of [BM07]. The proof of Theorem 1.4 can therefore be reformulated as: $(i)$ either the pair of $\widehat{d+1}$-transversals $(T, T')$ forms a octahedron covering $\mathbb{S}^{d-1}$, or $(ii)$ $\mathbf{0}$ belongs to a colourful simplex having $\mathrm{conv}(T)$ as a facet.

**Corollary 4.1.** *Given $d+1$ sets, or colours, $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_{d+1}$ of points in $\mathbb{R}^d$, if there is a pair of transversals forming an octahedron covering $\mathbb{S}^{d-1}$, then there are at least* $\min_{i \neq j} |\mathbf{S}_i \cup \mathbf{S}_j| - 2$ *colourful simplices containing* $\mathbf{0}$.

*Proof.* Denote $\mathbf{S}_i = \{v_i^1, v_i^2, \ldots\}$. Without loss of generality, we can assume that the octahedron covering $\mathbb{S}^{d-1}$ is formed by a pair $(T_1, T_2)$ of $\widehat{d+1}$-transversals such that $T_1 = \{v_1^1, v_2^1, \ldots, v_d^1\}$ covers a point in $\mathbf{S}_{d+1}$. Applying Proposition 2.1 $(I-2)$ times with $\mathbf{S}_i^* = \{v_i^1, v_i^k\}$ we obtain, for each $k = 3, \ldots, I$, an additional distinct colourful simplex containing $\mathbf{0}$ distinct from the $|\mathbf{S}_{d+1}|$ colourful simplices containing $\mathbf{0}$ given by the assumption that $(T_1, T_2)$ covers $\mathbb{S}^{d-1}$. □

As noted in [DSX11], the condition $\bigcap_i \mathrm{conv}(\mathbf{S}_i) \neq \emptyset$ implies the existence of at least one octahedron covering $\mathbb{S}^{d-1}$. Thus, setting $|\mathbf{S}_i| = d+1$, Corollary 4.1 yields $\mu(d) \geq 2d$ as proven in [DHST06].

4.2. **Argument to handle degeneracy for Theorem 1.3.** As noticed in Remark 1.6, Theorem 1.4, and therefore Theorem 1.3, holds if, for each transversal $T_i$, there exists a hyperplane containing $T_i$ but not $\mathbf{0}$. Actually, one can check that the proof of Theorem 1.4 still works if there is at least one transversal $T$ such that $\mathbf{0} \notin \mathrm{aff}(T)$ and such that the points of $T$ are affinely independent. Indeed, we can then choose a ray $\mathbf{r}$ such that, for any pair $(T, T')$ of transversals, $\mathbf{r} \cap \mathrm{aff}(T) = \mathbf{r} \cap \mathrm{aff}(T')$ if and only if $\mathrm{aff}(T) = \mathrm{aff}(T')$. Without loss of generality, we choose a transversal $T_{d+1} = \{v_1, \dots, v_d\}$ such that $\mathrm{aff}(T_{d+1})$ is the first affine subspace spanned by a transversal intersected by $\mathbf{r}$. Then, setting $j = d+1$, the condition of Theorem 1.3, namely the existence of a $k \in \{1, \dots, d\} \setminus \{i\}$ such that the convex hull of $\mathbf{S}_i \cup \mathbf{S}_{d+1}$ intersects the ray $\overrightarrow{v_k \mathbf{0}}$ in a point distinct from $v_k$, implies $(\mathbf{S}_i \cup \mathbf{S}_{d+1}) \cap H^+(T_{d+1}) \neq \emptyset$.

Therefore, an additional argument is needed only to handle the case when the maximum cardinality $a$ of an affinely independent colourful set whose affine hull does not contain $\mathbf{0}$ is at most $d-1$. We can choose a ray $\mathbf{r}$ such that the non-empty intersections with $\mathrm{aff}(A)$ for all colourful sets $A$ of cardinality $a$ are distinct. Let $A^0$ be a colourful set of cardinality $a$ such that $\mathrm{aff}(A^0)$ is the first intersected by $\mathbf{r}$. Without loss of generality, let $A^0 = \{v_1, \dots, v_a\}$ with $v_s \in \mathbf{S}_s$. Note that $\mathbf{S}_{a+1} \cup \ldots \cup \mathbf{S}_{d+1} \subset \mathrm{aff}(A^0 \cup \{\mathbf{0}\})$ as otherwise $\mathbf{0} \notin \mathrm{aff}(A^0 \cup \{v_j\})$ for $v_j \in \mathbf{S}_j$ with $j > a$ which contradicts the maximality of $a$. If there is a colourful simplex containing $\mathbf{0}$, we are done. Therefore, we can assume that, in $\mathrm{aff}(A^0 \cup \{\mathbf{0}\})$, we have an open half-space defined by $\mathrm{aff}(A^0)$ containing $\mathbf{0}$ but not $\mathbf{S}_{a+1} \cup \ldots \cup \mathbf{S}_{d+1}$, and will derive a contradiction.

