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Conditions for Sustainable Optimal Economic Development

*Y. Hossein Farzin**

Abstract

This paper shows that, for dynamic optimizing economies with different types of natural resource, environmental, and human-made capital stocks, a necessary and sufficient condition for permanently sustaining an optimal utility/consumption level is the *stationarity* of the current-value Hamiltonian. For economies whose development is not exogenously and directly affected by time (i.e., time-autonomous economies), this stationarity condition generalizes Dixit et al.'s (1980) "zero-net-aggregate-investment" rule of sustainability, which in turn generalizes Solow-Hartwick's sustainability rule. For non-autonomous economies, the stationarity condition is *not* generally fulfilled, and the current-value Hamiltonian *under* (*over*) estimates the true welfare level by an amount equal to the discounted value of the net "*pure time effect*." For the non-autonomous case of a *time-dependent utility discount rate,* a general condition on the discount rate function (of which the *hyperbolic* discount rate function is a special case) upholds the results obtained for autonomous cases. The paper concludes with a discussion of policies that promote both optimality and sustainability objectives.

1. Introduction

The search for an economic development path that is both optimal and permanently sustainable occupied economists as far back as Ramsey (1928), who sought an optimal path for the capital stock to converge to some positive level and remain permanently at that level (i.e., a steady-state level) *regardless* of the initial size of the capital stock. In Ramsey's problem, however, the focus was on the level of the capital stock rather than on sustainability of consumption or social utility level. In particular, *along an optimal path* to the steady state, the consumption or social utility path did not remain constant. Furthermore, the concept of capital stock was limited to manufactured capital. As such, the Ramsey problem did not deal with questions of intergenerational welfare equity and natural capital depletion.

On the other hand, over the past quarter of a century, concern about the long-run consequences of environmental and natural resource use has confronted economists with two important questions. First, how should the conventional measure of national income be modified to properly take account of depletion of natural resources and the consequent environmental quality degradation? Second, how do the concepts of economic welfare and intergenerational equity relate to the modified national income

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* *Y. Hossein Farzin,* Department of Agricultural and Resource Economics, University of California, Davis, CA 95616, USA. Tel: (530) 752-7610; Fax: (530) 752-5614; E-mail: Farzin@primal.ucdavis.edu. For helpful comments I am grateful to Ken Arrow, Michael Caputo, Kwan Choi, Larry Goulder, Geoff Heal, Larry Karp, Phil Martin, and Bob Solow. I also thank seminar participants at UC Berkeley, Stanford University, University of Washington, and Iowa State University. I alone, however, am responsible for any remaining error of omission or commission. Most of the results furnished in this paper were originally derived in an earlier expanded version titled "Sustainability, Hamiltonian Value, and NNP in a Cake-Eating Economy," written in 1999 and presented first at the Tenth Annual Conference of The European Association of Environmental and Resource Economists (EAERE), 2000, University of Crete, Rethymnon, Greece, June 30–July 2, 2000.

measure? In response to these concerns, a vast and growing literature provides many insights into both questions. The green national accounting issue, has been studied by Dasgupta and Heal (1979), Hartwick (1990, 1994), Mäler (1991), Dasgupta and Mäler (1991), Sefton and Weale (1996), and Heal and Kriström (1998). Economic welfare and sustainability was dealt with first by Solow (1974, 1986), Hartwick (1977), and Dasgupta and Mäler (1991), and then extended by Asheim (1994), Chichilnisky (1996), Heal (1998, 2000, 2003), and Farzin (2004).

The starting point for most of these contributions is Weitzman's (1976) seminal paper, which shows that, under some specific assumptions the optimal current-value Hamiltonian equals the economy's net national product (NNP) at any time. More important, Weitzman showed that, at any point in time, the optimal current-value Hamiltonian of a dynamically optimizing economy presents a (hypothetical) permanently constant consumption flow equivalent to the discounted value of the economy's optimal consumption path, so-called "stationary equivalence." Solow (1974) and Hartwick (1977) were the first to derive a condition for sustainability of a maximum constant consumption flow in the context of a closed economy using an exhaustible resource input and a reproducible capital with a constant technology to produce a consumption good. Their derived condition, known as Solow–Hartwick's sustainability rule, required that resource rents be reinvested in reproducible capital.

The concurrence of Weitzman's "stationary equivalence" result and Solow–Hartwick's sustainability rule have sometimes resulted in confusion about the relationships among the current-value Hamiltonian, NNP, and sustainability condition. A correct understanding of these relationships is crucial to the development of a sound theoretical basis and methods for green national accounting. This paper (i) dispels these misconceptions, (ii) generalizes some of the basic results, and (iii) provides further new insights into the relationships. Section 2 briefly reviews the characteristics of the optimal consumption policy for the special case of a purely exhaustible resource economy, highlighting the prevailing misconceptions and paradoxical results. Section 3 shows that, contrary to what is sometimes said due to misinterpretation of Weitzman's result (see Mäler (1991, p. 5) and Hartwick (1994, 2000, Ch. 3, p. 53), the current-value Hamiltonian does *not* represent the maximum sustainable constant utility (consumption) flow. Instead, I show that a necessary and sufficient condition for sustainability is that the current-value Hamiltonian must be *stationary*. Section 4 shows that the stationarity condition holds generally for the class of dynamically optimizing economies that are time-autonomous; that is, economies whose development is not exogenously and directly affected by mere passage of time. The optimal development of such economies is characterized by an infinite-horizon, time-autonomous optimal control problem, and the stationarity condition generalizes Dixit et al.'s (1980) "zero-net-aggregate-investment" rule which, in turn, is a generalization of Solow–Hartwick's "resource-rent-investment" rule of sustainability.

