Thompson 群 F 有限分解复杂度的相关问题

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摘要： 有限分解复杂度 (FDC) 是度量空间的大范围性质. 它推广了有限渐进维且应用到一大类群中. 为了使这个性质数量化, 在具有 FDC 的度量空间上定义一个可数序数即“复杂度”. 本文证明了 Thompson 群 F 的子群 $\mathbb{Z}\wr\mathbb{Z}\in\mathcal{D}_\omega$, 其中 ω 是最小的无限序数. 而且还证明了 Thompson 群 F 赋予关于无限生成集 $\{x_0, x_1, \cdots x_n, \cdots\}$ 的词度量后形成的度量空间不具有有限分解复杂度.

关键词： Thompson 群 F; 有限分解复杂度; 词度量; 正合度量空间; 约化森林图

中图分类号： O177

On finite decomposition complexity of Thompson's group F

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Abstract: Finite decomposition complexity (FDC) is a large scale property of a metric space. It generalizes finite asymptotic dimension and applies to a wide class of groups. To make the property quantitative, a countable ordinal "the complexity" can be defined for a metric space with FDC. This paper proves that the subgroup $\mathbb{Z}\wr\mathbb{Z}$ of Thompson's group F belongs to $\mathcal{D}_\omega$ exactly, where ω is the smallest infinite ordinal number. And it shows that F equipped with the word-metric with respect to the infinite generating set $\{x_0, x_1, \cdots x_n, \cdots\}$ does not have finite decomposition complexity.

Key words: Thompson's group F; Finite decomposition complexity; Word-metric; Exact metric spaces; Reduced forest diagram

0 Introduction

Inspired by the property of finite asymptotic dimension of Gromov [1], a geometric concept of finite decomposition complexity is recently introduced by E.Guentner, R.Tessera and G.Yu. Roughly speaking, a metric space has finite decomposition complexity when there is an

基金项目： Shanghai NSF (09ZR1402000), NNSFC(No.10731020) and the Ministry of Education, P.R.China

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algorithm to decompose the space into nice pieces in certain asymptotic way. It turned out that many groups have finite decomposition complexity and these groups satisfy strong rigidity properties including the stable Borel conjecture. In [2], E.Guentner, R.Tessera and G.Yu show that the class of groups with finite decomposition complexity includes all linear groups, subgroups of almost connected Lie groups, hyperbolic groups and elementary amenable groups and is closed under extensions, free amalgamated products, HNN-extensions and inductive limits.

Thompson's group F was discovered by Richard Thompson in 1965, initially used to construct finitely presented groups with unsolvable word problems. It is a long-standing open problem to determine whether F is amenable. Brin and Squier proved ([3], Corollary 4.9) that F contains no free subgroups of rank greater than 1. Hence if F is not amenable, then F is a finitely-presented counterexample to von Neumann's conjecture: a discrete group is not amenable if and only if it contains a subgroup which is free of rank 2. It is known that F is not an elementary amenable group [3]. So if F is amenable, then F is a finitely-presented counterexample to the conjecture that every discrete amenable group is an elementary amenable group. Both conjectures are false for finitely-generated groups. The study of finite decomposition complexity of F is partially inspired by the question of amenability of F.

1 Preliminaries

Recall that a collection of subspaces $\{Z_i\}$ of a metric space Z is *r-disjoint* if for all $i \neq j$ we have $d(Z_i, Z_j) \geq r$. To express the idea that Z is the union of subspaces Z_i and that the collection of these subspaces is r-disjoint, we write

$$Z = \bigsqcup_{r\text{-disjoint}} Z_i.$$

A family of metric spaces $\{Z_i\}$ is *bounded* if there is a uniform bound on the diameter of the individual Z_i:

$$\sup \operatorname{diam}(Z_i) < \infty.$$

Definition 1.1. Let X be a metric space. We say that the *asymptotic dimension* of X does not exceed n and write $asdimX \leq n$ if for every $r > 0$, the space X may be written as a union of $n+1$ subspaces, each of which may be further decomposed as a r-disjoint union:

$$X = \bigcup_{i=0}^{n} X_i, X_i = \bigsqcup_{r-\text{disjoint}} X_{ij} \text{ and } \sup_{i,j} \operatorname{diam} X_{ij} < \infty.$$

In the same spirit, we introduce our notion of finite decomposition complexity not for a metric space, but rather for a countable family of metric spaces. Throughout this paper we view a metric space as a singleton family.

Definition 1.2. [2] A metric family $\mathcal{X}$ is *r-decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition

$$X = X_0 \cup X_1, X_i = \bigsqcup_{r-\text{disjoint}} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. It is denoted by $\mathcal{X} \xrightarrow{r} \mathcal{Y}$.

Definition 1.3. [2]

(1) Let $\mathcal{D}_0$ be the collection of bounded families: $\mathcal{D}_0 = \{\mathcal{X} : \mathcal{X} \text{ is bounded }\}$.

