The global strong solution for Navier-Stokes equations in 3D thin domains with Navier-friction boundary conditions *

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Abstract

In this paper, we consider Navier-Stokes equations in thin 3D thin domain with more general Navier-friction boundary conditions (2.4) (compare with boundary condition in [1]). We prove the global existence of strong solutions for the initial data and external forces are in larger sets, and existence of attractor of strong solutions, we generalize the results in [1].

Keywords: 3D Navier-Stokes equations, thin domains, Navier-friction boundary conditions, attractors.

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1 Introduction

As is well known, the existence and uniqueness theory of 2D Navier-Stokes equations is well understood with suitable assumptions on the initial data and forces. However, there is no general existence and uniqueness result for strong solutions to 3D Navier-Stokes equations. In general, we have a global solution for small data, or a short-time solution for arbitrary data (see [10]). Thus a natural question arises, namely can we use the thinness of the three-dimensional domain in order to improve the global existence results of strong solution? The study of global existence of strong solutions of Navier-stokes equations on the thin domains originates in a series of papers [7] [8] by Hale and Raugel. In thin 3D domains, inspired by the methods developed in [7] [8], Raugel and Sell [3] [4] proved global existence of strong solutions for large initial data and forcing terms in the case of periodic boundary conditions (PP) or mixed boundary conditions (PD).i.e. periodic conditions

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in the vertical thin direction and homogeneous Dirichlet conditions on the lateral boundary. Raugel and Sell use, as an essential tool, the vertical mean operator M, which allows to decompose every function u into the sum of a function Mu which does not depend on the vertical variable x_3 and a function $(I-M)u$ with vanishing vertical mean and thus to apply more precise Sobolev and Poincaré inequalities. In fact, Raugel and Sell demonstrate that the initial data and forcing are allowed to grow at a faster rate as $\varepsilon \to 0$ than the classical results would allow. After these initial results, a series of papers which improved the results on the size of the initial data and external forces was made in [9,10,13,14].

For all the above boundary conditions,the mean vertical operator has advantage of commutativity with differential operators $\frac{\partial}{\partial x_i}, i = 1, 2, 3$, the Stokes operator A_ε. However, in the case of Navier-friction boundary conditions these underlying properties do not hold true.

The Navier boundary conditions appear already in the original paper of Navier [15], who claimed that the tangential component of the viscous stress at the boundary should be proportional to the tangential velocity. It was rigorously justified as a homogenization of the no-slip condition on a rough boundary (see [12]). In [16] the author introduce an elementary derivation of an explicit form of the Navier boundary condition for general regions Ω, that is,

$$u = -k(x)wN(x) \quad \text{on } \partial\Omega \tag{1.1}$$

where w is the vorticity matrix $w^{i,k} = u^j_{x_k} - u^k_{x_j}$ and $N(x)$ is the unit outer normal on $\partial\Omega$. In [17] Lions,Temam and Wang introduce the Navier conditions in terms of an interface condition(also see [2] for the study of such an interface condition in the case of a thin product domain). The authors in [1] consider the Navier-friction boundary conditions,

$$\begin{cases} u_3 = 0, \frac{1}{\varepsilon}\frac{\partial u_\alpha}{\partial x_3} + u_\alpha = 0 & on\ \Gamma_t, \\ u_3 = 0, \frac{\partial u_\alpha}{\partial x_3} = 0 & on\ \Gamma_b, \alpha = 1, 2, \\ u \text{ is periodic in the directions } x_1, x_2 \text{ with period } \omega. \end{cases}$$

by constructing the new average operator $\mathcal{M}_\varepsilon$ and its complement $\mathcal{N}_\varepsilon$ based on the spectral decomposition for the corresponding Stokes operator A_ε. The authors in [2] proposed to leave the traditional framework to dealing with the Stokes operator A_ε with divergence free constraint for its domain .i.e.$divu = 0$,and to work with operator $D_\varepsilon = -\nu\Delta_\varepsilon$ with the prescribed boundary conditions. However, In [1] [2],constructing the eigenvalue and eigenfunction of Stokes operator A_ε is required. The author consider the more general boundary $\Gamma_\varepsilon = (x_h, \varepsilon g(x_h))$ in [6].

In this paper, we consider the global-in-time of the strong solutions to the 3D Navier-stokes equations in thin 3D domains $\Omega_\varepsilon = \Omega \times (0, \varepsilon) = (0, l_1) \times (0, l_2) \times (0, \varepsilon), 0 < \varepsilon \leq 1$, with periodic conditions on the lateral boundary $\Gamma_l = \partial\Omega \times (0, \varepsilon)$ and friction boundary conditions on the thin vertical direction $\Gamma_t = \Omega \times \{x_3 = \varepsilon\}, \Gamma_b = \Omega \times \{x_3 = 0\}$. We consider the more general Navier-friction boundary conditions (2.4) which is equivalent to (1.1). For more general Navier-friction boundary conditions,it is difficult to utilize the technique in [1][2] to construct the spectrum of the corresponding Stokes operator A_ε with Navier-friction boundary conditions. Inspiried by the methods in [6] and [2],we define the new mean operator of vector field as (2.10), we deduce a "good" estimate of the trilinear term $|\int_{\Omega_\varepsilon}(u\cdot\nabla u)\cdot\triangle u dx|$ by decomposing the vector u into $v+w$ and by using the smallness properties of w as well as the fact that v depends only on the horizontal variable x', we improve the results in [1].

This paper is arranged as follows. In section 2, we recall the 3D Navier-stokes equations and its mathematical setting and the new vertical mean operators M_ε and its properties. In section 3, we deduce some essential auxiliary inequality.In section 4 , we deduce a "good" estimate of the trilinear term. Section 5 gives the proof of the global existence of strong solutions of (2.1)-(2.4) for large initial data and forcing term. Finally, in section 6, we show the existence of a compact local attractor to system(2.1)-(2.4).

2 Preliminaries

Let $\Omega_\varepsilon \subseteq \mathbb{R}^3$ and $\Omega_\varepsilon = \Omega\times(0,\varepsilon) = (0,l_1)\times(0,l_2)\times(0,\varepsilon), 0<\varepsilon<1$, we consider the following Navier-Stokes equations in Ω_ε,

$$\frac{\partial u}{\partial t} - \nu\triangle u + (u\cdot\nabla)u + \nabla p = f(x,t) \quad \text{in } \Omega_\varepsilon\times(0,\infty), \tag{2.1}$$

$$\text{div} u = 0, \quad \text{in } \Omega_\varepsilon\times(0,\infty), \tag{2.2}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega_\varepsilon\times\{t=0\}, \tag{2.3}$$

where $\nu>0$ is the kinematic viscosity and $f(x,t)$ is the body force, $u=(u_1,u_2,u_3)$ is the velocity vector at point x and time t, $p(x,t)$ is the pressure. To specify boundary conditions, we separate the boundary of Ω_ε as follows,

$$\Gamma_l = \partial\Omega\times(0,\varepsilon), \Gamma_t = \Omega\times\{\varepsilon\}, \Gamma_b = \Omega\times\{0\}.$$

We will assume the Navier-friction boundary conditions as follows,

$$\begin{cases} (u_1,u_2,u_3) = -k(x')\Big(\frac{\partial u_1}{\partial x_3},\frac{\partial u_2}{\partial x_3},0\Big), \text{ on } \Gamma_t, \\ (u_1,u_2,u_3) = k(x')\Big(\frac{\partial u_1}{\partial x_3},\frac{\partial u_2}{\partial x_3},0\Big), \text{ on } \Gamma_b, \\ u \text{ is perodic in the directions } x_1,x_2.i.e.\text{on } \Gamma_l. \end{cases} \tag{2.4}$$

where $x'\in\Omega$. On the other hand, we assume on $k(x')$ as follows,

$$\begin{cases} k(x') \text{ is periodic with respect to } x', \\ C_1\varepsilon \le |\nabla k^{-1}(x')| \le C_2\varepsilon, C_3\varepsilon \le k^{-1}(x') \le C_4\varepsilon. \end{cases} \tag{2.4$'$}$$

Where C_1, C_2, C_3, C_4 are independent of ε. We require in addition that the initial u_0 and $f(x,t)$ satisfy

(H) $\int_{\Omega_\varepsilon} udx = \int_{\Omega_\varepsilon} fdx = 0.$

For the mathematical setting of the Navier-Stokes equations, we consider the Hilbert space $H_\varepsilon \subseteq \mathbb{L}^2(\Omega_\varepsilon)$, such that

$$H_\varepsilon = \{u \in \mathbb{L}^2(\Omega_\varepsilon) : \ \text{div}u = 0, u \text{ is periodic in } x_1 \text{ and } x_2, \ u \text{ satisfies the contions(H)}\}.$$

The scalar inner product on H_ε is denoted by $(\cdot,\cdot)$, the associated norms is a general L^2 norm and is denoted by $\|\cdot\|_{L^2(\Omega_\varepsilon)}$.

We now define the space

$$\mathcal{H}^1 = \{u \in \mathbb{H}^1(\Omega_\varepsilon) : \text{div}u = 0, u \text{ satisfies (2.4) and the conditions(H)}\}.$$

We also define V_ε, a closed subspace of $\mathbb{H}^1(\Omega_\varepsilon)$ as follows

$$V_\varepsilon = \mathcal{H}^1 \cap \mathbb{H}^1(\Omega_\varepsilon)$$

.

Let $\mathbb{P}_\varepsilon$ denote the classical Helmholtz-Leray (orthogonal) projection of $\mathbb{L}^2(\Omega_\varepsilon)$ onto H_ε.