Section 5 considers the sustainability condition for more general cases of non-autonomous economies whose development not only depends on economic decisions, but is also exogenously—positively or negatively—affected by time. In such cases, Weitzman's "stationary equivalence" result no longer holds, and the current-value Hamiltonian *deviates* from the true welfare level by an amount equal to the discounted value of the flow of net "pure time effect." Furthermore, in non-autonomous cases, the stationarity of the current-value Hamiltonian no longer implies a constant utility (consumption) level unless the net pure time effect also remains constant over time. Section 6 addresses the sustainability condition for a special case of non-autonomous problems; namely, when the utility discount rate is time dependent. I obtain a new result, showing that the specific condition for the discount rate function that ensures Weitzman's "stationary equivalence" result, Dixit et al.'s rule, and hence Solow–Hartwick's rule, all carry over from autonomous problems to such non-autonomous cases. Section 7 summarizes the results and offers concluding remarks regarding some of the major obstacles to meet the sustainability conditions and also broad policies that can promote both optimality and sustainability, particularly in the context of developing countries.

2. The Exhaustible Resource Economy

Although it is obvious that for a purely exhaustible resource economy it is simply not feasible to permanently sustain a positive constant flow of resource consumption, to sharpen the general analytical results to be obtained subsequently, it is instructive to begin our analysis with this simple special case.[1] Thus, consider a purely exhaustible resource economy and, following Hotelling (1931), assume that: (i) it has a fully known and fixed initial stock of the resource of size $S_0 > 0$; (ii) the resource can be extracted costlessly; (iii) no technological change; (iv) population size remains constant; and (v) citizens' preferences are identical and presented by the representative consumer's utility function, $u(c)$, which is a twice differentiable, increasing, and strictly concave function of the resource consumption rate (i.e., $u'(c) > 0$, $u''(c) < 0$ for all $c \geq 0$ $c \geq 0$), with $\lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$. The utilitarian social planner uses a social welfare function defined as the discounted sum of the representative consumer's utility flow and her objective is to plan a path of resource extraction and consumption that maximizes this social welfare function given the resource stock constraint. Formally, she plans to

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c(t))\,dt \tag{1a}$$

$$\text{s.t.} \quad \dot{S}(t) = -c(t) \geq 0, \quad S(t) \geq 0, \quad S_0\,(\text{given}) \tag{1b}$$

where $\rho > 0$ is the social time preference rate, assumed constant. Assuming the constraint $S(t) \geq 0$ holds, the current-value Hamiltonian of this problem is

$$H(c(t), S(t), \lambda(t)) = u(c(t)) - \lambda(t)c(t) \tag{2}$$

where $\lambda(t)$ is the utility shadow price of the resource stock. The first-order conditions for an interior optimal path are

$$\frac{\partial H}{\partial C} = u'(c(t)) - \lambda(t) = 0 \tag{3}$$

$$-\frac{\partial H}{\partial S} = 0 = \dot{\lambda}(t) - \rho\lambda(t) \tag{4}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)S(t) = 0. \tag{5}$$

Differentiating (3) with respect to time, using (4), and denoting the elasticity of marginal utility of consumption by $\eta(c) = -cu''(c)/u'(c)$, the optimal consumption path is characterized by the familiar condition

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\rho}{\eta(c)}. \tag{6}$$

It is immediate from (6) that, in general, the optimal policy for an exhaustible resource economy does *not* sustain a positive constant flow of consumption and hence utility. In fact, for the class of isoelastic utility function, $u(c) = c^{1-\eta}/1 - \eta, 0 < \eta < \infty$, along the optimal path, the consumption level declines exponentially over time at the constant rate of ρ/η. That is,

$$c(t) = c(0)e^{-\frac{\rho}{\eta}t} \tag{7}$$

where from the resource stock constraint $\int_0^\infty c(t)dt = S_0$ and (7) one obtains $c(0) = \rho/\eta S_0$, so that (7) can be rewritten as

$$c(t) = \frac{\rho}{\eta} S_0 e^{-\frac{\rho}{\eta}t}, \forall t \in [0, \infty). \tag{8}$$

It is important to note that for an optimal policy to exist it is necessary that $\rho > 0$. In particular, in the limiting cases of no utility discounting, $\rho = 0$, or a pure egalitarian social welfare function where $\eta \to \infty$, a positive constant consumption path ($c(t) = \bar{c} > 0, \forall t \geq 0$), as implied by (6) for a general utility function, $u(c)$, cannot be sustained permanently by an exhaustible resource economy. On the other hand, the constant zero consumption path ($c(t) = 0, \forall t \geq 0$) implied by (8) for these limiting cases and when the utility function is isoelastic is evidently not optimal.

3. Sustainability and Current-Value Hamiltonian

In his classic paper, Weitzman (1976) investigated the welfare significance of NNP for a dynamic competitive economy that produced a single composite consumption good by utilizing services of capital, defined broadly to include a set of stocks of exhaustible natural resources and various kinds of reproductive capital stocks. A basic insight from that paper is that in a dynamically optimizing economy, along the optimal path, the current-value Hamiltonian *at time t*, $H^*(t)$, is related to the optimal utilitarian welfare/consumption *path*, $u(c^*(\tau))$, $\tau \in (t, \infty)$, according to the following relationship[2]

$$\int_t^\infty e^{-\rho(\tau-t)} H^*(t) d\tau = \frac{H^*(t)}{\rho} = \int_t^\infty e^{-\rho(\tau-t)} u(c^*(t)) d\tau. \tag{9}$$

Unfortunately, this relationship is sometimes misunderstood by thinking that $H^*(t)$ measures the maximum *sustainable* level of utility (consumption). This misunderstanding becomes evident from a seeming paradox of the exhaustible resource economy analyzed in the previous section. For that economy, using (8), (4) and (3) in (2), it is easy to calculate that $H^*(t) = \frac{\eta}{1-\eta}\left(\frac{\rho S_0}{\eta}\right)^{1-\eta} e^{-\rho\left(\frac{1}{\eta}-1\right)t} > 0$ for $\eta < 1$. But, as was noted in the previous section, there is *no* sustainable positive consumption, and hence utility, level.

The explanation for this paradox lies in a correct understanding of what $H^*(t)$ precisely measures: in utility units, $H^*(t)$ is the "*stationary equivalent*" of the optimal welfare path.[3] In other words, it is the hypothetical maximum *constant* utility/consumption path whose *time-t discounted value* is equivalent to that of the (generally non-constant) optimal path, $u(c^*(\tau))$, $\tau \in (t, \infty)$. But, "stationary equivalence" does not mean "sustainability." That is, it does *not* imply, as it is often misunderstood, that our economy can actually enjoy a constant utility/consumption equal to $H^*(t)$ forever.