Let $B_0 = \{a+1, \dots, d+1\}$. We remark that, for all $i, j \in B_0$ with $i \neq j$, the $k$, such that $\mathrm{conv}(\mathbf{S}_i \cup \mathbf{S}_j)$ intersects $\overrightarrow{x_k \mathbf{0}}$ in a point distinct from $x_k$, satisfies $k \in B_0$ since $\mathbf{S}_i \cup \mathbf{S}_j$ are separated from $\mathbf{0}$ by $\mathrm{aff}(A^0)$ in $\mathrm{aff}(A^0 \cup \{\mathbf{0}\})$; and therefore we have $|B_0| \geq 3$. We can now define the following set map:

$$\mathscr{F}(B) = \begin{cases} \{k : \exists (i,j) \in B \times B,\, i \neq j, \forall x_k \in \mathbf{S}_k,\, \mathrm{conv}(\mathbf{S}_i \cup \mathbf{S}_j) \cap \overrightarrow{x_k \mathbf{0}} \subsetneq \{x_k\}\} & \text{if } |B| \geq 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

We have $\mathscr{F}(B) \subset \mathscr{F}(B')$ if $B \subset B'$. Let $B_\ell = \mathscr{F}(B_{\ell-1})$ for $\ell = 1, 2, \ldots$ As remarked above $B_1 \subset B_0$ and, by induction, $B_\ell \subset B_{\ell-1}$ for $\ell \geq 1$. Thus, the sequence $(B_\ell)$ converges towards a set $B^*$ satisfying $\mathscr{F}(B^*) = B^*$. Finally, note that, by induction, $|B_\ell| \geq 3$: The base case holds as $|B_0| \geq 3$, and a pair $i, j \in B_\ell$ with $i \neq j$ yields a $k \in B_{\ell+1}$, then $i, k$ yields an additional $k'$ in $B_{\ell+1}$, which in turn, with $k$, yields a third element in $B_{\ell+1}$; and thus $|B^*| \geq 3$.

For any $v \in \bigcup_{k \in B^*} \mathbf{S}_k$, the ray $\overrightarrow{v\mathbf{0}}$ intersects the convex hull of $\bigcup_{k \in B^*} \mathbf{S}_k$ in a point distinct from $v$ since $\mathscr{F}(B^*) = B^*$. It contradicts the fact that $\mathrm{aff}(A^0)$ separates $\mathbf{0}$ from $\mathbf{S}_{a+1} \cup \ldots \cup \mathbf{S}_{d+1}$ in $\mathrm{aff}(A^0 \cup \{\mathbf{0}\})$ by the following argument. There exists at least one facet of $\mathrm{conv}(\bigcup_{k \in B^*} \mathbf{S}_k)$ whose supporting hyperplane separates $\mathbf{0}$ from $\mathrm{conv}(\bigcup_{k \in B^*} \mathbf{S}_k)$ and, for a vertex $v$ of this facet, we have $\mathrm{conv}(\bigcup_{k \in B^*} \mathbf{S}_k) \cap \overrightarrow{v\mathbf{0}} = \{v\}$, which is impossible.

## 5. Algorithm to find a colourful simplex

We present an algorithm based on the proof of Proposition 2.1 finding a colourful simplex containing $\mathbf{0}$ for sets satisfying the conditions of Theorem 1.4, and, therefore, for sets satisfying the condition of Theorem 1.3 and the general position assumption.

**Algo1.** Take any colourful $(d-1)$-simplex $\sigma$ whose vertices form, without loss of generality, a $\widehat{d+1}$-transversal $T = \{v_1, \ldots, v_d\}$, and a ray $\mathbf{r}$ intersecting $\sigma$ in its interior. Let $H^+(T)$ be the open half-space delimited by $\operatorname{aff}(T)$ and containing $\mathbf{0}$. Check if there is a colourful $d$-simplex having $\sigma$ as a facet and either containing $\mathbf{0}$ or having a facet $\tau$ intersecting $\mathbf{r}$ before $\sigma$. If there is none, we obtain $d$ new vertices forming a $\widehat{d+1}$-transversal $T' = \{v'_1, \ldots, v'_d\}$ in $H^+(T)$, see Section 4. Take a point $x \notin \bigcup_i \mathbf{S}_i$ in $\operatorname{aff}(\mathbf{r})$ such that $\mathbf{0} \in \operatorname{conv}(v_1, \ldots, v_d, x)$, and choose any point $v'_{d+1} \in \mathbf{S}_{d+1}$. We can use Proposition 2.1 and its constructive proof with $\mathbf{S}_i = \{v_i, v'_i\}$ for $i = 1, \ldots, d$ and $\mathbf{S}_{d+1} = \{x, v'_{d+1}\}$ to obtain a new colourful simplex containing $\mathbf{0}$ with at least one vertex in $T'$. If $v'_{d+1}$ is a vertex of the new simplex, we do have a colourful simplex containing $\mathbf{0}$. Otherwise, the facet of the simplex not containing $x$ is a colourful $(d-1)$-simplex $\tau$ intersecting $\mathbf{r}$ before $\sigma$ since $\operatorname{aff}(T)$ forms the boundary of $H^+(T)$.