In this paper, we will use various norm,such as $\|\cdot\|_{L^2}, \|\cdot\|_{H^1}, \|\cdot\|_{H^2}, \|\cdot\|_{L^q}$,etc.The subscripts should read as: $L^2 = L^2(\Omega_\varepsilon)^3, H^1 = H^1(\Omega_\varepsilon)^3, \ H^2 = H^2(\Omega_\varepsilon)^3$ etc. For functions $f = f(x,t) \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$, we define the norm $\|f\|_\infty$ by

$$\|f\|_\infty = \sup_{t>0} |f(\cdot,t)|,$$

where one uses the essential superum.

We also introduce the bilinear form $E(\cdot,\cdot)$ on $V_\varepsilon \times V_\varepsilon$ as follows

$$E(u,u^*) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla u^* dx + \int_{\Gamma_t \cup \Gamma_b} k^{-1}(x')u \cdot u^* dx', \quad \text{for all } u, u^* \in V_\varepsilon.$$

We have the following result.

Theorem 2.1. There exist c_0, c_0^* such that, for any $\varepsilon \in (0,1]$,

$$c_0\|u\|_{H^1}^2 \le E(u,u) \le c_0^*\|u\|_{H^1}^2, \quad \text{for all } u \in \mathcal{H}^1. \tag{2.5}$$

Proof. We remark that

$$E(u,u) = \|\nabla u\|_{L^2}^2 + \int_{\Gamma_t \cup \Gamma_b} k^{-1}(x')u^2 dx'. \tag{2.6}$$

Using lemma 3.4 and lemma 3.5 and $k(x')$ satisfies conditions required above, we easily deduce the right side of (2.5). The left side of (2.5) is the directly result lemma 3.1 and (2.6). The proof of lemma is completed.

Since by Theorem 2.1, the bilinear form $E(\cdot,\cdot)$ is a coercive continuous symmetric bilinear form on the space V_ε, one can define by using the essential function theory the Stokes operator A_ε as the isomorphism from V_ε onto the dual V_ε^* of V_ε.

$$\langle A_\varepsilon u, v\rangle_{V_\varepsilon^*,V_\varepsilon} = E(u,v), \quad \forall v \in V_\varepsilon.$$

One can also extend A_ε as a linear operator on H_ε. The domain $D(A_\varepsilon) \equiv \{u \in V_\varepsilon : A_\varepsilon u \in H_\varepsilon\}$ is exactly the space $\mathbb{H}^2(\Omega_\varepsilon) \cap V_\varepsilon$. the Stokes operator A_ε is given by

$$A_\varepsilon u = -\mathbb{P}_\varepsilon \Delta u \quad \forall u \in D(A_\varepsilon).$$

It is evident that A_ε is self-adjoint, and also we can claim that A_ε has compact bounded inverse in H_ε, which is a direct consequence of the compactness of embedding V_ε into H_ε. therefore, by essential function analysis, the set of all its eigenfunctions constructs a complete normal base of H_ε and we can define its fractional powers. For $0 \le s \le 2$, we denote by V_ε^s the space $D(A_\varepsilon^{\frac{s}{2}})$, equipped with the natural norm $\|\cdot\|_{V_\varepsilon^s} \equiv \|\cdot\|_s \equiv \|A_\varepsilon^{\frac{s}{2}}\cdot\|_{L^2(\Omega_\varepsilon)}$, in fact, $V_\varepsilon = D(A_\varepsilon^{\frac{1}{2}})$, the following equality holds

$$(A_\varepsilon u, u)_{L^2} = \frac{1}{2}E(u,u), \quad \text{for any } u \in D(A_\varepsilon) \tag{2.7}$$

.

Let B_ε be the bilinear form on V_ε defined, for $(u,v) \in V_\varepsilon \times V_\varepsilon$, by

$$\langle B_\varepsilon(u,v), w\rangle_{V_\varepsilon^*,V_\varepsilon} = \int_{\Omega_\varepsilon} (u\cdot\nabla)v\cdot w dx, \quad \forall w \in V_\varepsilon.$$

With these notions we can write the Navier-Stokes equations as a differential equation in V_ε^*:

$$\frac{\partial u}{\partial t} + \nu A_\varepsilon u + B_\varepsilon(u,u) = P_\varepsilon f, \quad u(0) = u_0. \tag{2.8}$$

Here $\frac{\partial u}{\partial t}$ denotes the derivative (in the sense of distributions) of u with respect to t.

We now recall the mean value operator M in the vertical direction which acts on the scalar functions defined on Ω_ε and is given by (see [3,4]):

$$(Mf)(x_1,x_2) = \frac{1}{\varepsilon}\int_0^\varepsilon f(x_1,x_2,s)ds, \quad \forall f \in L^2(\Omega_\varepsilon). \tag{2.9}$$

We remark that, for $1 \le p \le \infty$, $M : L^p(\Omega_\varepsilon) \to L^p(\Omega_\varepsilon)$ is a bounded linear operator of norm 1.We also recall that M is an orthogonal projection of $L^2(\Omega_\varepsilon)$ (for this property as well as for other properties, see [3] [4]).

Next we introduce the notion of mean value for vectors.If $u = (u_1,u_2,u_3)$ belongs to H_ε, then u is tangent to the boundary and, in particular, $u_3|_{\Gamma_t\cup\Gamma_b} = 0$, which implies by the poincaré inequality that $\|u_3\|_{L^2} \le \varepsilon\|u_3\|_{H^1}$. This indicates that taking the vertical mean value of the third component is not of real interest. Since the third component is small of order ε, therefore, we introduce the following mean value operator M_ε acting on $L^1(\Omega_\varepsilon)^3$:

$$M_\varepsilon u(x') = (Mu_1, Mu_2, 0), \quad \text{for all } u \in L^1(\Omega_\varepsilon)^3. \tag{2.10}$$

We remark that, as above, $M_\varepsilon : L^p(\Omega_\varepsilon)^3 \to L^p(\Omega_\varepsilon)^3$ is a bounded linear operator of norm 1. Clearly, M_ε and $I - M_\varepsilon$ are orthogonal projections in $(L^2(\Omega_\varepsilon))^3$. Moreover, from boundary conditions (2.4), we obtain that: $M_\varepsilon H_\varepsilon \subset H_\varepsilon$(in particular,$M_\varepsilon u$ is divergence free and tangent to the boundary). however, it do not commute with $\frac{\partial}{\partial x_3}$ and A_ε. Using these properties and the fact that $\mathbb{P}_\varepsilon$ is an orthogonal projection onto H_ε, one shows that

$$M_\varepsilon \mathbb{P}_\varepsilon u = \mathbb{P}_\varepsilon M_\varepsilon u, \quad \forall u \in (L^2(\Omega_\varepsilon))^3. \tag{2.11}$$

3 The several auxiliary estimates:

We denote C by positive constants thorough this paper, which is independent of ε and may change from line to line, the constants are fixed once and for all. We start with a series of simple preliminary lemma,first of all, let us observe that the Poincar$\acute{e}$ inequality holds true with constants.

Lemma 3.1. There exist positive constants C, which is independent of ε, such that for every $\varphi \in H^1(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} \varphi dx = 0$, the following inequality hold true:

$$\|\varphi\|_{L^2} \le C\|\nabla\varphi\|_{L^2}, \quad \text{for all } \varepsilon \in (0, 1]. \tag{3.1}$$

Proof. Let $\varphi_\varepsilon(x) = \varphi(x', \varepsilon x_3)$, then, φ_ε is a function defined on domain $\Omega_1 = \Omega \times (0, 1)$ and has vanishing mean, the standard Poincar$\acute{e}$ inequality therefore gives

$$\|\varphi_\varepsilon\|_{L^2(\Omega_1)} \le C\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_1)}$$

for some constant C independent of ε. Expressing this relation in terms of φ, we obtain

$$\varepsilon^{-\frac{1}{2}}\|\varphi\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{-\frac{1}{2}}\Big(\|\partial_1\varphi\|^2_{L^2(\Omega_\varepsilon)} + \|\partial_2\varphi\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2\|\partial_3\varphi\|^2_{L^2(\Omega_\varepsilon)}\Big)^{\frac{1}{2}},$$

which implies (3.1) , the proof of lemma is completed.

Lemma 3.2. Let φ be a function in $H^1(\Omega_\varepsilon)$ satisfy $M\varphi \equiv 0$, then,

$$\|\varphi\|_{L^2} \le \varepsilon\|\partial_3\varphi\|_{L^2}, \quad \text{for all } \varepsilon \in (0, 1].$$

Proof. We notice that $\int_{\Omega_\varepsilon} \varphi(x', x_3)dx_3dx' = 0$, which implies that

$$\begin{aligned}
\|\varphi\|^2_{L^2} &= \int_{\Omega_\varepsilon} \varphi(x', x_3)\int_0^{x_3} \partial_3\varphi(x', y_3)dy_3dx_3dx' \\
&\le \Big(\int_{\Omega_\varepsilon} \varphi^2(x', x_3)dx_3Dx'\Big)^{\frac{1}{2}}\Big(\int_{\Omega_\varepsilon}(\int_0^{x_3} \partial_3\varphi(x', y_3)dy_3)^2dx_3dx'\Big)^{\frac{1}{2}} \\
&\le \varepsilon\|\varphi\|_{L^2}\|\partial_3\varphi\|_{L^2}.
\end{aligned}$$

which completes the proof.

