Periodic and solitary wave solutions in quadratic nonlinear media*

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By using the theory of dynamical systems, solitary and periodic travelling wave solutions for a coupled quadratic nonlinear system are studied. Under different parameter conditions, explicit formulas of solitary wave solutions and periodic wave solutions are obtained. Moreover, Some known results in the literature are generalized.

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1. Introduction

Recently, an important result in nonlinear optics was obtained that a classical nonlinear optical effect in the form of the second harmonic generation (SHG in short) induced by a quadratic nonlinear medium rather than cubic ones. That is, a pump wave at the fundamental harmonic (FH) wave generates its second harmonic (SH) wave with the double frequency. Solitary waves in quadratic nonlinear materials have attracted growing attention because of the potential applications in many branches , such as switching devices and signal routing , and laser systems containing quadratic nonlinear crystal, and so on[1-3]. The SHG process can be derived from Maxwell's equations in the quadratic nonlinear medium. In the spatial case, the simplest mathematical model about SHG in a 1-D medium is described by the following normalized system [4]

$$\begin{cases} i\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} - u + u^* v = 0, \\ 2i\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{u^2}{2} = 0, \end{cases} \tag{1}$$

where $u(x,z), v(x,z)$ are the envelope functions of the FH and SH waves, respectively. The asterisk $*$ means complex conjugate, x and z are the transverse and propagating coordinate, respectively. α corresponds to normalized wave number mismatch.

Some stationary soliton solutions and elliptic function solutions of the system (1) have been obtained by various powerful methods, For example, variational approach [5], the direct trial method [6-7] and Lie group method [8] etc. By using the gauge transformation [1], some soliton wave solutions and periodic travelling wave solutions have been obtained. Recently, Lin and his co-workers [9] obtained some explicit periodic and solitary wave solutions for system (1) by applying the Bäcklund transformation and the trial method. Unfortunately, all the above results obtained are not complete since they did not study the bifurcation behavior of phase portraits for the corresponding travelling wave equations. In this paper, we consider bifurcation problem of solitary waves and periodic waves for system (1), by applying the theory of dynamical systems [10-14]. Under different parameter conditions, all explicit formulas of periodic and solitary wave solutions can be easily obtained.

2. Preliminaries

To look for the travelling wave solutions of system (1), we take the following transformation

$$\begin{cases} u(x,z) = U(\xi)e^{i\eta}, \\ v(x,z) = \phi(\xi)e^{2i\eta}, \\ \xi = kx - wz, \eta = k_1 - w_1 z, \end{cases} \tag{2}$$

Substituting (2) into (1), we can obtain

$$\begin{cases} k^2\frac{d^2U}{d\xi^2} + (w_1 - k_1^2 - 1)U + U\phi = 0, \\ k^2\frac{d^2\phi}{d\xi^2} + (4w_1 - 4k_1^2 - \alpha)\phi + \frac{U^2}{2} = 0, \end{cases} \tag{3}$$

Moreover, we have the parameter relation $w = 2k_1 k$.

We assume that $U = a\phi$, where a is a constant,

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then under the constraint conditions

$$a^2 = 2, w_1 - k_1^2 = \frac{\alpha - 1}{3}, \tag{4}$$

Eq. (3) becomes an equation

$$\frac{d^2\phi}{d\xi^2} + \frac{\alpha - 4}{3k^2}\phi + \frac{1}{k^2}\phi^2 = 0. \tag{5}$$

Denote that $\frac{d\phi}{d\xi} = y$, then Eq. (5) becomes the following two-dimensional dynamical system

$$\begin{cases} \frac{d\phi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{4-\alpha}{3k^2}\phi - \frac{1}{k^2}\phi^2. \end{cases} \tag{6}$$

So under the parameter conditions (4), the phase orbits defined by the vector fields of system (6) determine all travelling wave solutions of (1). That is, a homoclinic orbit of system (6) corresponds to a solitary wave solution of system (1),a heteroclinic orbit of system (6) corresponds to a kink (or antikink) wave solution. Similarly, a periodic orbit of system (6) corresponding to a periodic travelling wave solution of system (1). Thus, to investigate all bifurcations of solitary waves, kink (or anti-kink) waves and periodic waves of (1), we should find all bounded solutions of system (6) depending on the parameter space of this system. The bifurcation theory of dynamical systems [11] will play an important role in our study.