(2) Let α be an ordinal greater than 0, let $\mathcal{D}_\alpha$ be the collection of metric families decomposable over $\bigcup_{\beta<\alpha} \mathcal{D}_\beta$:

$$\mathcal{D}_\alpha = \{\mathcal{X} : \forall r > 0,\ \exists \beta < \alpha,\ \exists \mathcal{Y} \in \mathcal{D}_\beta,\ \ \text{such that } \mathcal{X} \xrightarrow{r} \mathcal{Y}\}.$$

We have two immediate observations.

(i) For any $\beta < \alpha, \mathcal{D}_\beta \subseteq \mathcal{D}_\alpha$.

(ii) asdim $X = 1$ if and only if $X \in \mathcal{D}_1$ exactly. i.e., $X \in \mathcal{D}_1$ and $X \bar{\in} \mathcal{D}_0$.

Moreover, by [2], we have known that X has finite asymptotic dimension if and only if X belongs to $\mathcal{D}_n$ for some $n \in \mathbb{N}$.

Definition 1.4. [2] Let $\mathfrak{U}$ be a collection of metric families. A metric family $\mathcal{X}$ is *decomposable* over $\mathfrak{U}$ if for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{U}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{U}$ is *stable under decomposition* if every metric family which decomposes over $\mathfrak{U}$ actually belongs to $\mathfrak{U}$.

Definition 1.5. [2] The collection $\mathcal{D}$ of metric families with *finite decomposition complexity* is the minimal collection of metric families containing bounded families and stable under decomposition. We abbreviate membership in $\mathcal{D}$ by saying that a metric family in $\mathcal{D}$ has FDC.

Proposition 1.1. ([2], Theorem 2.3.2) *A metric family $\mathcal{X}$ has finite decomposition complexity if and only if there exists a countable ordinal α such that $\mathcal{X} \in \mathcal{D}_\alpha$.*

Definition 1.6. Let G be a countable discrete group. A *length function* $l : G \longrightarrow \mathbb{R}_+$ on G is a function satisfying: for all $g, f \in G$,
(1) $l(g) = 0$ if and only if g is the identity element of G,
(2) $l(g^{-1}) = l(g)$,
(3) $l(gf) \leq l(g) + l(f)$.

A length function l is called *proper* if for all $C > 0, l^{-1}([0, C]) \subset G$ is finite.

Let S be a generating set for a group G, for any $g \in G$, define $l(g)$ to be the length of the shortest word representing g in elements of the generating set S. Then we say that l is *word-length function* for G with respect to S.

Definition 1.7. If $f : X \longrightarrow Y$ is a map of metric spaces, it is said to be:

- *bornologous* if for all $R > 0$ there exists $S > 0$ such that $d(x_1, x_2) < R$ implies $d(f(x_1), f(x_2)) < S$.
- *effectively proper* if for all $R > 0$ there exists $S > 0$ such that for all $x \in X$, $f^{-1}(B(f(x), R)) \subseteq B(x, S)$.

A *coarse embedding* is an effectively proper, bornologous map. Two maps $f, g : X \longrightarrow Y$ are *close* if $\{d(f(x), g(x)) : x \in X\}$ is a bounded set. If $f : X \longrightarrow Y$ is a coarse embedding and there exists a coarse embedding $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are close to the identities on X and Y respectively, then f is called a *coarse equivalence*.

Recall that a countable discrete group admits a proper length function l and that any two metrics defined from proper length functions by the formula

$$d(s, t) = l(s^{-1}t)$$

are coarsely equivalent(in fact, the identity map is a coarse equivalence).(cf.[4], Proposition 2.3.3) On the other hand, finite decomposition complexity is a coarsely invariant property of metric spaces([2], Theorem 3.1.3). As a consequence, we say that *a discrete group has finite decomposition complexity* if it is a metric space having finite decomposition complexity equipped with a metric induced by a proper length function.

Example 1.1. Let $G = \bigoplus \mathbb{Z}$(countable infinite direct sum),

$$\forall g = (\cdots, g(n), \cdots), h = (\cdots, h(n), \cdots) \in G, d_1(g, h) = \sum_{n \in \mathbb{Z}} |\ n\ ||\ g(n) - f(n)\ |.$$

Note that d_1 is a proper left-invariant metric. It was proved that $(G, d_1) \in \mathcal{D}_\omega$(cf.[2], Example 2.3.4), where ω is the smallest infinite ordinal number. Moreover, for any $\alpha < \omega$, $(G, d_1) \bar{\in} \mathcal{D}_\alpha$.