Lemma 3.3. Let φ be a function in $H^1(\Omega_\varepsilon)$, then,

$$\|\varphi\|_{L^2} \leq \|M\varphi\|_{L^2} + \varepsilon\|\partial_3\varphi\|_{L^2}, \quad \text{for all } \varepsilon \in (0,1].$$

Proof. The proof of lemma follows directly from the lemma 3.2:

$$\begin{aligned}
\|\varphi\|_{L^2} &\leq \|M\varphi\|_{L^2} + \|(I-M)\varphi\|_{L^2} \\
&\leq \|M\varphi\|_{L^2} + \varepsilon\|\partial_3\varphi\|_{L^2} \\
&= \|M\varphi\|_{L^2} + \varepsilon\|\partial_3\varphi\|_{L^2}.
\end{aligned}$$

Lemma 3.4. For any $\varepsilon \in (0,1]$ and for any $\varphi \in H^1(\Omega_\varepsilon)$ and satisfy (2.4), we have

$$\|\varphi(x', \varepsilon)\|_{L^2(\Gamma_t)} \leq 2\varepsilon^{-\frac{1}{2}}\|M\varphi\|_{L^2(\Omega_\varepsilon)} + \sqrt{5}\varepsilon^{\frac{1}{2}}\|\partial_3\varphi\|_{L^2(\Omega_\varepsilon)}, \tag{3.2}$$

$$\|\varphi(x', 0)\|_{L^2(\Gamma_b)} \leq 2\varepsilon^{-\frac{1}{2}}\|M\varphi\|_{L^2(\Omega_\varepsilon)} + \sqrt{5}\varepsilon^{\frac{1}{2}}\|\partial_3\varphi\|_{L^2(\Omega_\varepsilon)}. \tag{3.3}$$

Proof. We first note that

$$\begin{aligned}
\|\varphi(x', \varepsilon)\|^2_{L^2(\Gamma_t)} &= \int_\Omega \varphi^2(x', \varepsilon)dx' \\
&= \varepsilon^{-1}\int_\Omega\int_0^\varepsilon \partial_{x_3}(x_3\varphi^2(x))dx_3dx'.
\end{aligned}$$

Using lemma 3.3 and the Young inequality, we infer from the previous equality that

$$\begin{aligned}
\|\varphi(x', \varepsilon)\|^2_{L^2(\Gamma_t)} &= \varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)} + 2\varepsilon^{-1}\int_{\Omega_\varepsilon} x_3\varphi(x)\partial_{x_3}\varphi(x)dx \\
&\leq 2\varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_3\varphi\|^2_{L^2(\Omega_\varepsilon)} \\
&\leq 4\varepsilon^{-1}\|M\varphi\|^2_{L^2(\Omega_\varepsilon)} + 5\varepsilon\|\partial_3\varphi\|^2_{L^2(\Omega_\varepsilon)}
\end{aligned}$$

which implies the inequality (3.2).

To prove the second inequality,we only instead of $\partial_{x_3}(x_3\varphi^2(x))$ with $\partial_{x_3}((x_3-\varepsilon)\varphi^2(x))$ in the proof of (3.2), the detail is omitted. The proof of lemma is completed.

Remark 3.1. Using the same method, for $\varphi \in H^2(\Omega_\varepsilon)$, we also easily deduce that

$$\int_{\Gamma_t\cup\Gamma_b} |\nabla\varphi|^2dx' \leq C\varepsilon^{-1}\|\nabla\varphi\|^2_{L^2} + C\varepsilon\|\varphi\|^2_{H^2}.$$

Lemma 3.5. For $\varepsilon \in (0,1]$, for any $\varphi \in H^2(\Omega_\varepsilon)$ and for $i = 1, 2$, one has

$$\|\partial_i\varphi\|_{L^2} \leq \|M\varphi\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\partial_i\partial_3\varphi\|_{L^2(\Omega_\varepsilon)}.$$

Proof. We note that $M\partial_i\varphi = \partial_iM\varphi$, for $i = 1, 2$, the proof of lemma is direct result of lemma 3.3.

Lemma 3.6. There exist a positive constant C, which is independent of ε, such that,for all $\varepsilon \in (0,1]$ and for any function $\varphi \in H^2(\Omega_\varepsilon)$, one has

$$\|M\varphi\|_{L^2(\Omega_\varepsilon)} \leq \|\varphi\|_{L^2(\Omega_\varepsilon)},\ \|M\varphi\|_{H^1} \leq C\|\varphi\|_{H^1},\ \|M\varphi\|_{H^2} \leq C\|\varphi\|_{H^2}.$$

Proof. The first relation follows simply by noting that the operator M is nothing else but the L^2 orthogonal projection onto the space of functions independent of the x_3, to prove the last two inequalities, by using again the formula $\partial_i M\varphi = M\partial_i\varphi,\ \partial_i\partial_j M\varphi = M\partial_i\partial_j\varphi,\ i,j = 1,2,$ together with lemma 3.4, we easy deduce the conclusion.

Theorem 3.7. There exists a positive constant C, which is independent of ε, and $\varepsilon_0 \leq 1$ such that, for all $\varepsilon \in (0,\varepsilon_0]$, and for all $u \in V_\varepsilon$, the functions $w = (I - M_\varepsilon)u$ satisfy,

$$\|w\|_{L^2} \leq C\varepsilon\|\partial_3 w\|_{L^2}, \quad \text{for all} \quad \varepsilon \in (0,\varepsilon_0], \tag{3.4}$$

and, if moreover u belongs to $H^2(\Omega_\varepsilon)^3$ and satisfy (2.4), then

$$\|\nabla w\|_{L^2} \leq C\varepsilon\|\nabla^2 w\|_{L^2} + C\varepsilon\|u\|_{L^2}. \tag{3.5}$$

Proof. Inequality (3.4)follows directly from Lemma 3.2 since $Mw_1 \equiv Mw_2 \equiv 0$ and $w_3|_{\Gamma_t} \equiv u_3|_{\Gamma_t} \equiv w_3|_{\Gamma_b} \equiv u_3|_{\Gamma_b} \equiv 0$. To prove (3.5), we take different cases. First,for $i = 1,2$, we have that

$$\partial_{x_3} w_i = \partial_{x_3} u_i = \int_0^{x_3} \partial_{x_3}^2 u_i(x', \xi)d\xi + k^{-1}(x')u_i(x', 0),$$

where we have used the boundary conditions (2.4).By using $(2.4)'$,we find

$$\begin{aligned}\|\partial_3 w_i\|_{L^2}^2 &\leq C\varepsilon^2\|\partial_3^2 w_i\|_{L^2}^2 + \int_{\Omega_\varepsilon} k^{-2}(x')u_i^2(x',0)dx \\ &\leq C\varepsilon^2\|\partial_3 w_i\|_{L^2}^2 + C\varepsilon^3\int_{\Gamma_b} u_i^2(x',0)dx'\end{aligned}$$

From Lemma 3.4, we infer from the previous inequality that

$$\|\partial_3 w_i\|_{L^2}^2 \leq C\varepsilon^2\|\partial_3^2 w_i\|_{L^2}^2 + C\varepsilon^2\|u_i\|_{L^2}^2 + C\varepsilon^4\|\partial_3 w_i\|_{L^2}^2; \tag{3.6}$$

On the other hand, we remark that

$$\begin{aligned}\partial_{x_i} w_3 &= \partial_{x_i} u_3 - \partial_{x_i} u_3(x', 0) \\ &= \int_0^{x_3} \partial_{x_3}\partial_{x_i} u_3 d\xi.\end{aligned}$$

Therefore we obtain

$$\|\partial_{x_i} w_3\|_{L^2}^2 \leq C\varepsilon^2\|\partial_3\partial_{x_i} u_3\|_{L^2}^2. \tag{3.7}$$

Next, for $i,j \in \{1,2\}$ since $M\partial_j w_i \equiv \partial_j M w_i \equiv 0$,we deduce from Lemma 3.2 that

$$\|\partial_j w_i\|_{L^2} \leq C\varepsilon\|\partial_{x_j}\partial_3 w_i\|_{L^2} \leq C\varepsilon\|\nabla^2 w\|. \tag{3.8}$$

Finally,the estimate for $\partial_3 w_3$ follows immediately from the above bounds together with the divergence free condition

$$\|\partial_3 w_3\|_{L^2} = \|\partial_1 w_1 + \partial_2 w_2\|_{L^2} \le C\varepsilon\|\nabla^2 w\|_{L^2}. \tag{3.9}$$

Therefore (3.6)-(3.9) imply (3.5), the proof of theorem is completed.

The next result is a regularity type estimate for u.

Theorem 3.8. There exist a positive constant ε_0, with $0 < \varepsilon_0 < 1$ and a positive constant C, which is independent of ε,such that,for all $u \in V_\varepsilon \cap H^2(\Omega_\varepsilon)^3$ satisfying the Navier friction boundary conditions (2.4), one has

$$\|u\|_{H^2} \le C\|\triangle u\|_{L^2} + C\|u\|_{H_1}, \quad \text{for all } \varepsilon \in (0, \varepsilon_0]. \tag{3.10}$$

Proof. First of all, we deduce the estimate of $\|\nabla^2 u\|_{L^2}^2$,

$$\begin{aligned}
\|\nabla^2 u\|_{L^2}^2 &= \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \partial_i\partial_j u \cdot \partial_i\partial_j dx \\
&= -\sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \partial_j u \cdot \partial_i^2 \partial_j u dx + \sum_{j=1}^{3} \int_{\Gamma_t \cup \Gamma_b} N_3 \partial_j u \cdot \partial_3 \partial_j u ds_x \\
&= \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \partial_j^2 u \cdot \partial_i^2 u dx - \sum_{i=1}^{3} \int_{\Gamma_t \cup \Gamma_b} N_3 \partial_3 u \cdot \partial_i^2 u ds_x + \sum_{j=1}^{3} \int_{\Gamma_t \cup \Gamma_b} N_3 \partial_j u \cdot \partial_3 \partial_j u ds_x \\
&= \|\triangle u\|_{L^2}^2 - \sum_{i=1}^{2} \int_{\Gamma_t \cup \Gamma_b} N_3 \partial_3 u \partial_i^2 u ds_x + \sum_{i=1}^{2} \int_{i=1}^{2} N_3 \partial_i u \partial_3 \partial_i u ds_x \\
&= \int_{\Omega_\varepsilon} |\triangle u|^2 dx + I_1 + I_2,
\end{aligned}$$

where we integrated twice by parts and used the boundary conditions (2.4) , where N_3 is the unit exterior normal on Γ_t and Γ_b respectively. We now give the estimate of I_1 and I_2 respectively.