In this letter, we first consider the dynamical behaviour of system (6) and then obtain two class of explicit parametric representations of homoclinic orbits and periodic orbits of system (6) under different parameter conditions. At last, we will give the exact explicit formulas of periodic and solitary wave solutions for system (1).

3. The parametric representations of phase orbits of system (6)

In this section, We consider the dynamics of phase orbits of system (6) in its parameter space. Obviously, there exists two critical points of system (6) at $O(0, 0)$ and $A(\frac{4-\alpha}{3}, 0)$. Moreover, system (6) is a Hamiltonian system with Hamiltonian

$$H(\phi, y) = \frac{y^2}{2} + \frac{\alpha - 4}{6k^2}\phi^2 + \frac{\phi^3}{3k^2} = h, \tag{7}$$

where h is a Hamiltonian constant.

Combining the theory of dynamical system with Jacobian elliptic functions[15], we can obtain the following conclusions.

(1) Assume that $\alpha < 4$

In this case, the origin O and A are saddle point and center of system (6), respectively. We see from (6) and (7) that corresponding to the level curves defined by $H(\phi, y) = h = 0$, a homoclinic orbit connecting the origin of system (6) has the following parametric representation:

$$\phi(\xi) = \frac{4 - \alpha}{2} sech^2(\frac{\sqrt{12 - 3\alpha}}{6|k|}\xi). \tag{8}$$

When $h \in (\frac{(\alpha-4)^3}{162k^2}, 0)$, there exists a family of periodic closed orbits of system (6) having the level curves $H = h$. Denote that

$$y^2 = 2h + \frac{4 - \alpha}{3k^2}\phi^2 - \frac{2}{3k^2}\phi^3 = \frac{2}{3k^2}(r_1 - \phi)(\phi - r_2)(\phi - r_3), \tag{9}$$

where $h \in (\frac{(\alpha-4)^3}{162k^2}, 0)$, and $r_j (j = 1, 2, 3)$ are functions of h. Thus, from (6) and (7) ,we can obtain that the family of periodic closed orbits which defined by $H = h$ have the following parametric representations:

$$\phi(\xi) = r_1 - (r_1 - r_2) sn^2[\frac{\sqrt{r_1 - r_3}}{\sqrt{6}|k|}\xi, m]. \tag{10}$$

where $sn(x, m)$ is the Jacobian elliptic function with the modulus $m \in (0, 1)$ and $m = \sqrt{\frac{r_1-r_2}{r_1-r_3}}$. Clearly, $\phi(\xi)$ is a periodic function with period $T(m) = \frac{2\sqrt{6}|k|}{\sqrt{r_1-r_3}}K(m)$, where $K(m)$ is the complete elliptic integral of the first kind.

(2) Assume that $\alpha > 4$

In this case, the origin O and A are center and saddle points of system (6), respectively. We see from (6) and (7) that corresponding to the level curves defined by $H(\phi, y) = h = \frac{(\alpha-4)^3}{162k^2}$, a homoclinic orbit connecting the saddle point A of system (6) has the following parametric representation:

$$\phi(\xi) = \frac{\alpha - 4}{6}[1 - 3tanh^2(\frac{\sqrt{3\alpha - 12}}{6|k|}\xi)]. \tag{11}$$

When $h \in (0, \frac{(\alpha-4)^3}{162k^2})$, there exists a family of periodic closed orbits of system (6) having the level curves $H = h$. Denote that

$$y^2 = 2h + \frac{4 - \alpha}{3k^2}\phi^2 - \frac{2}{3k^2}\phi^3 = \frac{2}{3k^2}(\beta_1 - \phi)(\phi - \beta_2)(\phi - \beta_3), \tag{12}$$

where $h \in (0, \frac{(\alpha-4)^3}{162k^2})$, and $\beta_j (j = 1, 2, 3)$ are functions of h. Thus, similar to case (1), we can obtain that the family of periodic closed orbits which defined by $H = h$ have the following parametric representations:

$$\phi(\xi) = \beta_1 - (\beta_1 - \beta_2) sn^2[\frac{\sqrt{\beta_1 - \beta_3}}{\sqrt{6}|k|}\xi, m_0]. \quad (13)$$