2 Finite decomposition complexity of some groups

Let G and N be finitely generated groups and let $1_G \in G$ and $1_N \in N$ be their units. The *support* of a function $f : N \to G$ is the set

$$\text{supp}(f) = \{x \in N | f(x) \neq 1_G\}.$$

The direct sum $\bigoplus_N G$ of groups G (or restricted direct product) is the group of functions

$$C_0(N, G) = \{f : N \to G \text{ with finite support}\}.$$

There is a natural action of N on $C_0(N, G)$:

$$a(f)(x) = f(xa^{-1}) \text{ for all } a \in N, x \in N \text{ and } f \in C_0(N, G).$$

The semidirect product $C_0(N,G)\rtimes N$ is called *restricted wreath product* and is denoted as $G\wr N$. We recall that the product in $G\wr N$ is defined by the formula

$$(f,a)(g,b)=(fa(g),ab).$$

Let S and T be finite generating sets for G and N, respectively. Let $1\in C_0(N,G)$ denotes the constant function taking value 1_G, and let $\delta_v^b : N\to G, v\in N, b\in G$ be the δ-function, i.e.,

$$\delta_v^b(v)=b \text{ and } \delta_v^b(x)=1_G \text{ for } x\neq v.$$

Note that $a(\delta_v^b)=\delta_{va}^b$ and hence $(\delta_v^b,1_N)=(1,v)(\delta_{1_N}^b,1_N)(1,v^{-1})$. Since every function $f\in C_0(N,G)$ can be presented $\delta_{v_1}^{b_1}\cdots\delta_{v_k}^{b_k}$,

$$(f,1_N)=(\delta_{v_1}^{b_1},1_N)\cdots(\delta_{v_k}^{b_k},1_N) \text{ and } (f,u)=(f,1_N)(1,u).$$

The set $\widetilde{S}=\{(\delta_{1_N}^s,1_N),(1,t)|s\in S,t\in T\}$ is a generating set for $G\wr N$. We will use abbreviations f for $(f,1_N)$ and t for $(1,t)$ for elements of the group $G\wr N$. So we denote $(f,t)=(f,1_N)(1,t)$ by ft.

Lemma 2.1. ([5], Proposition 2.4) *Let $x=(f,n)\in H\wr\mathbb{Z}$, $m=\min\{k\in\mathbb{Z}\mid f(k)\neq 1_H\}$, $M=\max\{k\in\mathbb{Z}\mid f(k)\neq 1_H\}$, then the length of x satisfies:*

$$|x|=\begin{cases}|n| & \text{if } f=e.\\ \sum_{i\in\mathbb{Z}}|f(i)|+L_{\mathbb{Z}}(x). & \text{otherwise.}\end{cases}$$

where e is the identity of $\bigoplus_{l\in\mathbb{Z}} H$, $L_{\mathbb{Z}}(x)$ denotes the length of the shortest path starting from 0, ending at n and passing through m and M in the (canonical) Cayley graph of $\mathbb{Z}$.

Lemma 2.2. *Let X be a metric space with a left-invariant metric and $\{X_i\}_i$ be a sequence of subspaces of X with the induced metric. If $\{X_i\}_i\in\mathcal{D}_\alpha$, then $\{gX_i\}_{g,i}\in\mathcal{D}_\alpha$, where $gX_i=\{gh|h\in X_i\}$.*

Proof. We will prove it by induction on α. First when $\alpha=0$, we have $\sup_i \operatorname{diam} X_i<\infty$. Since the metric is left-invariant, $\operatorname{diam} gX_i=\operatorname{diam} X_i$. Then $\sup_{g,i}\operatorname{diam} gX_i=\sup_{g,i}\operatorname{diam} X_i<\infty$. i.e., the result is true for $\alpha=0$. Now assume that for any $\beta<\alpha$, if $\{X_i\}_i\in\mathcal{D}_\beta$, then $\{gX_i\}_{g,i}\in\mathcal{D}_\beta$. If $\{X_i\}_i\in\mathcal{D}_\alpha$, then for every $r>0$, there exist $\beta<\alpha$ and $\mathcal{Y}\in\mathcal{D}_\beta$, such that $\{X_i\}\xrightarrow{r}\mathcal{Y}$. So we get a decomposition:

$$X_i=X_{i0}\cup X_{i1}, X_{ij}=\bigsqcup_{r-\text{disjoint}} X_{ijk}, \text{where } \{X_{ijk}\}\in\mathcal{D}_\beta.$$

Then we have:

$$gX_i=gX_{i0}\cup gX_{i1}, gX_{ij}=\bigsqcup_{r-\text{disjoint}} gX_{ijk}.$$

By assumption, $\{gX_{ijk}\}\in\mathcal{D}_\beta$. Hence, $\{gX_i\}\in\mathcal{D}_\alpha$. $\square$

Theorem 2.1. *Let H be a countable group and $H^m = \underbrace{H \times H \times \cdots \times H}_{m}$. For every $r \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and a metric family $\mathcal{Y}$ such that*

(1) $H \wr \mathbb{Z} \xrightarrow{r} \mathcal{Y}$,

(2) there is a coarse embedding from $\mathcal{Y}$ to $\{gH^m\}_{g \in \bigoplus H}$.

In particular, $\mathbb{Z} \wr \mathbb{Z} \in \mathcal{D}_\omega$, and for any $\alpha < \omega, \mathbb{Z} \wr \mathbb{Z} \bar{\in} \mathcal{D}_\alpha$.