Estimate of I_1.Using boundary conditions (2.4) and the conditions $(2.4)'$, one has

$$|I_1| \le |\sum_{i,k=1}^{2} \int_{\Gamma_t} \partial_3 u_k \partial_i^2 u_k dx'| + |\sum_{i,k=1}^{2} \int_{\Gamma_b} \partial_3 u_k \partial_i^2 u_k dx'| = I_{11} + I_{12}$$

It suffices to estimate I_{11},

$$\begin{aligned}
|I_{11}| &= |\sum_{i,k=1}^{2}\int_{\Gamma_t} k^{-1}(x^{'})u_k\partial_i^2 u_k dx^{'}| \\
&\le |\sum_{i,k=1}^{2}\int_{\Gamma_t} \partial_i u_k \partial_i(k^{-1}(x^{'})u_k)dx^{'}| \\
&\le \sum_{i,k=1}^{2}\int_{\Gamma_t} |\partial_i(k^{-1}(x^{'}))||\partial_i u_k||u_k|dx^{'} + \int_{\Gamma_t} k^{-1}(x^{'})|\partial_i u_k|^2 dx^{'} \\
&\le C\varepsilon\int_{\Gamma_t}|\partial_i u_k|^2 dx^{'} + C\varepsilon\int_{\Gamma_t} u_k^2 dx^{'} \\
&\le C\varepsilon\|u\|_{H^2}^2 + C\|u\|_{H^1}^2,
\end{aligned}$$

where we have used Lemma 3.4.

Estimate of I_2.

$$\begin{aligned}
|I_2| &\le |\sum_{i,k=1}^{2}\int_{\Gamma_t}\partial_i u_k\partial_i\partial_3 u_k dx^{'}| + |\sum_{i,k=1}^{2}\int_{\Gamma_b}\partial_i u_k\partial_i\partial_3 u_k dx^{'}| \\
&\le \sum_{i,k=1}^{2}\int_{\Gamma_t}|\partial_i u_k\partial_i(k^{-1}(x^{'})u_k)|dx^{'} + \int_{\Gamma_b}|\partial_i u_k\partial_i(k^{-1}(x^{'})u_k)|dx^{'}
\end{aligned}$$

Using again Lemma 3.4 and the conditions $(2.4)^{'}$, we easily deduce from inequality that

$$|I_2| \le C\varepsilon\|u\|_{H^2}^2 + C\|u\|_{H^1}^2.$$

From the estimates of I_1 and I_2, we obtain

$$\|\nabla^2 u\|_{L^2}^2 \le \|\triangle u\|_{L^2}^2 + C_0\varepsilon\|u\|_{H^2}^2 + C\|u\|_{H^1}^2$$

where C_0 and C are positive constants which do not depend on ε, One then finds that

$$\|u\|_{H^2}^2 \le \|\triangle u\|_{L^2}^2 + C_0\varepsilon\|u\|_{H^2}^2 + C\|u\|_{H^1}^2$$

We can choose ε_1 to satisfy $0<\varepsilon_1\le 1$ and $C_0\varepsilon_1 \le \frac{1}{2}$, One then obtains, for $0<\varepsilon\le\varepsilon_1$,

$$\frac{1}{2}\|u\|_{H^2}^2 \le \|\triangle u\|_{L^2}^2 + C\|u\|_{H^1}^2,$$

Which implies (3.10).

We now go back to the study of the Stokes operator $A_\varepsilon = -\mathbb{P}_\varepsilon\triangle$, we remark that $-\mathbb{P}_\varepsilon\triangle \ne -\triangle$ in general. The estimates we give below will be used later for deriving a priori estimates of the strong solutions to problem (2.1)-(2.4).

Lemma 3.9. There exists a positive constant C, which is dependent of ε, such that,for all $\varepsilon \in (0,1]$, for any vector $u \in H_\varepsilon \cap H^2(\Omega_\varepsilon)^3$ that satisfies the boundary conditions (2.4), one has

$$\|(I-\mathbb{P}_\varepsilon)\triangle u\|_{L^2} \leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1} + C\varepsilon^{\frac{3}{2}}\|u\|_{H^2}, \quad \text{for all } \varepsilon \in (0,1]. \tag{3.11}$$

Proof. By the properties of the Leray projection $\mathbb{P}_\varepsilon$, we know that there exits some scalar function $q \in L^2(\Omega_\varepsilon)$ such that

$$\triangle u - \mathbb{P}_\varepsilon \triangle u = \nabla q.$$

We can assume without loss of generality that q has vanishing mean on Ω_ε (if not, we set $\widetilde{q} = q - \int_{\Omega_\varepsilon} q dx$).Clearly, q is periodic in $x'-$ direction and satisfies the relation

$$\triangle q = 0, \quad N \cdot \nabla q|_{\partial\Omega_\varepsilon} \equiv N \cdot \triangle u|_{\partial\Omega_\varepsilon}.$$

where N is the unit exterior normal of $\partial\Omega_\varepsilon$. Duo to the characterization of $D(A_\varepsilon)$, we have also that $\nabla q \in L^2(\Omega_\varepsilon)^3$, after some simply calculation we find that

$$\triangle u_3 = -(\partial_1\partial_3 u_1 + \partial_2\partial_3 u_2), \quad \text{on } \Gamma_t \cup \Gamma_b.$$

Using the Navier-friction boundary conditions (2.4) for this u, we obtain

$$\partial_3 q(x) = \partial_1(k^{-1}(x')u_1) + \partial_2(k^{-1}(x')u_2) \ \text{ on } \Gamma_t,$$
$$\partial_3 q(x) = -(\partial_1(k^{-1}(x')u_1) + \partial_2(k^{-1}(x')u_2)) \ \text{ on } \Gamma_t.$$

We can now go to the estimate of ∇q.

We integrate by parts, using the boundary conditions of q to deduce that

$$\begin{aligned}\|\nabla q\|_{L^2}^2 = \int_{\Omega_\varepsilon} \nabla q \cdot \nabla q dx &= -\int_{\Omega_\varepsilon} \triangle q \cdot q dx + \int_{\Gamma_t\cup\Gamma_b} q[\partial_1(k^{-1}(x')u_1)) + \partial_2(k^{-1}(x')u_2)]dx' \\ &= \int_{\Gamma_t\cup\Gamma_b} q[\partial_1(k^{-1}(x')u_1)) + \partial_2(k^{-1}(x')u_2)]dx'.\end{aligned} \tag{3.12}$$

Therefore, using the trace theorem and Lemma 3.1, we obtain that

$$\|q\|_{L^2} \leq C\|q\|_{H^1} \leq C\|\nabla q\|_{L^2}, \tag{3.13}$$

where C is independent of ε. To estimate (3.12), it suffices to estimate the term $\int_{\Gamma_t} |\partial_i(k^{-1}(x')u_i))|^2 dx'$, for $i \in \{1,2\}$.

$$\begin{aligned}\int_{\Gamma_t} |\partial_i(k^{-1}(x')u_i))|^2 dx' &\leq 2\int_{\Gamma_t} |\partial_i(k^{-1}(x'))|^2|u_i|^2 dx' + 2\int_{\Gamma_t} k^{-2}(x')|\partial_i u_i|^2 dx' \\ &= I_1 + I_2.\end{aligned} \tag{3.14}$$

From Lemma 3.4 , Lemma 3.5, Remark 3.1 and the conditions $(2.4)'$, we deduce that

$$I_1 \leq C\varepsilon(\|u\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2), \tag{3.15}$$

$$I_2 \leq C\varepsilon\|u\|_{H^1}^2 + C\varepsilon^3\|u\|_{H^2}^2. \tag{3.16}$$

Thanks to (3.12)-(3.16) , we infer from (3.11) that,

$$\|\nabla q\|_{L^2}^2 \le C(\varepsilon^{\frac{1}{2}}\|u\|_{H^1} + \varepsilon^{\frac{3}{2}}\|u\|_{H^2})\|\nabla q\|_{L^2},$$

which imply (3.11),the proof of lemma is completed.

We next recall three auxiliary inequalities, which are very useful for constructing the strong solutions of problem (2.1)-(2.4).