For the latter to be the case, $H^*(t)$ must satisfy an additional condition: it must be time invariant (or *stationary*). Otherwise, it does *not* represent an actually *sustainable* constant consumption level.[4] The important point to note is that even for autonomous optimal control problems, which characterize most of economic problems studied in the literature, the optimal current-value Hamiltonian need *not* be constant over time. In fact, for the economy analyzed in Weitzman (1976), which presents an example of such problems, we can prove the following proposition, which to our knowledge has not been shown in the previous literature.

Proposition 1. *For Weitzman's economy, the stationarity of the optimal current-value Hamiltonian is a necessary and sufficient condition for permanently sustaining a constant utility/consumption path.*

Proof. Differentiating the second equation in (9) w.r.t. t, and using (9) again, one has

$$\dot{H}^*(t)=\rho\left[\rho\int_t^{\infty}e^{-\rho(\tau-t)}u(c(\tau))d\tau-u(c(t))\right]=\rho\left[H^*(t)-u(c(t))\right].$$

Sufficient condition: recalling that $u'(c)>0,\ \forall c\geq 0$ it immediately follows that $\dot{H}^*(t)=0,\ \forall t\geq 0 \Rightarrow H^*(t)=H^*(\text{cons.})=u(c(t)),\ \forall t\geq 0 \Rightarrow c(t)=u^{-1}(H^*)=\text{cons.},\ \forall t\geq 0.$

Necessary condition: letting $c(\tau)=\bar{c}\geq 0, \forall\tau\geq 0$, so that $u(c(\tau))=u(\bar{c})\geq 0, \forall\tau\geq 0$, and performing the integral yields $\dot{H}^*(t)=0,\ \forall t\geq 0$. Q.E.D.

In the special case of our exhaustible resource economy, it is easy to verify that

$$\dot{H}^*(t)=-\rho\left(\frac{\rho S_0}{\eta}\right)^{1-\eta}e^{-\rho\left(\frac{1}{\eta}-1\right)t}<0,\quad \forall t\geq 0.$$

That is, the stationarity condition is not satisfied, thus confirming that there is no sustainable consumption (utility) path for that economy.

4. Sustainability Condition: Generalization

It is quite tempting to go beyond Proposition 1 to explore if the *stationarity* of $H^*(t)$ is a general sustainability condition for any dynamically optimizing economy characterized by an infinite-horizon optimal control problem in which the instantaneous value function may take the most general form of $u(\mathbf{c}(t), \mathbf{s}(t), t)$, where $\mathbf{c}(t)$ is the vector of n control variables $c_i(t)$, $i = 1, 2, \ldots, n$, denoting final consumption goods, $\mathbf{s}(t)$ is the vector of m state variables, $s_j(t)$, $j = 1, 2, \ldots, m$, denoting various types of renewable and non-renewable natural and environmental capital stocks as well as human-made reproducible capital stocks (including human/knowledge capital), and the differential equations constraints take the general form of $\dot{s}_j = g_j(\mathbf{c}(t), \mathbf{s}(t), t)$, $j = 1, 2, \ldots, m$. Obviously, a dynamic economy so characterized is general enough to present almost any interesting case that one may come across in the literature. For example, it includes cases where the utility derives not only from consumption of goods or resources, but also from capital stocks (for instance, the amenity values of environmental and natural resource stocks such as forests for their recreational or carbon sequestering services, stock of biodiversity, landscape, the atmosphere as stock of clean air or pollution sink, and so on[5]). It also includes cases where there is an exogenous flow of population growth, technological change, or positive or negative externalities over time.

Formally, let us consider the general optimal control problem[6]

$$\begin{aligned}
&\underset{\mathbf{c}(t)}{\text{Maximize}}\ V=\int_0^\infty e^{-\rho t}u(\mathbf{c}(t),\mathbf{s}(t),t)dt \\
&\text{s.t.} \qquad \dot{s}_j=g_j(\mathbf{c}(t),\mathbf{s}(t),t), \qquad j=1,2,\ldots,m, \\
&\qquad\qquad s_j(0)=s_{j0}(\text{given}) \qquad j=1,2,\ldots,m.
\end{aligned} \tag{10}$$

Let $\mathbf{c}^*(t)$, $\mathbf{s}^*(t)$, $\boldsymbol{\lambda}^*(t)$ be the solution to this problem, where $\boldsymbol{\lambda}^*(t)$ is the vector of co-state variables. Then the current-value Hamiltonian $H(\mathbf{c}, \mathbf{s}, \lambda, t) = u(\mathbf{c}, \mathbf{s}, t) + \sum_{j=1}^{m}\lambda_j(t)g_j(\mathbf{c}, \mathbf{s}, t)$ is maximized along the optimal paths. In general, the total time derivative of the current-value Hamiltonian is (for notational convenience, superscript *, denoting the optimal paths, is suppressed)

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum_{i=1}^{n}\frac{\partial H}{\partial c_i}\dot{c}_i+\sum_{j=1}^{m}\frac{\partial H}{\partial s_j}\dot{s}_j+\sum_{j=1}^{m}\frac{\partial H}{\partial \lambda_j}\dot{\lambda}_j. \tag{11}$$

Recalling that along the optimal path

$$\frac{\partial H}{\partial c_i}\dot{c}_i=0, \quad \forall i=1,2,\ldots,n \tag{12a}$$

(as either $\partial H/\partial c_i = 0$ for an interior solution or $\dot{c}_i = 0$ for a boundary solution),

$$-\frac{\partial H}{\partial s_j}=\dot{\lambda}_j-\rho\lambda_j, \quad \forall j=1,2,\ldots,m, \tag{12b}$$

$$\dot{s}_j=\frac{\partial H}{\partial \lambda_j}, \quad \forall j=1,2,\ldots,m \tag{12c}$$

and substituting from (12a)–(12c) in (11), we have

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\rho\sum_{j=1}^{m}\lambda_j\dot{s}_j. \tag{13}$$

Along an optimal path, equation (13) holds generally for *both* non-autonomous and autonomous cases, Weitzman's economy being a special case of the latter. It enables us to state the following proposition, which has not appeared in the previous literature.