Proof. For every $r \in \mathbb{N}$, $i \in \mathbb{Z}$, let $A_i = [2i(r+1), (2i+1)(r+1)] \cap \mathbb{Z}$ and $B_i = [(2i+1)(r+1), (2i+2)(r+1)] \cap \mathbb{Z}$, then we have a decomposition:

$$\mathbb{Z} = A \cup B, \text{ where } A = \bigsqcup_{r-\text{disjoint}} A_i \text{ and } B = \bigsqcup_{r-\text{disjoint}} B_i.$$

For every $i \in \mathbb{Z}$, choose $a_i \in A_i$, $b_i \in B_i$ and choose $n > 3r + 2$. Let

$$G_{n,a_i} = \{f \in \bigoplus_{l \in \mathbb{Z}} H |\ f(j) = 1_H, \quad \forall j > n + a_i \text{ or } j < -n + a_i\}.$$

It is a subgroup of $\bigoplus_{l \in \mathbb{Z}} H$ and $\bigoplus_{l \in \mathbb{Z}} H$ admits a decomposition into cosets of G_{n,a_i}, i.e.,

$$\bigoplus_{l \in \mathbb{Z}} H = \bigcup_{g \in \bigoplus H} gG_{n,a_i}.$$

Similarly, we can define G_{n,b_i} and obtain $\bigoplus_{l \in \mathbb{Z}} H = \bigcup_{g \in \bigoplus H} gG_{n,b_i}$. Therefore, as a set, $H \wr \mathbb{Z}$ is

$$\begin{aligned}\left(\bigoplus H, \mathbb{Z}\right) &= \left(\bigcup_{i \in \mathbb{Z}} (\bigoplus H, A_i)\right) \bigcup \left(\bigcup_{i \in \mathbb{Z}} (\bigoplus H, B_i)\right) \\ &= \left(\bigcup_{i \in \mathbb{Z}, g \in \bigoplus H} (gG_{n,a_i}, A_i)\right) \bigcup \left(\bigcup_{i \in \mathbb{Z}, g \in \bigoplus H} (gG_{n,b_i}, B_i)\right)\end{aligned}$$

Next we will show that $\bigcup_{i \in \mathbb{Z}, g \in \bigoplus H} (gG_{n,a_i}, A_i)$ and $\bigcup_{i \in \mathbb{Z}, g \in \bigoplus H} (gG_{n,b_i}, B_i)$ are r-disjoint unions. Assume that $(g_1, a) \in (g_1 G_{n,a_i}, A_i), (g_2, a') \in (g_2 G_{n,a_j}, A_j)$ and $(g_1 G_{n,a_i}, A_i) \neq (g_2 G_{n,a_j}, A_j)$. We need to show that $d((g_1, a), (g_2, a')) > r$.

- **Case 1**. If $i \neq j$, then $d((g_1, a), (g_2, a')) = |(a^{-1}(g_1^{-1} g_2), a^{-1} a')| \geq |a^{-1} a'| = d(a, a') \geq d(A_i, A_j) > r$.
- **Case 2**. If $i = j$, since $(g_1 G_{n,a_i}, A_i) \neq (g_2 G_{n,a_j}, A_j)$, we have $g_1 G_{n,a_i} \neq g_2 G_{n,a_i}$, i.e., $g_1^{-1} g_2 \bar{\in} G_{n,a_i}$. By the definition of G_{n,a_i}, we have

$$\exists j > n + a_i \text{ or } j < -n + a_i, \text{s.t. } g_1(j) \neq g_2(j).$$

It follows that

$$\exists j > n \text{ or } j < -n, \text{s.t. } (a_i^{-1} g_1 a_i)(j) = g_1(j + a_i) \neq g_2(j + a_i) = (a_i^{-1} g_2 a_i)(j).$$

By Lemma 2.1, we obtain that $d(a_i^{-1} g_1 a_i, a_i^{-1} g_2 a_i) > 2n$, which implies that $d((g_1, a), (g_2, a')) > r$.

In fact, if $d((g_1, a), (g_2, a')) \leq r$, then

$$\begin{aligned} d(a_i^{-1} g_1 a_i, a_i^{-1} g_2 a_i) &= d(g_1 a_i, g_2 a_i) \\ &\leq d(g_1 a_i, g_1 a) + d(g_1 a, g_2 a') + d(g_2 a_i, g_2 a') \\ &= d(a_i, a) + d(g_1 a, g_2 a') + d(a_i, a') \\ &\leq (r+1) + r + (r+1) = 3r + 2 < n. \qquad \text{Contradiction!} \end{aligned}$$

Therefore, $\bigcup_{i\in\mathbb{Z}, g\in\bigoplus H} (gG_{n,a_i}, A_i)$ are r-disjoint unions. We can similarly show that $\bigcup_{i\in\mathbb{Z}, g\in\bigoplus H} (gG_{n,b_i}, B_i)$ are r-disjoint unions. Let

$$G_n = \{f \in \bigoplus H \mid f(j) = 1_H \text{ for every } j > n \text{ or } j < -n\} \cong H^{2n+1},$$

we define a map

$$\begin{array}{rcl} \rho : (gG_{n,a_i}, A_i) & \to & \widetilde{g}G_n \\ (g_1, a) & \mapsto & a_i^{-1} g_1 a_i \end{array}$$

where $\widetilde{g} = a_i^{-1} g a_i$. It is easy to see ρ is well defined.