In other words, given a colourful $(d-1)$-simplex $\sigma$ intersecting $\mathbf{r}$, Algo1 finds either a colourful simplex containing $\mathbf{0}$, or a colourful $(d-1)$-simplex $\tau$ intersecting $\mathbf{r}$ before $\sigma$. Since there is a finite number of colourful $(d-1)$-simplices, the algorithm eventually finds a colourful simplex containing $\mathbf{0}$.

Bárány [Bár82] and Bárány and Onn [BO97] introduced geometric algorithms finding a colourful simplex containing $\mathbf{0}$ for sets satisfying the conditions of Theorem 1.1. These algorithms and some other methods, including multi-update modifications, are studied and benchmarked in [DHST08]. Note that the complexity of this challenging problem is still an open question.

## 6. Proof of Theorem 1.5 in the plane using graphs

In this section we present a proof of Theorem 1.5 for the planar case which also provides a alternative and possibly more combinatorial proof of Theorem 1.4 in the plane. Consider the graph $G = (V, E)$ with $V = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3$ and where a pair of nodes are adjacent if and only if they have different colours. We get a directed graph $D = (V, A)$ by orienting the edges of $G$ such that $\mathbf{0}$ is always on the right side of any arc, i.e. on the right side of the line extending it, with the induced orientation. Since $\operatorname{conv}(\mathbf{S}_i \cup \mathbf{S}_j) \cap \operatorname{aff}(x_k, \mathbf{0}) \neq \emptyset$ with $i, j, k$ pairwise distinct and $x_k \in \mathbf{S}_k$, we have $\deg^+(v) \geq 1$ and $\deg^-(v) \geq 1$ for all $v \in V$. It implies that there exists at least one circuit in $D$, and we consider the shortest circuit $C$. We first show that the length of $C$ is at most 4 since any circuit of length 5 or more has necessarily a chord. Indeed, take a vertex $v$, there is a vertex $u$ on the circuit at distance 2 or 3 having a colour distinct from the colour of $v$, and thus the arc $(u, v)$ or $(v, u)$ exists in $D$. Therefore, the length of $C$ must be 3 or 4. If the length is 3, we are done as the 3 vertices of $C$ form a colourful triangle containing $\mathbf{0}$. If the length is 4, the circuit $C$ is 2-coloured as otherwise we could again find a chord. Consider such a 2-coloured circuit $C$ of length 4 and take any generic ray originating from $\mathbf{0}$. We recall that given an oriented closed curve $\mathscr{C}$ in the plane, with $k_+$, respectively $k_-$, denoting the number of times a generic ray intersects $\mathscr{C}$ while entering by the right, respectively left, side, the quantity $k_+ - k_-$ does not depend on the ray. Considering the realization of $C$ as a curve $\mathscr{C}$, we have $k_- = 0$ by definition of the orientation of the arcs. Since we can choose a ray intersecting $C$ at least once, $k_+$ remains constant and non-zero. Take now a vertex $w$ of the missing colour, and take the ray originating from $\mathbf{0}$ in the opposite direction. This ray intersects an arc of $C$ since $k_+ \neq 0$, and the endpoints of the arc together with $w$ form a colourful triangle containing $\mathbf{0}$. □

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UNIVERSITÉ PARIS EST, CERMICS, 6-8 AVENUE BLAISE PASCAL, CITÉ DESCARTES, 77455 MARNE-LA-VALLÈE, CEDEX 2, FRANCE
*E-mail address*: frederic.meunier@cermics.enpc.fr

ADVANCED OPTIMIZATION LABORATORY, DEPARTMENT OF COMPUTING AND SOFTWARE, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, CANADA, *and* EQUIPE COMBINATOIRE ET OPTIMISATION, UNIVERSITÉ PIERRE ET MARIE CURIE, PARIS, FRANCE
*E-mail address*: deza@mcmaster.ca