PROPOSITION 2. *For any dynamic economy characterized by an* autonomous *infinite-horizon control problem, the stationarity of the current-value Hamiltonian is a necessary and sufficient condition for sustainability of a constant* utility *path.*

PROOF. It suffices to show that Weitzman's fundamental relationship (9) holds true for any autonomous infinite-horizon control problem, so that the proof of Proposition 1 can be invoked.

For an autonomous economy, the functions u or g_js take the form of $u(\mathbf{c}(t), \mathbf{s}(t))$ and $\dot{s}_j = g_j(\mathbf{c}(t), \mathbf{s}(t))$, so the current-value Hamiltonian is $H(\mathbf{c}(t), \mathbf{s}(t), \boldsymbol{\lambda}(t)) = u(\mathbf{c}(t), \mathbf{s}(t)) + \sum_{j=1}^{m}\boldsymbol{\lambda}_j(t)g_j(\mathbf{c}(t), \mathbf{s}(t))$. Since for such cases $\partial H/\partial t = 0$, $\forall t \geq 0$, (11) reduces to

$$\frac{dH}{dt}=\rho\sum_{j=1}^{m}\lambda_j\dot{s}_j. \tag{13a}$$

Recalling that $\dot{s}_j = g_j$, substituting (13a) in the expression for the optimal current-value Hamiltonian and rearranging terms yields the differential equation

$$\frac{dH(t)}{dt} = \rho[H(t) - u(\mathbf{c}(t), \mathbf{s}(t))] \tag{14}$$

which can be solved to give

$$\int_{t=\tau}^{\infty} e^{-\rho(t-\tau)} H(\tau) dt = \frac{H(\tau)}{\rho} = \int_{t=\tau}^{\infty} e^{-\rho(t-\tau)} u(\mathbf{c}(t), \mathbf{s}(t)) dt \tag{15}$$

for any $\tau \geq 0$ *along the optimal path.*[7] Q.E.D.

Remark 1 It should be noted that in the general case of Proposition 2 where $u(\mathbf{c}(t), \mathbf{s}(t))$ is a vector-valued function of the flows of various consumption goods, sustainability is defined only in terms of a permanently constant *utility* path, and not of constant consumption paths. Accordingly, in invoking the proof of Proposition1 only the constancy of utility flow is relevant.

Remark 2 Recalling that $\sum_{j=1}^{m} \lambda_j \dot{s}_j$ is the value of *net aggregate investment* along the optimal path at any time, it is interesting to note from (13a) that for any $\rho > 0$

$$\frac{dH}{dt} = 0, \quad \forall t \geq 0 \Leftrightarrow \sum_{j=1}^{m} \lambda_j \dot{s}_j = 0, \forall t \geq 0. \tag{13b}$$

That is, our stationarity condition ($dH/dt = 0, \forall t \geq 0$) for sustainability of autonomous dynamic economies generalizes the familiar "zero-net-aggregate-investment" rule which was originally derived by Dixit et al. (1980) only as a *sufficient* condition for sustainability[8] (see also Solow (1986), Hartwick (1977) and Mäler (1991) among others). In turn, the latter rule generalized Solow–Hartwick's sustainability rule of investing resources rents in a reproducible capital.[9] It is important to note that our stationarity condition is *both* a necessary and sufficient condition for sustaining a constant optimal utility path.

Remark 3 Interpreting the value of the integral $W_t \equiv \int_t^{\infty} e^{-\rho(\tau-t)} u(c^*(\tau)) \delta\tau$ in (9), or its generalized version $W_t \equiv \int_{t=\tau}^{\infty} e^{-\rho(\tau-t)} u(\mathbf{c}(t), \mathbf{s}(t)) dt$ for the class of time-autonomous economies in (15), as economy's stock of "total wealth" (measured in utility units) at any time t, we arrive at another basic and familiar insight from Weitzman's fundamental relationship (reflected by the second equality in (9), or from its generalized form here for autonomous economies (reflected by the second equality in (15). That is, along the optimal path, at any time the current-value Hamiltonian is the imputed "interest" on the economy's stock of wealth (Solow, 1986; Hartwick, 1994, and others). Now, according to Proposition 2 for autonomous economies, *only under the condition of stationarity of the current-value Hamiltonian* ($dH/dt = 0, \forall t \geq 0$), the utility level along the optimal path remains permanently constant ($u(\mathbf{c}(t), \mathbf{s}(t) = \bar{u}, \forall t \geq 0$), implying in turn that the value of wealth remains intact ($W_t = \bar{u}/\rho \equiv \overline{W}, \forall t \geq 0$). In that case, the optimal current-value Hamiltonian may be interpreted as *Hicksian* income, in utility terms; that is, the maximum constant utility level (equal to interest on wealth, $H = \rho\overline{W} = \bar{u}$) that can be permanently sustained. It is important to re-emphasize here that while for all autonomous economies the current-value Hamiltonian can be interpreted as interest on total wealth, it represents the *sustainable* constant utility (consumption) level *if, and only if*, it is time stationary. Unfortunately, the neglect of the latter condition in the literature has led to the common mistake of interpreting the current-value Hamiltonian as the sustainable constant utility (consumption) level (see, for example, Mäler (1991) and Hartwick (1994, 2000, Ch. 3, p. 53). Although, under the specific assumptions of Weitzman's model, the optimal current-value Hamiltonian at any time equals NNP, it does *not*, contrary to what is sometimes incorrectly believed, equal Hicksian income unless the current-value Hamiltonian is stationary.

Remark 4 In the special case of a purely exhaustible resource economy, since by definition there is no accumulable capital stock and since no optimal policy exists for $\rho = 0$, it follows from (13a) that

$$\frac{dH}{dt} = \rho\lambda(t)\,\dot{S}(t) = -\rho\lambda(t)\,c(t) < 0 \tag{13c}$$

i.e., the stationarity condition for sustainability is never met and hence there exists no sustainable (positive) constant utility (consumption) level. This reconfirms and generalizes the result in the previous section for the isoelastic utility function. Note that, in fact, for such an economy, along the optimal path the level of well being *declines* over time.