Lemma 3.10. There exist positive constants ε_0 and C (independent of ε) such that, for any $\varepsilon \in (0, \varepsilon_0]$, the following inequalities hold. For any $\varphi \in H^1(\Omega_\varepsilon)$ such that $M\varphi \equiv 0$,

$$\|\varphi\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}\|\nabla\varphi\|_{L^2}, \quad \text{for all } q \in [2, 6]. \tag{3.17}$$

In particular, for $w = (I - M_\varepsilon)u$, where $u \in V_\varepsilon$, one has,

$$\|w\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}\|\nabla w\|_{L^2}, \quad \text{for all } q \in [2, 6]. \tag{3.18}$$

Moreover, if $u \in H^2(\Omega_\varepsilon)^3 \cap V_\varepsilon$ satisfies the boundary conditions (2.4), we have,

$$\|\nabla w\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}(\|w\|_{H^2} + \|u\|_{L^2}), \quad \text{for all } q \in [2, 6]. \tag{3.19}$$

Proof. Inequality (3.17) with $q = 2$ follows from Lemma 3.2. Consider the case when $q = 6$. We recall the anisotropic Ladyzhenskaya's inequality in [9]. Let $\Omega = \Pi_{i=1}^3(a_i, b_i)$, there exists an absolute constant C_0 such that for all $u \in H^1(\Omega)$

$$\|u\|_{L^6(\Omega)} \le C_0\Pi_{i=1}^3\Big(\frac{1}{b_i - a_i}\|u\|_{L^2(\Omega)} + \|\frac{\partial u}{\partial x_i}\|_{(L^2(\Omega))^3}\Big)^{\frac{1}{3}}. \tag{3.20}$$

Apply (3.20) to w, we obtain

$$\begin{aligned}\|w\|_{L^6(\Omega_\varepsilon)} &\le C_1\Big(\frac{1}{\varepsilon}\|w\|_{L^2(\Omega_\varepsilon)} + \|\frac{\partial w}{\partial x_3}\|_{(L^2(\Omega_\varepsilon))^3}\Big)^{\frac{1}{3}}\Pi_{j=1}^2\Big(\|w\|_{L^2(\Omega_\varepsilon)} + \|\frac{\partial w}{\partial x_j}\|_{(L^2(\Omega_\varepsilon))^3}\Big)^{\frac{1}{3}} \\ &\le C_1\|\nabla w\|_{L^2}, \quad \forall u \in V_\varepsilon,\end{aligned}$$

where we have used Lemma 3.1 and Lemma 3.2, C_1 is independent of ε. By interpolation between $L^2(\Omega_\varepsilon)$

and $L^6(\Omega_\varepsilon)$, we obtain the inequality (3.17). The inequality (3.18) is an obvious consequence of (3.17).

Using (3.18), we write,for $i, j \in \{1, 2, 3\}$,

$$\begin{aligned}\|\partial_i w_j\|_{L^q} &\le C(\|M\partial_i w_j\|_{L^q} + \|(I - M)\partial_i w_j\|_{L^q}) \\ &\le C(\|M\partial_i w_j\|_{L^q} + \varepsilon^{\frac{3}{q}-\frac{1}{2}}\|w\|_{H^2}).\end{aligned} \tag{3.21}$$

By the two-dimensional Gagliarddo-Nirenberg inequality and Lemma 3.5 and (3.5) of Theorem 3.7, we obtain that

$$\begin{aligned}\|M\partial_i w_j\|_{L^q} &\le C\varepsilon^{\frac{1}{q}}\|M\partial_i w_j\|_{L^2(\Omega)}^{\frac{2}{q}}\|M\partial_i w_j\|_{H^1(\Omega)}^{1-\frac{2}{q}}\\ &\le C\varepsilon^{\frac{1}{q}-\frac{1}{2}}\|M\partial_i w_j\|_{L^2(\Omega_\varepsilon)}^{\frac{2}{q}}\|M\partial_i w_j\|_{H^1(\Omega_\varepsilon)}^{1-\frac{2}{q}}\\ &\le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}(\|w\|_{H^2}+\|u\|_{L^2}).\end{aligned}\tag{3.22}$$

The inequalities (3.21) and (3.22) imply the inequality (3.19).

Lemma 3.11. (Agmon's inequality) There exist positive constants ε_0 and C (independent of ε), such that for all $\varepsilon\in(0,\varepsilon_0]$, the following inequality holds for $w=(I-M_\varepsilon)u$,where $u\in H^2(\Omega_\varepsilon)^3\cap V_\varepsilon$,

$$\|w\|_{L^\infty}\le C\varepsilon^{\frac{1}{2}}\|w\|_{H^2}+C\varepsilon^{\frac{1}{2}}\|u\|_{L^2}.\tag{3.23}$$

Proof. We recall the Agmon's inequality from the anisotropic Agmon's inequality [9]. Let $\Omega=\Pi_{i=1}^3(a_i,b_i)$, there exists an absolute constant C_0 such that for all $u\in H^2(\Omega)$

$$\|u\|_{L^\infty(\Omega)}\le C_0\|u\|_{L^2(\Omega)}^{\frac{1}{4}}\Pi_{i=1}^3\Big(\frac{1}{(b_i-a_i)^2}\|u\|_{L^2(\Omega)}+\frac{1}{b_i-a_i}\|\frac{\partial u}{\partial x_i}\|_{L^2(\Omega)}+\|\frac{\partial^2 u}{\partial x_i^2}\|_{L^2(\Omega)}\Big)^{\frac{1}{4}}.\tag{3.24}$$

Apply (3.7) to $w_i,\ i=1,2,3$,

$$\begin{aligned}\|w_i\|_{L^\infty} &\le C_1\|w_i\|_{L^2(\Omega_\varepsilon)}^{\frac{1}{4}}\Big(\frac{1}{\varepsilon^2}\|w_i\|_{L^2(\Omega_\varepsilon)}+\frac{1}{\varepsilon}\|\frac{\partial w_i}{\partial x_3}\|_{L^2(\Omega_\varepsilon)}+\|\frac{\partial^2 w_i}{\partial x_3^2}\|_{L^2(\Omega_\varepsilon)}\Big)^{\frac{1}{4}}\\ &\times \Pi_{j=1}^2\Big(\|w_i\|_{L^2(\Omega_\varepsilon)}+\|\frac{\partial w_i}{\partial x_j}\|_{L^2(\Omega_\varepsilon)}+\|\frac{\partial^2 w_i}{\partial x_j^2}\|_{L^2(\Omega_\varepsilon)}\Big)^{\frac{1}{4}}\\ &\le C_1\varepsilon^{\frac{1}{2}}\|w\|_{H^2}+C\varepsilon^{\frac{1}{2}}\|u\|_{L^2},\end{aligned}\tag{3.25}$$

where we have used Lemma 3.2 and Theorem 3.7, C_1 is independent of ε. The proof of Lemma is completed.

Lemma 3.12. There exist positive constants ε_0 and C (independent of ε), such that for all $\varepsilon\in(0,\varepsilon_0]$, the following estimate holds for any $v=M_\varepsilon u$, where $u\in H^2(\Omega_\varepsilon)^3\cap V_\varepsilon$,

$$\|v\|_{L^4}\le C\varepsilon^{-\frac{1}{4}}\|v\|_{L^2}^{\frac{1}{2}}\|v\|_{H^1}^{\frac{1}{2}},\quad\text{and}\quad \|\nabla v\|_{L^4}\le C\varepsilon^{-\frac{1}{4}}\|v\|_{H^1}^{\frac{1}{2}}\|v\|_{H^2}^{\frac{1}{2}}.\tag{3.26}$$

Proof. By the two-dimensional Gagliardo-Nirenberg inequality we infer that

$$\|v\|_{L^4(\Omega_\varepsilon)}=\varepsilon^{\frac{1}{4}}\|v\|_{L^4(\Omega)}\le C\varepsilon^{\frac{1}{4}}\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(\Omega)}^{\frac{1}{2}}=C\varepsilon^{-\frac{1}{4}}\|v\|_{L^2(\Omega_\varepsilon)}^{\frac{1}{2}}\|v\|_{H^1(\Omega_\varepsilon)}^{\frac{1}{2}}.$$

The proof of the second inequality of (3.26) follows in the same way.

4 Estimates for trilinear form:

In this section,we will deduce a "good" estimate of the trilinear term $|\int_{\Omega_\varepsilon}(u\cdot\nabla u)\cdot\triangle u dx|$ by decomposing the vector u into $v+w$ and by using the smallness properties of w as well as the fact that v depends only on the horizontal variable $x^{'}$. we start with the following simple result.

Lemma 4.1 There exist positive constants ε_0 and C such that,for all $\varepsilon\in(0,\varepsilon_0]$, for any $U\in L^2(\Omega_\varepsilon)^3$, any $U^*\in H^1(\Omega_\varepsilon)^3$, and any $w=(I-M_\varepsilon)u$ with $u\in D(A_\varepsilon)$,one has

$$|\int_{\Omega_\varepsilon}(w\cdot\nabla U^*)Udx|\leq C\varepsilon^{\frac{1}{2}}\|U^*\|_{H^1}\|w\|_{H^2}\|U\|_{L^2}+C\varepsilon^{\frac{1}{2}}\|U^*\|_{H^1}\|u\|_{L^2}\|\|U\|_{L^2},\quad \text{for all } \varepsilon\in(0,\varepsilon_0]. \tag{4.1}$$

Proof. Applying the Hölder inequality and Lemma 3.11, we deduce that

$$|\int_{\Omega_\varepsilon}(w\cdot\nabla U^*)Udx|\leq\|w\|_{L^\infty}\|\nabla U^*\|_{L^2}\|U\|_{L^2}\leq C\varepsilon^{\frac{1}{2}}\|U^*\|_{H^1}\|w\|_{H^2}\|\|U\|_{L^2}+C\varepsilon^{\frac{1}{2}}\|U^*\|_{H^1}\|u\|_{L^2}\|\|U\|_{L^2},$$

which proves the inequality (4.1).