We claim that ρ is a coarse embedding.

In fact, for any $R_1 > 0, R_2 > 0$, there exist $S_1 = R_1 + 2(r+1), S_2 = R_2 + 2(r+1)$ such that

(1) if $d((g_1, a), (g_2, a')) \leq R_1$, then

$$d(\rho((g_1, a)), \rho((g_2, a'))) = d(a_i^{-1} g_1 a_i, a_i^{-1} g_2 a_i) \leq d(a_i, a) + d(g_1 a, g_2 a') + d(a_i, a') \leq R_1 + 2(r+1) = S_1.$$

(2) Conversely, if $d(\rho((g_1, a)), \rho((g_2, a'))) \leq R_2$, then

$$d((g_1, a), (g_2, a')) \leq d(g_1 a, g_1 a_i) + d(g_1 a_i, g_2 a_i) + d(g_2 a', g_2 a_i) \leq R_2 + 2(r+1) = S_2.$$

Hence, we can get a coarse embedding from the metric family $\mathcal{Y}$ to $\{\widetilde{g}G_n\}_{\widetilde{g}}$. To complete the proof, we only need to take $m = 2n + 1$. In particular, when $H = \mathbb{Z}$, by Lemma 2.2, $\{g\mathbb{Z}^m\}_{g\in\bigoplus\mathbb{Z}} \in \mathcal{D}_m$. Hence, for every $r > 0$, there exist $m \in \mathbb{N}$ and $\mathcal{Y} \in \mathcal{D}_m$ such that $\mathbb{Z}\wr\mathbb{Z} \xrightarrow{r} \mathcal{Y}$. Therefore, $\mathbb{Z}\wr\mathbb{Z} \in \mathcal{D}_\omega$. On the other hand, since for any $\alpha < \omega$, $\bigoplus\mathbb{Z} \bar{\in} \mathcal{D}_\alpha$ and $\bigoplus\mathbb{Z}$ is a subgroup of $\mathbb{Z}\wr\mathbb{Z}$, we have $\mathbb{Z}\wr\mathbb{Z} \bar{\in} \mathcal{D}_\alpha$. □

3 Decomposition complexity and Thompson's group F

We present a brief introduction to Thompson's group F and refer the interested readers to [6] and [3] for more detailed discussions. Thompson's group F has been studied for several decades. It can be described as the group of piecewise-linear homeomorphisms of the unit interval, all of whose derivatives are integer powers of 2 and with a finite number of break points which are all dyadic rational numbers. It can also be described as the group with the following infinite presentation:

$$\langle x_0, x_1, \cdots, x_n, \cdots | x_n x_k = x_k x_{n+1}\ \forall k < n \rangle$$

From this presentation, we may see $x_{n+1} = x_0^{-1} x_n x_0$ for $n \geq 1$, thus F is finitely generated by $\{x_0, x_1\}$. However, it is still useful to consider the infinite generating set $\{x_0, x_1, \cdots, x_n, \cdots\}$.

We define *a caret* to be a vertex of the tree together with two downward oriented edges, which we refer to as the left and right edges of the caret. Every caret has the form of the rooted tree in Figure 1.

图 1: A caret

Elements of F can be viewed as pairs of finite binary rooted trees, each with the same number of carets, called *tree diagrams*. A *binary forest* is a sequence $(T_0, T_1, \cdots)$ of finite binary trees. A binary forest is *bounded* if only finitely many of the trees T_i are nontrivial. *Forest diagram*, which represents an element of F as a pair of bounded binary forests is another useful diagram representation for F. A forest diagram (or a tree diagram) is *reduced* if it does not have any opposing pairs of carets.

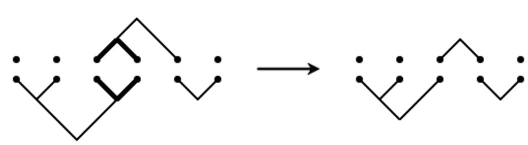

图 2: An example of an unreduced and a reduced forest diagrams representing the same element in F

Lemma 3.1. ([6], Proposition 2.2.4) *Every element of Thompson's group F has a unique reduced forest diagram.*

It is easy to translate between tree diagrams and forest diagrams [6]. Given a tree diagram, we simply remove the right stalk of each tree to get the corresponding forest diagram, see Figure 3

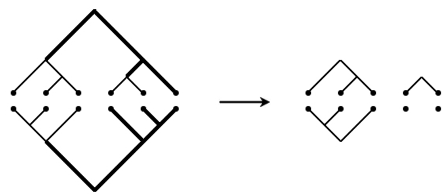

图 3: A tree diagram be translated into a forest diagram

Recall that a metric space is *proper* if every closed ball is compact.