5. Sustainability Condition: Non-autonomous Cases

We now return to problem (10) and invoke equation (13) to examine the sustainability condition for the more general case of time *non-autonomous* economies where at least one of the functions $u(\mathbf{c}(t), \mathbf{s}(t), t)$ or $g_j(\mathbf{c}(t), \mathbf{s}(t), t)$ depends explicitly on t. Examples of situations giving rise to non-autonomous cases include exogenous changes over time in population size, in taste and preferences (habit formation), in the state of technology, in the terms of trade of a small open economy, in the rate of physical stock depreciation or growth (for instance, the decay of the CO_2 stock in the atmosphere or growth of forest stocks with time, or additions to reserves of mineral deposits due to exogenous new discoveries).

As in problem (10), we continue to assume a constant discount rate $\rho > 0$. Thus, along an optimal path, one has

$$\frac{\partial H}{\partial t} = \frac{\partial u}{\partial t} + \sum_{j=1}^{m} \lambda_j(t)\frac{\partial g_j}{\partial t} \tag{16}$$

which measures the *net* change in the optimal current-value Hamiltonian at time t due purely to passage of time alone. We may term this as net "*pure time effect*," which may be positive (for example in the case of exogenous technological progress alone) or negative (for example when there is exogenous population growth or when the rate of stock depreciation changes with time).

Recalling that $\dot{s}_j = g_j$ and substituting for $\sum_{j=1}^{m}\lambda_j \dot{s}_j$ from the Hamiltonian expression into (13), one has along the optimal path

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \rho[H(t) - u(t)]. \tag{17}$$

Solving the differential equation (17) yields, for any $\tau \geq 0$

$$\rho\int_{\tau}^{\infty} e^{-\rho(t-\tau)} H(\tau) dt = H(\tau) = \rho\int_{\tau}^{\infty} e^{-\rho(t-\tau)} u(t) dt - \int_{\tau}^{\infty} e^{-\rho(t-\tau)}\frac{\partial H(t)}{\partial t}dt \tag{18}$$

where $\partial H/\partial t$ is given by (16).

Relationship (18) is a general result, leading to further important insights.[10]

First, since

$$\frac{\partial H}{\partial t}=\frac{\partial u}{\partial t}+\sum_{j=1}^{m}\lambda_j(t)\frac{\partial g_j}{\partial t}$$

is not identically equal to zero *for all* $t \geq 0$, the second integral on the RHS of (18) does not vanish *for all* $\tau \geq 0$, so that, on comparing (18) with (15) or with (9), we have

Proposition 3. *The "stationary equivalence" property of the current value Hamiltonian (Weitzman's fundamental relationship) can be generalized only for time-autonomous dynamic economies, but does* not *hold for non-autonomous cases.*

It then immediately follows from (18)

Corollary 1. *The interpretation of the optimal current-value Hamiltonian as interest (return) on economy's wealth (see Remark 3 above), and hence as NNP, does* not *hold for time non-autonomous economies. For these cases, at any time t, the current-value Hamiltonian will under (over) estimate the true welfare level by an amount equal to the discounted value of the net "pure time effect"* ($\int_{\tau}^{\infty} e^{-\rho(t-\tau)}\partial H(t)/\partial t dt$) *if this effect is positive (negative).*

Second, by (17), one has

$$\frac{dH}{dt}=0,\quad \forall t\geq 0 \Rightarrow u(\mathbf{c}(t),\mathbf{s}(t),t)=\overline{H}+\frac{1}{\rho}\frac{\partial H}{\partial t},\quad \forall t\geq 0. \tag{18a}$$

So that,

Corollary 2. *In contrast to the case of time-autonomous economies, for non-autonomous cases the stationarity of the current-value Hamiltonian is not a sufficient condition for sustainability of a constant utility (consumption) level unless in the exceptional case where the net "pure time effect," ∂H/∂t, also remains constant (including 0) over time.*

Third, it also follows from (13) that

Corollary 3. *For time non-autonomous economies, Dixit et al.'s "zero-net-aggregate-investment" rule, and a fortiori Solow-Hartwick's "resource-rent-investment" rule, is not a sufficient condition for sustaining a constant utility (consumption) path.*

Notice that for the non-autonomous case, the stationarity of the current-value Hamiltonian implies that Dixit et al.'s "zero-net-aggregate-investment rule" needs to be modified according to

$$\sum_{j=1}^{m}\lambda_j\dot{s}_j=-\frac{1}{\rho}\frac{\partial H}{\partial t}=-\int_t^{\infty}e^{-\rho(\tau-t)}\frac{\partial H(t)}{\partial t}d\tau.$$

Accordingly, at any time, the net aggregate investment can be *negative* (positive) as long as the disinvestment (investment) in aggregate capital stocks is exactly made up for by a constant positive (negative) flow of "pure time effect" of equivalent (discounted) value. Roughly speaking, this means that the economy can afford to let its national wealth run down (and hence raise its consumption level) provided it enjoys a free (windfall) flow of benefits (for example due to exogenous technological progress) of the same discounted value. Conversely, it should *optimally* make up for exogenous losses (for example due to transboundary environmental externalities or an exogenous deterioration in its terms of trade) by building up the aggregate capital stock.