Where the modulus $m_0 = \sqrt{\frac{\beta_1-\beta_2}{\beta_1-\beta_3}}$.Clearly, the period of $\phi(\xi)$ is $T(m_0) = \frac{2\sqrt{6}|k|}{\sqrt{\beta_1-\beta_3}}K(m_0)$.

4. Main results

Based on the above analysis , we can obtain the main results of this paper as follows.

Theorem Suppose that the conditions

$$w = 2k_1 k, \quad w_1 - k_1^2 = \frac{\alpha - 1}{3}, \quad (14)$$

hold, and $r_i, \beta_i (i = 1, 2)$ are defined by (9) and (12) , then the system (1) have the following exact and explicit formulas of bounded travelling wave solutions

(1) When $\alpha < 4$, we have

(i)Solitary wave solutions

$$\begin{cases} u(x,z) = \pm\frac{\sqrt{2}(4-\alpha)}{2} sech^2(\frac{\sqrt{12-3\alpha}}{6|k|}\xi)e^{i\eta}, \\ v(x,z) = \frac{4-\alpha}{2} sech^2(\frac{\sqrt{12-3\alpha}}{6|k|}\xi)e^{2i\eta}, \end{cases} \quad (15)$$

(ii)periodic wave solutions

$$\begin{cases} u(x,z) = \pm\sqrt{2}[r_1 - (r_1 - r_2)sn^2(\frac{\sqrt{r_1-r_3}}{\sqrt{6}|k|}\xi, m)]e^{i\eta}, \\ v(x,z) = r_1 - (r_1 - r_2)sn^2[\frac{\sqrt{r_1-r_3}}{\sqrt{6}|k|}\xi, m]e^{2i\eta}. \end{cases} \quad (16)$$

where $\xi = kx - wz, \eta = k_1 - w_1 z, m = \sqrt{\frac{r_1-r_2}{r_1-r_3}}$.

(2) When $\alpha > 4$, we have

(i)Solitary wave solutions

$$\begin{cases} u(x,z) = \pm\frac{\sqrt{2}(\alpha-4)}{6}[1 - 3tanh^2(\frac{\sqrt{3\alpha-12}}{6|k|}\xi)]e^{i\eta}, \\ v(x,z) = \frac{\alpha-4}{6}[1 - 3tanh^2(\frac{\sqrt{3\alpha-12}}{6|k|}\xi)]e^{2i\eta}, \end{cases} \quad (17)$$

(ii)periodic wave solutions

$$\begin{cases} u(x,z) = \pm\sqrt{2}[\beta_1 - (\beta_1 - \beta_2)sn^2(\frac{\sqrt{\beta_1-\beta_3}}{\sqrt{6}|k|}\xi, m_0)]e^{i\eta}, \\ v(x,z) = \beta_1 - (\beta_1 - \beta_2)sn^2[\frac{\sqrt{\beta_1-\beta_3}}{\sqrt{6}|k|}\xi, m_0]e^{2i\eta}. \end{cases} \quad (18)$$

where $\xi = kx - wz, \eta = k_1 - w_1 z, m_0 = \sqrt{\frac{\beta_1-\beta_2}{\beta_1-\beta_3}}$.

To our knowledge, the exact explicit solitary and periodic solutions (15)-(18) we obtained in this paper have not been found before.

5. Conclusions

In summary, we have derived many families of exact travelling wave solutions of the quadratic nonlinear system (1), including exact solitary and periodic waves, based on the method of dynamical system and bifurcation theory. The paper is shown that the method is suitable to seek exact travelling wave solutions of non-integrable coupled nonlinear partial differential equations. From the above analysis, we believe that this method can be extended and applied to other nonlinear evolution equations.

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