The action of the generators $\{x_0, x_1, \cdots x_n, \cdots\}$ on forest diagrams is particularly nice:

Lemma 3.2. ([6], Proposition 2.3.1 and Proposition 2.3.4) *Let $\mathfrak{f}$ be a forest diagram for some $f \in F$, then*

(1) a forest diagram for $x_n f$ can be obtained by attaching a caret to the roots of trees n and $(n+1)$ in the top forest of $\mathfrak{f}$.

(2) a forest diagram for $x_n^{-1} f$ can be obtained by "dropping a negative caret" at position n. If tree n is nontrivial, the negative caret cancels with the top caret of this tree. If the tree n is trivial, the negative caret "fall through" to the bottom forest, attaching to the specified leaf.

Remark 3.1. *Note that the forest diagram given for $x_n f$ may not be reduced, even if we started with a reduced forest diagram $\mathfrak{f}$. In particular, the caret that was created could oppose a caret in the bottom forest. In this case, left-multiplication by x_n effectively "cancels" the bottom caret.*

Example 3.2. Let $f \in F$ has the reduced forest diagram in Figure 4, then $x_0 f, x_1 f$ have reduced forest diagrams in Figure 5 and $x_0^{-1} f, x_1^{-1} f$ have reduced forest diagrams in Figure 6.

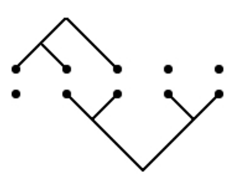

图 4: The reduced tree diagram for f

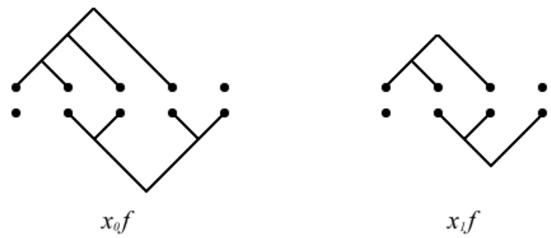

图 5: The reduced tree diagrams for x_0f, x_1f

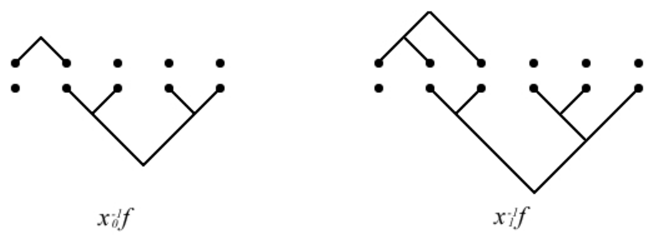

图 6: The reduced tree diagrams for $x_0^{-1}f, x_1^{-1}f$

Let S be a rooted binary tree, the right side of S is the maximal path of right edges in S which begins at the root of S. Define the exponents of S as follows: let $I_0, \cdots, I_n$ be the leaves of S in order. For every integer k with $0 \leq k \leq n$, let a_k be the length of the maximal path of left edges in S which begins at I_k and which does not reach the right side of S. Then a_k is the k^{th} exponent of S.

Example 3.3. The right side of the rooted binary tree S in Figure 7 is highlighted. Its leaves are labeled 0,$\cdots$,5 in order and the exponents of S in order are 2,1,0,0,0,0.

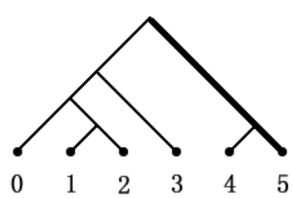

图 7: An rooted binary tree

Lemma 3.3. ([3], Normal Form) *Let f be a non-trivial element of F with the reduced tree diagram (R, S). Let $a_0, \cdots, a_n$ be the exponents of R and $b_0, \cdots, b_n$ be the exponents of S.*

Then f can be expressed uniquely in the form: $f = x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}$ such that

(1) exactly one of a_n and b_n is nonzero,

(2) for every integer i with $0 \leq i < n$, if $a_i > 0$ and $b_i > 0$, then either $a_{i+1} > 0$ or $b_{i+1} > 0$.

In this case, we say $f = x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}$ is the normal form for f.

Lemma 3.4. ([7]) *Let G be a group with the generating set S, and let $l : G \to \mathbb{N}$ be a function. Then l is the word-length function for G with respect to S if and only if:*

(1) $l(e) = 0$, where e is the identity of G.

(2) $\mid l(sg) - l(g) \mid \leq 1$ for all $g \in G$ and $s \in S$.

(3) For $g \in G \setminus \{e\}$, there exists $s \in S \cup S^{-1}$ such that $l(sg) < l(g)$.

Recall that a metric space has *bounded geometry* if for every $r > 0$, there exists an $N = N(r)$ such that every ball of radius r contains at most N points.

Lemma 3.5. ([4]) Let X be a discrete metric space, the following are equivalent:

(1) For every $R > 0, \epsilon > 0$, there exist $\xi : X \to l_1(X)_{1,+}$and $S > 0$ such that

 (a) $\|\xi(x) - \xi(y)\|_1 \leq \epsilon$ whenever $d(x, y) \leq R$.