6. Sustainability Condition: Time-dependent Discount Rate

A special non-autonomous case is when the instantaneous discount rate $\rho(t)$ varies with time, so that, denoting by $\psi(t) \equiv \int_0^t \rho(s)ds$ the discount rate over the interval of time $(0, t]$, the discount factor at any time t is $e^{-\psi(t)}$. As is familiar, in this case the current-value Hamiltonian expression remains as before, but equations (12b) and (13) are modified as

$$-\frac{\partial H}{\partial s_j} = \dot{\lambda}_j - \rho(t)\lambda_j, \quad \forall j = 1, 2, \ldots, m \tag{19}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \rho(t)\sum_{j=1}^{m}\lambda_j \dot{s}_j. \tag{20}$$

Concentrating on cases where, as in the general autonomous problem, none of the functions u or g_js depends explicitly on t, so that $\partial H/\partial t \equiv 0$, (20) simplifies to

$$\frac{dH}{dt} = \rho(t)\sum_{j=1}^{m}\lambda_j \dot{s}_j \tag{20a}$$

which is the analog of (13a) for the case of constant discount rate. Following the same steps leading to (14), one obtains the modified version of (14) as

$$\frac{dH(t)}{dt} - \rho(t)H(t) = -\rho(t)u(\mathbf{c}(t), \mathbf{s}(t)). \tag{21}$$

Solving this differential equation yields *for all* $\tau \geq 0$

$$H(\tau) = \int_\tau^\infty \rho(t)\left[e^{-(\psi(t)-\psi(\tau))}u(\mathbf{c}(t), \mathbf{s}(t))\right]dt + \lim_{t\to\infty}\left[e^{-(\psi(t)-\psi(\tau))}H(\tau)\right]. \tag{22}$$

Condition (22) establishes a new result in the literature and is important in two respects. First, it is noted from (22) that the assumption of bounded current-value Hamiltonian along the optimal path does not ensure that the second term on the RHS of (22) vanishes. For this to be the case, the instantaneous discount rate function $\rho(s)$ must satisfy the following condition

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} \int_0^t \rho(s)ds = +\infty. \tag{23}$$

Thus, from a purely technical viewpoint, with a time-dependent utility discount rate, one can no longer necessarily use the well-known result of Michel (1982), showing that the present value Hamiltonian corresponding to a well defined optimal control problem approaches zero when time goes to infinity. Instead, the result must be defined conditional on the assumption, which in the context of the present paper is equivalent to assuming that the sum of utility discount rates approaches infinity when time goes to infinity. Accordingly, condition (23) extends Michel's result for the case of time-dependent discount rate.

Second, the economic interpretation of condition (22) reveals that, if the utility discount rate is time dependent, then in general the current-value Hamiltonian along the optimal trajectory no longer represents the discounted value of the imputed interest income (in utility terms), but that *plus* the limit of the Hamiltonian value as time approaches infinity. Consequently, for the case of time-dependent discount rate Proposition 3 is modified as

COROLLARY 4. *When the discount rate varies with time, the "stationary equivalence" property of the current value Hamiltonian, i.e., Weitzman's fundamental result generalized as*

$$\int_{\tau}^{\infty} \rho(t)e^{-(\psi(t)-\psi(\tau))}H(\tau)dt = \int_{\tau}^{\infty} \rho(t)e^{-(\psi(t)-\psi(\tau))}u(\mathbf{c}(t),\mathbf{s}(t))dt$$

holds if and only if the discount rate function satisfies the condition $\lim_{t\to\infty}\int_0^t \rho(s)ds = +\infty$.

This is an important result for it modifies the claims in the literature (see, for example, Svensson (1986) and Asheim (1994, p. 261)) that Weitzman's fundamental result does not hold without the assumption of a constant utility discount rate. It shows that the result holds provided the discount rate function satisfies the condition $\lim_{t\to\infty}\int_0^t \rho(s)ds = +\infty$, which is obviously the case as long as the discount rate does not decline too fast with time. One example of such a discount rate function which has recently received considerable attention in the economic literature (Liabson, 1996, 1997, among others) is the hyperbolic discount function. Presented in the form of

$$\rho(t) = \frac{k_1}{1+k_2 t}, \quad (0 < k_1 < 1, k_2 > 0),$$

it is readily checked that

$$\lim_{t\to\infty} \int_0^t \rho(s)ds = \frac{k_1}{k_2}\lim_{t\to\infty} \ln(1+k_2 t) = \infty.$$

Note that this condition is not satisfied, for example, by the exponentially declining function $\rho(t) = k_1 e^{-k_2 t}$, for which $\lim_{t\to\infty}\psi(t) = k_1/k_2$. However, it should be noted that even if the utility discount rate function satisfies condition (23), the implied optimal sustainable consumption path will be time inconsistent (Strotz, 1956), unless the social planner can somehow precommit to it.

Further, it is noted that the integral $\int_\tau^\infty \rho(t)e^{-(\psi(t)-\psi(\tau))}\boldsymbol{u}(\mathbf{c}(t), \mathbf{s}(t))dt$ can no longer be interpreted as the interest on stock of wealth in the same precise sense as in the case of constant discount rate $\rho(t) = \rho$ (see *Remark 3* above), for it now presents the discounted value of the stream of interests on the optimal utility path. Thus, by (22) and (23), we can state

COROLLARY 5. *When the discount rate varies with time, the optimal current-value Hamiltonian (or NNP) does* not *in general represent the interest on the economy's wealth. It presents the discounted value of the flow of interest on the optimal utility path* only if *the discount rate function satisfies the condition* $\lim_{t\to\infty}\int_0^t \rho(s)ds = +\infty$.

This corollary has an important implication for green national accounting: while it cautions us against equating the interest on wealth as green NNP when the discount rate (or the consumption rate of interest) varies with time (as noted correctly by Svensson (1986, p. 155), Hung (1993, p. 381), and Asheim (1994, p. 261)), it also shows the condition under which such a practice would be valid.

Second, it easily follows from (20a) and (21) that

$$\frac{dH}{dt} = 0, \forall t \geq 0 \Rightarrow \sum_{j=1}^{m} \lambda_j \dot{s}_j = 0, \forall t \geq 0, \Rightarrow H(t) = \overline{H}(cons.) = u(t), \forall t \geq 0.$$

That is, as in the case of constant discount rate, the stationarity of the current-value Hamiltonian, and hence the "zero-net-aggregate-investment" rule is still sufficient for sustainability of a constant positive utility level (equal to the constant Hamiltonian value). However, contrary to the case of constant discount rate, the reverse is no longer generally true. This latter is seen by noting from (22) that for a constant utility flow, $u(t) \equiv \overline{u} > 0$, one has for all $\tau \geq 0$ (recalling that $\dot{\psi}(t) = \rho(t)$)

$$H(\tau) = \overline{u} + \lim_{t\to\infty} e^{-(\psi(t)-\psi(\tau))}(H(t) - \overline{u}). \tag{24}$$