First, we estimate the term $|\int_{\Omega_\varepsilon}(v\cdot\nabla u)\cdot\triangle u dx|$. Integrating this term by parts, we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon}(v\cdot\nabla u)\cdot\triangle u dx = -\sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}v_i\partial_i\partial_k u_j\partial_k u_j dx \;-\; \sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i u_j\partial_k u_j dx\\ +\sum_{i,j=1}^{3}\int_{\Gamma_t\cup\Gamma_b}v_i\partial_i u_j\partial_3 u_j N_3 ds_x.\end{aligned} \tag{4.2}$$

We claim that the first term in the right hand side of (4.2) vanish. Indeed, integrating by parts, using the facts that $v\cdot N=0$ on $\partial\Omega_\varepsilon$ and that the divergence of v vanishes in Ω_ε, we obtain that

$$\sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i u_j\partial_k u_j dx=-\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_i v_i(\partial_k u_j)^2dx=0.$$

Thus (4.2) reduces to the equality

$$\int_{\Omega_\varepsilon}(v\cdot\nabla u)\cdot\triangle u dx=-\sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i u_j\partial_k u_j dx+\sum_{i,j=1}^{3}\int_{\Gamma_t\cup\Gamma_b}v_i\partial_i u_j\partial_3 u_j N_3 ds_x. \tag{4.3}$$

We now estimate each term in the right hand side of (4.3) as follows.

$$\begin{aligned}\sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i u_j\partial_k u_j dx &= \sum_{i,j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i(v_j+w_j)\partial_k(v_j+w_j)dx\\ &= \sum_{i,j,k=1}^{2}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i v_j\partial_k v_j dx+\sum_{i,j,k=1}^{2}\int_{\Omega_\varepsilon}\partial_k v_i(\partial_i v_j\partial_k w_j+\partial_i w_j\partial_k v_j)dx\\ &+\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i w_j\partial_k w_j dx,\end{aligned} \tag{4.4}$$

where we have used $v_3 = 0$. We claim that the first term in the right hand side of (4.4) vanishes. Indeed, using $\mathrm{div} v = \partial_1 v_1 + \partial_2 v_2 = 0$, we easy obtain that

$$\begin{aligned}\sum_{i,j,k=1}^{2}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i v_j\partial_k v_j dx &= \int_{\Omega_\varepsilon}[(\partial_1 v_1)^3+\partial_1 v_1(\partial_1 v_2)^2+\partial_1 v_2\partial_2 v_1\partial_1 v_1+(\partial_1 v_2)^2\partial_2 v_2\\ &+ (\partial_2 v_2)^3+\partial_2 v_2(\partial_2 v_1)^2+\partial_2 v_1\partial_1 v_2\partial_2 v_2+(\partial_2 v_1)^2\partial_1 v_1]dx=0. \quad (4.5)\end{aligned}$$

By using Theorem 3.7 and Lemma 3.12 and Hölder inequality, we find

$$\begin{aligned}|\sum_{i,j,k=1}^{2}\int_{\Omega_\varepsilon}\partial_k v_i(\partial_i v_j\partial_k w_j+\partial_i w_j\partial_k v_j)dx| &\le C\|\nabla v\|^2_{L^4(\Omega_\varepsilon)}\|\nabla w\|_{L^2}\\ &\le C\varepsilon^{\frac12}\|v\|_{H^1}\|v\|_{H^2}(\|w\|_{H^2}+\|u\|_{L^2}). \quad (4.6)\end{aligned}$$

By using (3.19)(with $q = 4$) and Hölder inequality, we obtain that

$$\begin{aligned}|\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\Omega_\varepsilon}\partial_k v_i\partial_i w_j\partial_k w_j dx| &\le C\|v\|_{H^1}\|\nabla w\|^2_{L^4}\\ &\le C\varepsilon^{\frac12}\|v\|_{H^1}(\|w\|^2_{H^2}+\|u\|^2_{L^2}). \quad (4.7)\end{aligned}$$

Next, we estimate term $|\sum_{i,j=1}^{3}\int_{\Gamma_t\cup\Gamma_b}v_i\partial_i u_j\partial_3 u_j N_3 ds_x|$. By using $\partial_i u_3|_{\Gamma_t\cup\Gamma_b}=0$, for $i = 1, 2$ and the boundary conditions (2.4), we find

$$|\sum_{i,j=1}^{3}\int_{\Gamma_t\cup\Gamma_b}v_i\partial_i u_j\partial_3 u_j N_3 ds_x| \le |\sum_{i,j=1}^{2}\int_{\Gamma_t}v_i\partial_i u_j k^{-1}(x')u_j dx'|+|\sum_{i,j=1}^{2}\int_{\Gamma_b}v_i\partial_i u_j k^{-1}(x')u_j dx'|. \quad (4.8)$$

We now only estimate the first term in the right side of (4.8), the estimate of the second term follows in the same way.

$$\begin{aligned}\int_{\Gamma_t}v_i\partial_i u_j k^{-1}(x')u_j dx' &= \varepsilon^{-1}\int_{\Gamma_t}\int_0^{\varepsilon}v_i\partial_3(x_3\partial_i u_j u_j)k^{-1}(x')dx_3dx'\\ &= \varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x')v_i\partial_i u_j u_j dx+\varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x')x_3 v_i\partial_3\partial_i u_j u_j dx\\ &+ \varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x')x_3 v_i\partial_i u_j\partial_3 u_j dx, \quad i,j\in\{1,2\}. \quad (4.9)\end{aligned}$$

First, we estimate the first term in the right side of (4.9).

$$\begin{aligned}|\varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x')v_i\partial_i u_j u_j dx| &\le C\int_{\Omega_\varepsilon}|v_i\partial_i v_j v_j|dx+C\int_{\Omega_\varepsilon}|v_i\partial_i v_j w_j|dx\\ &+ C\int_{\Omega_\varepsilon}|v_i\partial_i w_j v_j|dx+C\int_{\Omega_\varepsilon}|v_i\partial_i w_j w_j|dx \quad (4.10)\end{aligned}$$

By using Lemma 3.12 and Hölder inequality, we find

$$\begin{aligned}\int_{\Omega_\varepsilon}|v_i\partial_i v_j v_j|dx &\le C\|v\|^2_{L^4}\|\nabla v\|_{L^2}\\ &\le C\varepsilon^{-\frac12}\|v\|_{L^2}\|v\|^2_{H^1}. \quad (4.11)\end{aligned}$$

Applying Lemma 3.2, Lemma 3.12, Lemma 3.6 and Hölder inequality, we get

$$\begin{aligned}\int_{\Omega_\varepsilon}|v_i\partial_i v_j w_j|dx &\leq \|v\|_{L^4}\|\nabla v\|_{L^4}\|w\|_{L^2}\\ &\leq C\varepsilon^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|v\|_{H^1}\|v\|_{H^2}^{\frac{1}{2}}\|w\|_{H^1}\\ &\leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}^2\|u\|_{H^2}. \end{aligned} \tag{4.12}$$

From Lemma 3.12, Lemma 3.6,(3.8) and Hölder inequality, we deduce that

$$\begin{aligned}\int_{\Omega_\varepsilon}|v_i\partial_i w_j v_j|dx &\leq \|v\|_{L^4}^2\|\partial_i w_j\|_{L^2}\\ &\leq C\varepsilon^{\frac{1}{2}}\|v\|_{L^2}\|v\|_{H^1}\|w\|_{H^2}\\ &\leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}^2\|u\|_{H^2}. \end{aligned} \tag{4.13}$$

By using Lemma 3.12 Lemma 3.6 (3.8), (3.18) and Hölder inequality, we obtain that

$$\begin{aligned}\int_{\Omega_\varepsilon}|v_i\partial_i w_j w_j|dx &\leq \|v\|_{L^4}\|\partial_i w_j\|_{L^2}\|w\|_{L^4}\\ &\leq C\varepsilon^{\frac{3}{4}}\|v\|_{L^2}^{\frac{1}{2}}\|v\|_{H^1}^{\frac{1}{2}}\|w\|_{H^2}\|w\|_{H^1}\\ &\leq C\varepsilon^{\frac{3}{4}}\|u\|_{H^1}^2\|u\|_{H^2}. \end{aligned} \tag{4.14}$$

Next,we will give the estimates of the last two terms in the right side of (4.9) respectively.Arguing as in the proof of (4.13) (4.14), we easily obtain the following estimates,

$$\begin{aligned}|\varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x^{'})x_3v_i\partial_3\partial_i u_j u_j dx| &\leq C\varepsilon\int_{\Omega_\varepsilon}|v_i\partial_3\partial_i u_j v_j|dx + C\varepsilon\int_{\Omega\varepsilon}|v_i\partial_3\partial_i u_j w_j|dx\\ &\leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}^2\|u\|_{H^2}. \end{aligned} \tag{4.15}$$

$$\begin{aligned}|\varepsilon^{-1}\int_{\Omega_\varepsilon}k^{-1}(x^{'})x_3v_i\partial_i u_j\partial_3 u_j dx| &\leq C\varepsilon\int_{\Omega_\varepsilon}|v_i\partial_i v_j\partial_3 u_j|dx + C\varepsilon\int\Omega_\varepsilon|v_i\partial_i w_j\partial_3 u_j|dx\\ &\leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}^2\|u\|_{H^2}. \end{aligned} \tag{4.16}$$

In the next theorem ,we summarize all the estimates that we just have performed. Combining (4.3)-(4.16) and Lemma 3.6, we deduce the following result.

Theorem 4.2. There exist positive constants ε_0 and C (independent of ε), such that, for all $\varepsilon \in (0,\varepsilon_0]$, for any $u \in D(A_\varepsilon)$, with $v = M_\varepsilon u$, we have,

$$|\int_{\Omega_\varepsilon}(v\cdot\nabla u)\triangle u dx| \leq C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}(\|u\|_{H^2}+\varepsilon^{-1}\|u\|_{L^2}).$$

5 Global existence of strong solution:

We are now able to establish the global existence of the strong solutions to problem (2.1)-(2.4) for a large set of initial data and forcing terms.