 (b) $\mathrm{supp}\,\xi(x) \subseteq B(x, S)\,(= \{y \in X | d(x, y) \leq S\})$ for every $x \in X$.

(2) For every $R > 0, \epsilon > 0$, there exist a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X and a partition of unity $\{\phi_i\}_{i\in I}$ subordinate to $\mathcal{U}$ and $S > 0$ such that

 (c) $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$ whenever $d(x, y) \leq R$.

 (d) $\mathrm{diam}(U_i) \leq S$ for every $i \in I$.

Recall that we say a metric space X is exact if X satisfies the property in (2) .

Remark 3.2. *(1) If $Y \subseteq X$ and X is an exact metric space, then Y is an exact metric space.*

(2) If there is a coarse embedding $f : X \to Y$ and Y is an exact metric space, then X is an exact metric space. Therefore, if $f : X \to Y$ is a coarse equivalence, then X is an exact metric space if and only if Y is an exact metric space.

By the equivalence in Lemma 3.5 and use the same proof of Nowak in the Theorem 5.1 in [8], one can obtain the following proposition.

Proposition 3.1. ([8]) Let Γ be a finite group, d_n is l^1-metric for Γ^n, $\mathcal{X}_\Gamma = \bigsqcup_{n=1}^{\infty} \Gamma^n$ is a metric space with a metric d such that

- d restricted to Γ^n is d_n,
- $d(\Gamma^n, \Gamma^{n+1}) \geq n+1$,
- if $n \leq m$, then we have $d(\Gamma^n, \Gamma^m) = \sum_{k=n}^{m-1} d(\Gamma^k, \Gamma^{k+1})$. Then $(\mathcal{X}_\Gamma, d)$ is not an exact metric space.

Corollary 3.1. *Let $G = \bigoplus_{n\geq 0} \mathbb{Z}$(countable infinite direct sum), let d_2 be the l^1-metric for $\bigoplus_{n\geq 0} \mathbb{Z}$, i.e.,*

$$\forall g = (\cdots, g(n), \cdots), h = (\cdots, h(n), \cdots) \in G, d_2(g,h) = \sum_{n\in\mathbb{Z}} |g(n) - f(n)|.$$

Then $(\bigoplus_{n\geq 0} \mathbb{Z}, d_2)$ is not an exact metric space.

Proof. Note that $\mathbb{Z}_2$ can be embedded isometrically into $\mathbb{Z}$ as metric spaces. Then define a map $\varphi : \bigsqcup_{n=1}^{\infty} \mathbb{Z}_2^n \to \bigoplus_{n\geq 0} \mathbb{Z}$ as follows: for every natural number $n \geq 1$, if $x = (x(1), x(2), \cdots, x(n)) \in \mathbb{Z}_2^n$, then define $\varphi(x) \in \bigoplus_{n\geq 0} \mathbb{Z}$, let

$$\varphi(x)(k) = \begin{cases} x\left(k - \frac{n^2-n}{2}\right) & \frac{n^2-n}{2} + 1 \leq k \leq \frac{n^2+n}{2}, \\ \frac{(n-1)(n+2)}{2} & k = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\varphi(\mathbb{Z}_2) = (0, \mathbb{Z}_2, 0, \cdots), \quad \varphi(\mathbb{Z}_2^2) = (2, 0, \mathbb{Z}_2, \mathbb{Z}_2, 0, \cdots), \quad \varphi(\mathbb{Z}_2^3) = (5, 0, 0, 0, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, 0, \cdots), \quad \cdots$$

Define a metric d for $\bigsqcup_{n=1}^{\infty} \mathbb{Z}_2^n$ by

$$d(x,y) = d_2(\varphi(x), \varphi(y)), \quad \forall\, x, y \in \bigsqcup_{n=1}^{\infty} \mathbb{Z}_2^n.$$

Then it is easy to check that

- d restricted to $\mathbb{Z}_2^n$ is d_n, which is l^1-metric for $\mathbb{Z}_2^n$,
- $d(\mathbb{Z}_2^n, \mathbb{Z}_2^{n+1}) = n+1$,
- if $n \leq m$, then we have $d(\mathbb{Z}_2^n, \mathbb{Z}_2^m) = \sum_{k=n}^{m-1} d(\mathbb{Z}_2^k, \mathbb{Z}_2^{k+1})$.