So that unless $\overline{u} = \lim_{t\to\infty} H(t)$ or condition (23) is met, $H(\tau) \neq \overline{u}$ for all $\tau \geq 0$, i.e., a constant utility level does not generally imply a constant current-value Hamiltonian (equal to the constant utility level). We can therefore state the following

Proposition 4. *Even when the discount rate varies with time, the stationarity of the current-value Hamiltonian, and hence Dixit et al.'s "zero-net-aggregate-investment" rule, (a fortiori Solow–Hartwick's "resource-rent-investment" rule) is still a sufficient condition for sustainability of a constant utility (consumption) path (equal to the optimal current-value Hamiltonian), but the converse is no longer true unless either* $\lim_{t\to\infty}\int_0^t \rho(s)ds = +\infty$ *or* $\lim_{t\to\infty} H(t) = \overline{u}$.

According to the first part of Proposition 4, it is incorrect to think that Dixit et al.'s rule, or Solow–Hartwick's rule, of sustainability is valid only if the utility discount rate is constant. The second part of the Proposition shows the specific condition under which the reverse of these rules also holds despite a variable discount rate. On both accounts, Proposition 4 weakens Svensson's (1986, p. 154, p. 155) claim of the contrary. As we have seen, in general, for any autonomous problem, the stationarity of the current-value Hamiltonian is a sufficient condition for sustainability regardless of whether the discount rate is constant or time-dependent. But, while for a constant discount rate, the stationarity is also a necessary condition, for a time-dependent discount rate it is so provided that as time goes to infinity, either the discount *factor* approaches zero or the optimal Hamiltonian approaches the constant utility level. Obviously, these results also extend to Dixit et al.'s and Solow–Hartwick's rules.

7. Concluding Remarks

This paper has examined the fundamental relationships between current-value Hamiltonian, sustainability and NNP, thus clarifying some of the misconceptions surrounding these relationships in the green accounting literature. It has also generalized and extended basic results obtained from the literature for special cases and provided new insights into the relationships.

The current-value Hamiltonian does *not* represent the maximum sustainable level of consumption (utility). Instead, in any dynamic optimizing economy presented by an autonomous optimal control problem, a necessary and sufficient condition for sustainability is that the current-value Hamiltonian should be *stationary* over time. Even when the optimal current-value Hamiltonian equals NNP, it is only under the stationarity condition that it can be interpreted as Hicksian income. For the more general case of time non-autonomous economies, characterized by exogenous changes in the economy over time, the "stationary equivalence" property of the current-value Hamiltonian does not carry over, which has two important implications. First, the optimal current-value Hamiltonian can no longer be interpreted as interest on the economy's wealth and, hence, as NNP. In fact, equating NNP with the current-value Hamiltonian will lead to an under-estimation (over-estimation) of the true level of well being if the net "pure time effect" is positive (negative). Second, the stationarity of the current-value Hamiltonian, and hence the "zero-net-aggregate-investment" rule, will no longer be a sufficient condition for permanently sustaining a constant utility (consumption) level. While these results pose conceptual and measurement difficulties for green national accounting, few economists may view continued exogenous changes, such as technological progress, population growth, preference shifts, or environmental externalities, as realistic possibilities. Interestingly, for one special, but important, non-autonomous case—namely, a time dependent discount rate—we have shown that the results obtained in the general autonomous case *do* hold, provided the discount rate function satisfies a certain mild condition; namely, the sum over time of the discount rates be unbounded as time goes to infinity.

This paper, thus, raises several important conceptual and policy issues. First, as we have noted, the necessary and sufficient conditions for an optimal development path to be sustainable are extremely stringent. In a time-autonomous economy, the optimized current value Hamiltonian must remain *constant* over time, a condition that even theoretically is a rare possibility. The economic policy implication of the condition, namely, the requirement that net aggregate investment to be held at zero at every point in time (condition (13b)), is also stringent for several reasons. (i) In practice it is extremely difficult to identify all the existing natural and man-made (including knowledge and human) capital stocks and have a good knowledge of the processes by which they render positive or negative flows of inputs and services. (ii) Even if all the capital stocks could be identified, it would be enormously difficult to measure the stocks and their changes over time. (iii) Even more complicated would be the task of economic evaluation of the identified stocks, as in any time period, they should all be optimally evaluated at their shadow (or social accounting) price, but for many capital stocks, particularly natural assets either there are no markets, or markets are distorted by imperfect competition, imperfect information, or public policy interventions. Furthermore, there are no theoretical insights, let alone a practical policy guidance, as to whether a period of negative (positive) net aggregate investments can be filled in by a subsequent period of positive (negative) net aggregate investments in order to fulfill the sustainability condition. Such flexibility would greatly enhance the task of devising policies to achieve a sustainable development.

Turning to the case of a non-autonomous economy (see Corollary 2), the condition for an optimal policy to be sustainable is even less likely to hold than in the case of an autonomous economy. In a non-autonomous economy, not only the net aggregate investment must remain at zero, but also the net "pure time effect" must remain constant. In fact, even in the simple non-autonomous case where the only direct dependence of the economy on time comes through a time-dependent social discount rate, there is little prospect for the sustainability of an optimal development path because, even if the discount rate function satisfies the required condition in

Proposition 4 (i.e., declines sufficiently slowly), it renders the optimal policy time inconsistent (Strotz, 1956). Thus, there must be either a credible commitment to the optimal policy or, if such a commitment device does not exists, it calls for additional policies to substitute for commitment.

These problems are more severe in developing countries because they are more dependent on environmental and natural capital bases for their well being. At the same time, developing countries are also more likely to experience missing or malfunctioning markets, absent or ill-defined and ill-enforced property rights, imperfect information, and market distorting public interventions. Furthermore, the "pure time effects" of the exogenous factors affecting the economy, whether positive (as in the case of disembodied technological change or knowledge spillover) or negative (as in the case of worsening of the terms of trade, or negative externalities associated with climate change), are likely to be more prevalent and pronounced in developing countries.