Theorem 5.1. There exist positive constants $\varepsilon_0, k_0, k_1, K_0, K_1$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, for any forcing term $f \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$, and for any initial datum $u_0 \in V_\varepsilon$, that satisfy

$$\begin{aligned} \|u_0\|_{H^1} \leq k_0\varepsilon^{-\frac{1}{2}}, \|M_\varepsilon u_0\|_{L^2} \leq k_1, \\ \|f\|_\infty \leq K_0\varepsilon^{-\frac{1}{2}}, \|M_\varepsilon f\|_\infty \leq K_1, \end{aligned} \tag{5.1}$$

the Navier-Stokes equations (2.1)-(2.4) have unique global strong solution $u(t)$ with

$$u \in C^0([0,\ \infty); V_\varepsilon) \cap L^\infty(0,\infty; V_\varepsilon) \cap L^2_{loc}(0,T; H^2(\Omega_\varepsilon))$$

and

$$\|u(t)\|_{H^1} \leq C^*\varepsilon^{-\frac{1}{2}}, \quad \text{for all } t \geq 0. \tag{5.2}$$

Remark 5.1. Since $f = \mathbb{P}_\varepsilon f + \nabla q$ and that we can replace ∇p by $\nabla p + \nabla q$ in the equation (2.1), we may assume without loss of generality that $f = \mathbb{P}_\varepsilon f$.

Proof. Let $u = u(t) \in C^0(0, T(u_0, f)); V_\varepsilon$ be the strong solutions of (2.1)-(2.4) which is assumed to exist on some maximal interval $[0T(u_0, f))$. The existence of such an interval for a given $u_0 \in V_\varepsilon$ can be proved as in the classical case of the Navier-Stokes equations , such the strong solution is unique on the time interval of existence within the class of weak solutions (see[10]). We recall that $u(t)$ also belongs to $L^2((0,T); D(A_\varepsilon))$ for any $0 < T < T(u_0, f)$. Our purpose is to show that $T(u_0, f)$ is actually equal to $+\infty$.

Taking the scalar product in $L^(\Omega_\varepsilon)$ of (2.1) and applying the Green formula , we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \nu E(u,u) = (f,u)_{L^2}. \tag{5.3}$$

Since M_ε is an orthogonal projection on $L^(\Omega_\varepsilon)^3$, one has

$$(f,u)_{L^2} = ((I - M_\varepsilon)f, (I - M_\varepsilon)u)_{L^2} + (M_\varepsilon f, M_\varepsilon u)_{L^2}.$$

Using the inequality (2.5), Lemma 3.6 , Theorem 3.7 and Hölder inequality, we deduce from (5.3) that, for $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \frac{c_0}{2}\|u\|^2_{H^1} &\leq \|w\|_{L^2}\|(I - M_\varepsilon)f\|_{L^2} + \|v\|_{L^2}\|M_\varepsilon f\|_{L^2} \\ &\leq C(\varepsilon\|u\|_{H^1}\|f\|_{L^2} + \|M_\varepsilon f\|_{L^2}\|u\|_{L^2}). \end{aligned}$$

Using the Young inequality, we infer from the above inequality that there exists positive constant C (independent of ε),such that, for $0 < \alpha \leq \frac{c_0}{4}$, and for $\varepsilon \in (0, \varepsilon_0]$,

$$\frac{d}{dt}\|u\|^2_{L^2} + \alpha\|u\|^2_{L^2} + \frac{c_0}{2}\|u\|^2_{H^1} \leq c_1(\varepsilon^2\|f\|^2_{L^2} + \|M_\varepsilon f\|^2_{L^2}). \tag{5.4}$$

Multiplying (5.4) by $e^{\alpha s}$ and integrating the result from 0 to t , we obtain, for $t \in [0, T)$,

$$\|u(t)\|_{L^2}^2 + \frac{c_0}{2}\int_0^t e^{\alpha(s-t)}\|u(s)\|_{H^1}^2 ds \le e^{-\alpha t}\|u_0\|_{L^2}^2 + c_1\alpha^{-1}(1-e^{-\alpha t})(\varepsilon^2\|f\|_\infty^2 + \|M_\varepsilon f\|_\infty^2). \quad (5.5)$$

We first point out that $A_\varepsilon^{\frac{1}{2}}u$ and $\frac{d}{dt}A_\varepsilon^{\frac{1}{2}}$ belongs to $L^2((0,T);V_\varepsilon$ and to $L^2((0,T);V_\varepsilon^*)$ respectively, and thus, by (2.7) and by Lemma 1.2 of Chapter 3 of [10], we have

$$\frac{d}{dt}E(u,u) = \frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}u\|_{L^2}^2 = 2(\frac{d}{dt}u, A_\varepsilon u)_{L^2}.$$

Taking the scalar product in $L^2(\Omega_\varepsilon)$ of (2.1) with $A_\varepsilon u = -\mathbb{P}_\varepsilon\triangle u$ and using the above equality, we find

$$\frac{1}{2}\frac{d}{dt}E(u,u) + \nu\|\mathbb{P}_\varepsilon\triangle u\|_{L^2}^2 = \int_{\Omega_\varepsilon}(u\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx - \int_{\Omega_\varepsilon} f\cdot\mathbb{P}_\varepsilon\triangle u dx. \quad (5.6)$$

Noting the decomposition

$$\int_{\Omega_\varepsilon}(u\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx = \int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx + \int_{\Omega_\varepsilon}(w\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx$$

and applying Lemma 4.1 together with the Young inequality, we obtain

$$|\int_{\Omega_\varepsilon}(u\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx| \le |\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx| + \frac{\nu}{8}\|\mathbb{P}_\varepsilon\triangle u\|_{L^2}^2 + C\varepsilon\|u\|_{H^1}^2(\|w\|_{H^2}^2 + \|u\|_{L^2}^2).$$

Therefore,(5.6) and the Young inequality imply that

$$\frac{1}{2}\frac{d}{dt}E(u,u) + \frac{3\nu}{4}\|\mathbb{P}_\varepsilon\triangle u\|_{L^2}^2 \le |\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx| + C\varepsilon\|u\|_{H^1}^2(\|w\|_{H^2}^2 + \|u\|_{L^2}^2) + C\|f\|_{L^2}^2, \quad (5.7)$$

or also, by the Theorem 3.9,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(u,u) + \frac{3\nu}{4}\|\triangle u\|_{L^2}^2 &\le |\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx| + C\varepsilon\|u\|_{H^1}^2(\|w\|_{H^2}^2 + \|u\|_{L^2}^2)\\ &+ C\varepsilon\|u\|_{H^1}^2 + C\varepsilon^3\|u\|_{H^2}^2 + C\|f\|_{L^2}^2, \quad (5.8)\end{aligned}$$

It remains to estimate the term $|\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx|$.Using the decomposition

$$\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx = \int_{\Omega_\varepsilon}(v\cdot\nabla u)(\mathbb{P}_\varepsilon - Id)\triangle u dx + \int_{\Omega_\varepsilon}(v\cdot\nabla u)\triangle u dx,$$

and applying the Lemma 3.9 and the Theorem 4.2, we find

$$\begin{aligned}|\int_{\Omega_\varepsilon}(v\cdot\nabla u)\mathbb{P}_\varepsilon\triangle u dx| &\le C\|v\cdot\nabla u\|_{L^2}(\varepsilon^{\frac{1}{2}}\|u\|_{H^1} + \varepsilon^{\frac{3}{2}}\|u\|_{H^2})\\ &+ C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}(\|u\|_{H^1} + \varepsilon^{-1}\|u\|_{L^2}). \quad (5.9)\end{aligned}$$

Applying Lemma 3.10 and 3.12, and using the interpolation inequality

$$\|v\|_{H^1} \le C\|v\|_{L^2}^{\frac{1}{2}}\|v\|_{H^2}^{\frac{1}{2}} \le C\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}},$$

, and using the decomposition $\nabla u = \nabla v + \nabla w$, we obtain

$$\begin{aligned}\|v\cdot\nabla u\|_{L^2} &\le C\|v\|_{L^4}(\|\nabla v\|_{L^4} + \varepsilon^{\frac14}\|w\|_{H^2} + \varepsilon^{\frac14}\|u\|_{L^2})\\ &\le C\varepsilon^{-\frac12}\|u\|_{L^2}\|u\|_{H^2} + C\|u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12}\|u\|_{H^2}.\end{aligned}\tag{5.10}$$

Using (5.9) and (5.10), we infer from (5.8) that, for $\varepsilon\in(0,\varepsilon_0]$,

$$\begin{aligned}\frac12\frac{d}{dt}E(u,u) + \frac{3\nu}{4}\|\triangle u\|_{L^2}^2 &\le C\varepsilon\|u\|_{H^1}^2\|u\|_{H^2}^2 + C\varepsilon\|u\|_{H^1}^2 + C\varepsilon^3\|u\|_{H^2}^2\\ &+ C\varepsilon^{\frac12}\|u\|_{H^1}\|u\|_{H^2}^2 + C\varepsilon^{-\frac12}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2} + C\|f\|_{L^2}^2,\end{aligned}\tag{5.11}$$