By Proposition 3.1, $(\bigsqcup_{n=1}^{\infty} \mathbb{Z}_2^n, d)$ is not an exact metric space. By the definition of the metric d, we have

$$\varphi : (\bigsqcup_{n=1}^{\infty} \mathbb{Z}_2^n, d) \to (\bigoplus_{n\geq 0} \mathbb{Z}, d_2)$$

is an isometric map. Therefore, $(\bigoplus_{n\geq 0} \mathbb{Z}, d_2)$ is not an exact metric space. □

In the following, we would also use:

Lemma 3.6. ([2], Theorem 4.3) *A metric space having finite decomposition complexity is exact.*

Theorem 3.1. *Let F be Thompson group, $A = \{x_0, x_1, \cdots x_n, \cdots\}$ be the generating set for F described above. For any $f \in F$, let*

$$f = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} x_n^{-b_n} \cdots x_0^{-b_0}$$

be the normal form for f. Define

$$l(f) = \sum_{k=0}^{n} \mid a_k \mid + \sum_{k=0}^{n} \mid b_k \mid,$$

let d be the metric induced by l, then

(1) l is the word-length function for F with respect to A.

(2) $(F, d) \bar{\in} \mathcal{D}$, i.e. the metric space (F, d) does not have finite decomposition complexity.

Note that here (F, d) is a metric space without bounded geometry.

Proof. First we are going to prove that l is the word-length function for F with respect to A. By Lemma 3.3, we can see $l(f)$ is equal to the sum of exponents of $\begin{pmatrix} R_1 \\ S_1 \end{pmatrix}$ which is the reduced tree diagram for f. By the translation between tree diagrams and forest diagrams, it is easy to see $l(f)$ is equal to the number of carets in the reduced forest diagram for f. Clearly, $l(1_F) = 0$. By the property of the action of x_n in Lemma 3.2, we have

$$l(x_n f) = l(f) \pm 1, \text{ for every } n \geq 0.$$

By Lemma 3.4, it suffices to show that

$$\text{for } f \in F \setminus \{1_F\}, \text{ there exists } s \in A \cup A^{-1} \text{ such that } l(sf) < l(f).$$

Indeed, let

$$f = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} x_n^{-b_n} \cdots x_0^{-b_0}$$

be the normal form and let $\begin{pmatrix} R_2 \\ S_2 \end{pmatrix}$ be the reduced forest diagram for f. By Lemma 3.2, if R_2 is non-trivial, assume that the m^{th} tree is non-trivial, then

$$l(x_m^{-1} f) = l(f) - 1 < l(f).$$

Otherwise, since R_2 is trivial, $a_n = 0$, then $b_n \neq 0$, By Remark 4.1, we have

$$l(x_n f) = l(f) - 1 < l(f).$$

Now we will show (F, d) does not have finite decomposition complexity. For any $k \geq 0$, let

$$t_k = x_k^2 x_{k+1}^{-1} x_k^{-1}.$$

Note that

$$\forall i \neq j,\ t_i t_j = t_j t_i.$$

Therefore $\{t_k\}_{k\geq 0}$ generates $\bigoplus_{n\geq 0} \mathbb{Z}$ with an isomorphism $t_k \mapsto e_k = (0, 0, \cdots, 1, 0, \cdots)$. By the reduced forest diagram of $t_k^n (n \in \mathbb{N}, n > 0)$ in Figure 8,

$$l(t_k^n) = 2(n + 1).$$

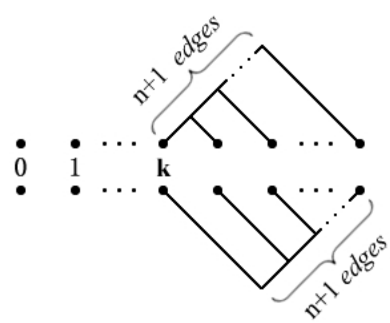

图 8: The reduced forest diagram for t_k^n

If $n < 0, t_k^n = \left(t_k^{|n|}\right)^{-1}$, then

$$l(t_k^n) = l(t_k^{|n|}) = 2(\mid n \mid + 1).$$

Therefore,

$$\forall k \geq 0, n \in \mathbb{Z} \text{ and } n \neq 0, \quad l(t_k^n) = 2(|n| + 1).$$

Note that if $n = 0$, $l(t_k^0) = l(1_F) = 0$.

It follows that

$$\forall x=(x(0),x(1),\cdots),y=(y(0),y(1),\cdots)\in\bigoplus_{n\geq 0}\mathbb{Z},\ d(x,y)=\sum_{n\geq 0,x(n)\neq y(n)}2(|\ x(n)-y(n)\ |+1).$$

Let d_2 be the l^1-metric for $\bigoplus_{n\geq 0}\mathbb{Z}$ and $id:(\bigoplus_{n\geq 0}\mathbb{Z},d)\to(\bigoplus_{n\geq 0}\mathbb{Z},d_2)$ be the identity map, it is easy to see that

$$d_2(x,y)\leq d(x,y)\leq 4d_2(x,y).$$

Therefore, the identity map is a coarse equivalence. By Corollary 3.1, $(\bigoplus_{n\geq 0}\mathbb{Z},d_2)$ is not an exact metric space. Hence, the subspace $(\bigoplus_{n\geq 0}\mathbb{Z},d)$ of (F,d) is not an exact metric space. Then (F,d) is not an exact metric space. By Lemma 3.6, the metric space (F,d) does not have finite decomposition complexity. □

Acknowledgements

We thank Guoliang Yu for many stimulating discussions.

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