This means that it is very difficult to have conditions for a development path to be both optimal and sustainable fulfilled in practice. If optimality and sustainability are incompatible objectives, which objective should be the priority of a development policy? This is essentially the dynamic counterpart of the basic "efficiency versus equity" question in static economic analyses. In the dynamic version, society faces a basic tradeoff between dynamic efficiency (optimality) gains, which can result from depleting some of the natural capital stocks and investing in other (natural and man-made) productive capital stocks, on the one hand, and intergenerational welfare equality on the other.

In principle, the objectives of intertemporal optimality and intergenerational equality need not be inconsistent. For example, societies could redistribute the efficiency gains across present and future generations and potentially make all generations better off than they would be if the efficiency gains were sacrificed in return for attaining intergenerational equality (as would be the case, for example, if one were to allocate capital stocks *equally* (and hence sub-optimally) across generations. The problem, however, is a lack of credible commitment devices whereby the optimality gains can *actually* be transferred to future generations. It is, perhaps, partly this lack of a commitment device that has prompted some ecologists and ecological economists to advocate "strong sustainability" criteria that would require the stocks of natural capital to be kept intact or, at least, not exploited beyond certain threshold levels. This may have some merit in cases where, due to the absence or failure of markets and other institutions, stocks of natural capital are likely to be over-depleted. However, resorting to such a crude means as a substitute for a more efficient commitment device may come at the cost of inflicting significant welfare losses on all generations. The problem of lack of a commitment device for intergenerational transfer of optimality gains, is likely to be particularly acute in developing countries, where political and social institutions may be weak or governments too corrupt to be trusted to act as the trustees and agents of such transfers.

Several policies can promote both optimality and sustainability. First, even if it is impossible to adopt a policy of "zero-net-aggregate-investment" at every point in time, it would still be prudent to maximize efficiency gains by optimally using natural resource stocks and investing the resource rents in other productive (both natural and man-made) capital stocks. This requires that changes in all the natural and man-made capital stocks that are used in production processes be accounted for and evaluated at correct (shadow) prices at any point in time which, in turn, means that policies must

(a) institute and strengthen property rights and their enforcement, and (b) internalize the externalities associated with utilization of natural and environmental resource stocks by pricing them at their marginal social costs/benefits.[11]

The proposed policy also requires that depreciation of natural capital stocks be made up for by investing adequately both in natural and man-made capital stocks, and particularly in irreversible physical infrastructure, knowledge/human capital, and social capital. Farzin (1999) shows that in many natural resource-based developing countries, actual savings and investment rates are far below the rates that would be needed to ensure that living standards would not decline over time. However, when natural resource stocks are owned by unrepresentative and corrupt governments, there is no assurance that resource rents will be reinvested so that the gains from investments will be transferred to future generations. In such cases, fighting corruption to institute and enforce laws that commit the governments to save and invest rents from natural capital stocks, or privatizing resource stocks, can help to achieve both sustainability and optimality.

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Notes

1. Interestingly, however, Farzin (2004) shows that while an exhaustible resource economy is not sustainable in the sense of permanently maintaining a positive consumption level, it *is* sustainable in the sense of maintaining the asset value of the resource stock intact if the society's preferences are represented by a logarithmic utility function, or equivalently, if the resource is extracted and consumed at rates that decline over time at the social discount rate.
2. Since $u(c)$ is a single-valued, monotonic function of c, sustainability can be equivalently defined in terms of a constant utility or consumption flow. In fact, Weitzman assumed a linear utility function of the form $u(c(t)) = c(t)$.
3. Note that the utility units of $H^*(t)$ can be readily converted into real consumption units by choosing a dated utility numeraire such as $u^*(c(0))$ or generally $u^*(c(t))$ for any $t \geq 0$.
4. The stationarity condition is also necessary and sufficient for time consistency of the optimal solution path; i.e., for the optimal policy to be a sub-game perfect Nash equilibrium of the intergenerational allocation game where each generation has to decide how much to consume and how much capital stock to leave for the future generations so that neither the present nor any of future generations will have an incentive to deviate from it.
5. For example, Heal (2000, 2003) utilizes a utility function that includes both the flow and stock of a single natural resource, but is not explicitly time dependent.
6. Without loss of generality, we could also have a set of, say r, inequality constraints of the form $g_k(\mathbf{c}(t), \mathbf{s}(t), t)) \geq 0$, $k = 1, 2, \ldots, r$, and h equality constraints of the form $g_l(\mathbf{c}(t), \mathbf{s}(t), t) = 0$, $l = 1, 2, \ldots, h$, on control variables, where these constraints would be assumed to satisfy the rank condition of the constraint qualifications; namely, that the matrix (of order pn) of partial derivatives of the $p(>h)$ binding constraints with respect to control variables be of rank p. For analytical

convenience and to focus on the question at hand, we ignore these additional constraints and assume that the optimal control problems we are examining are all concave problems. In particular, we assume that the functions $\mathbf{c}(t)$, u and g_j satisfy all the continuity and differentiability conditions for the existence and uniqueness of solution to problem (10).
7. Note that it is a necessary condition that along the optimal path $H(t)$ satisfies the condition that $\lim_{t\to\infty} e^{-\rho t}H(t) = 0$, see Michel (1982).
8. Dixit et al. (1980) derived their *sufficiency* condition in a less general framework than that analyzed here, although they did not assume a constant discount rate. In section 6, we obtain the general sustainability condition when the discount rate is time dependent.
9. Obviously, in an economy with heterogeneous capital stocks if net aggregate investment is always positive, net national product and hence the optimal utility level can rise over time.
10. To be sure, several interesting special cases of this general result have been studied in the literature. For instance, Weitzman (1997), Weitzman and Löfgren (1997), and Hartwick and Long (1999) have studied the conditions of a constant consumption path when technology, output prices, or interest rates change exogenously over time. The result furnished in (18) is, however, a more general and explicit one, embracing these and many other possible specific cases where the pure time effects are present.
11. An important step towards the latter goal would be levying charges and/or removing direct or indirect subsidies (particularly on energy, water, and other natural and environmental resources and their complementary inputs) to avoid overexploitation of un-priced or under-priced natural and environmental resources.