Using the young inequality several times,we deduce from Theorem 3.8 that there exist positive constants C_0 and C, which is independent of ε, such that,for $\varepsilon\in(0,\varepsilon_0]$,

$$\begin{aligned}\frac{d}{dt}E(u,u) + (\nu - C_0\varepsilon^{\frac12}\|u\|_{H^1} &- C_0\varepsilon\|u\|_{H^1}^2 - C_0\varepsilon)\|u\|_{H^2}^2\\ &\le C(\|u\|_{H^1}^2 + \varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2 + \|f\|_{L^2}^2).\end{aligned}\tag{5.12}$$

To prove global existence of the solution $u(t)$, we argue by contradiction. we assume that $\varepsilon_0\le\frac{\nu}{4C_0}$ and that, for $\varepsilon\in(0,\varepsilon_0]$, the initial data u_0 satisfy the following condition

$$\frac{\nu}{2} > C_0\varepsilon^{\frac12}\|u_0\|_{H^1} + C_0\varepsilon\|u_0\|_{H^1}^2 + C_0\varepsilon.\tag{5.13}$$

Next, we assume that there exists a time $T_0>0$ such that,

$$\begin{aligned}\frac{\nu}{2} &> C_0\varepsilon^{\frac12}\|u(t)\|_{H^1} + C_0\varepsilon\|u(t)\|_{H^1}^2 + C_0\varepsilon,\quad \text{for all}\ \ t\in[0,T_0),\ \text{and}\\ \frac{\nu}{2} &= C_0\varepsilon^{\frac12}\|u(T_0)\|_{H^1} + C_0\varepsilon\|u(T_0)\|_{H^1}^2 + C_0\varepsilon.\end{aligned}\tag{5.14}$$

We shall show by contradiction that $T_0=+\infty$.

Using the inequality (2.5), we deduce from (5.12) and (5.14) that, for $\varepsilon\in(0,\varepsilon_0]$ and $t\in[0,T_0]$,

$$\frac{d}{dt}E(u,u) + \alpha E(u,u) + \frac{\nu}{4}\|u\|_{H^2}^2 \le C(\|u\|_{H^1}^2 + \varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2 + \|f\|_{L^2}^2),\tag{5.15}$$

where $\alpha = \min(\frac{c_0}{4}, \frac{\nu}{4c_0^*})$.

Multiplying (5.15) by $e^{\alpha s}$, integrating the result from 0 to t, we obtain,for $t\in[0,T_0]$,

$$\begin{aligned}E(u,u) &+ \frac{\nu}{4}\int_0^t e^{\alpha(s-t)}\|u(s)\|_{H^2}^2ds \le e^{-\alpha t}E(u_0,u_0) + C\alpha^{-1}(1-e^{-\alpha t})\|f\|_\infty^2\\ &+ C\int_0^t e^{\alpha(s-t)}\|u(s)\|_{H^1}^2ds + C\varepsilon^{-1}\int_0^t e^{\alpha(s-t)}\|u(s)\|_{L^2}^2\|u(s)\|_{H^1}^2ds.\end{aligned}\tag{5.16}$$

The inequality (2.5) and (5.16) imply that, for $t\in[0,T_0]$,

$$\begin{aligned}\sup_{s\in[0,t]}\|u(s)\|_{H^1}^2 &\le C[\|u_0\|_{H^1}^2 + \|f\|_\infty^2 + C\int_0^t e^{\alpha(s-t)}\|u(s)\|_{H^1}^2ds\\ &+ C\varepsilon^{-1}\int_0^t e^{\alpha(s-t)}\|u(s)\|_{L^2}^2\|u(s)\|_{H^1}^2ds].\end{aligned}\tag{5.17}$$

The estimates (5.5)and (5.17) imply, for $t \in [0, T_0]$,

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^1}^2 \le C[\|u_0\|_{H^1}^2 + \|f\|_\infty^2 + \varepsilon^{-1}\|u_0\|_{L^2}^4 + \varepsilon^{-1}\|M_\varepsilon f\|_\infty^4 + \varepsilon^3\|f\|_\infty^4] \triangleq R_0^2(\varepsilon) \tag{5.18}$$

We remark that, due to Theorem 3.7, there exists a positive constant C such that

$$\|u_0\|_{L^2}^2 \le \|v_0\|_{L^2}^2 + C\varepsilon^2\|u_0\|_{H^1}^2. \tag{5.19}$$

Using the initial datum and (5.19), we deduce from (5.18) that,for $\varepsilon \in (0, \varepsilon_0]$,

$$R_0^2(\varepsilon) \le C_1(\varepsilon^{-1}(k_0^2 + K_0^2 + k_1^4 + K_1^4) + \varepsilon^3(k_0^4 + K_0^4)) \tag{5.20}$$

for some positive constant C_1 independent of ε. If k_0, k_1, K_0, K_1 small enough (and independent of ε) , it follows from (5.20) that,for $\varepsilon \in (0, \varepsilon_0]$,

$$C_0\varepsilon^{\frac{1}{2}}R_0(\varepsilon) + C_0\varepsilon R_0^2(\varepsilon) + C_0\varepsilon < \frac{\nu}{2}, \tag{5.21}$$

which contradicts the statement (5.14). It follows that $T_0 = +\infty$.Thus, the initial data u_0 and the forcing term $f(t)$ satisfy condition (5.1) Of Theorem 5.1 which implies that there exists a unique global solution $u(t)$ to problem (2.1)-(2.4). Moreover, we infer from (5.18), (5.20) that there exists positive constant C^*, such that, for $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_t \|u(t)\|_{H^1} \le C^*\varepsilon^{-\frac{1}{2}}, \tag{5.22}$$

which proves the theorem.

6 Local attractors

Like in [3],[4],we are going to introduce a local attractor and show that this local attractor is actually the compact global attractor of all Leray-Hopf solutions. In order to simplify the statements, we assume in this section that the forcing term f does not depend on the time variable and satisfies the conditions (5.1). We also assume that the constants given in (5.1) satisfy the condition (5.21) as well as the following additional condition

$$\frac{4c_1}{c_0}(K_0^2 + K_1^2) \le k_1^2, \tag{6.1}$$

where the constant c_1 is given in (5.5). According to Theorem 5.1, for any u_0 satisfy ing conditions (5.1) and $\varepsilon \in (0, \varepsilon_0]$, there exists a unique global strong solution $u(t) = S_\varepsilon(f;t)u_0 \in C^0([0,\ \infty); V_\varepsilon)$ of problem (2.1)-(2.4). We next define the sets

$$\begin{aligned} \mathcal{B}_{0,\varepsilon} &= \{u_0 \in V_\varepsilon : \|u_0\|_{H^1} \le k_0\varepsilon^{-\frac{1}{2}}; \|M_\varepsilon u_0\|_{L^2} \le k_1\}, \\ \mathcal{B}_\varepsilon &= \overline{\bigcup_{t\ge 0} S_\varepsilon(f,t)\mathcal{B}_{0,\varepsilon}}^{V_\varepsilon}. \end{aligned} \tag{6.2}$$

Duo to Theorem 5.1, the set $\mathcal{B}_\varepsilon$ is bounded in V_ε and is positively invariant under $S_\varepsilon(f,t)$,for $t \geq 0$. As in the case of the Navier-Stokes equations with classical boundary conditions,one shows that, for any $u_0 \in \mathcal{B}_\varepsilon, S_\varepsilon(f,t)u_0$ belongs to $C^0((0, \ \infty); D(A_\varepsilon))$. Since $D(A_\varepsilon)$ is compactly embed in V_ε,this means that, for $t > 0$,the mapping $S_\varepsilon(f,t)$ is compact from V_ε into itself.Thus, the $\omega-$limit set of $\mathcal{B}_\varepsilon$,

$$\mathcal{A}_\varepsilon = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S_\varepsilon(f,t)\mathcal{B}_{0,\varepsilon}}^{V_\varepsilon}$$

is well-define and non-empty,compact set and attracts $\mathcal{B}_\varepsilon$.The set $\mathcal{A}_\varepsilon$ is the compact global attractor of the restriction of $S_\varepsilon(f,t)$ to $\mathcal{B}_\varepsilon$. In fact, it is also a local attractor in V_ε and its basin of attraction contains $\mathcal{B}_\varepsilon$.

We next show, like in [3],[4],that $\mathcal{A}_\varepsilon$ is the global attractor of the weak Leray-Hopf solutions of (2.1)-(2.4).

We recall that $C^0_\omega([0,T]; H_\varepsilon)$ is a subspace of $L^\infty((0,T); H_\varepsilon)$ consisting of all functions which are weakly continuous,that is, for each $h \in H_\varepsilon$, the mapping $t \ \rightarrow \ (u(t), h)$ is continuous. In particular, the relation $u(0) = u_0$ is understood in this sense.

We recall that by a weak Leray-Hopf solution on the time interval $[0,T]$, we mean a function $u(\cdot) \in L^2((0, \ T); V_\varepsilon) \cap L^\infty((0,T); H_\varepsilon) \cap C^0_\omega([0,T]; H_\varepsilon)$ with $\partial_t u \in L^1((0,T); V^*_\varepsilon)$, such that $u(0) = u_0$ holds in the weak sense, the equation

$$(u(t) - u(t_0), u^*)_{L^2} + \int_{t_0}^{t} E(u,u^*)ds + \int_{t_0}^{t}(\sum_{j=1}^{3} u_j\partial_j u, u^*)_{L^2}ds = \int_{t_0}^{t}(f,u^*)ds, \tag{6.3}$$

is satisfied,for all $t \geq t_0 \geq 0$ and $u^* \in V_\varepsilon$, and the energy inequality

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_{t_0}^{t} E(u,u)ds \leq \frac{1}{2}\|u(0)\|^2_{L^2} + \int_{t_0}^{t}(f,u(s))_{L^2}ds \tag{6.4}$$

holds for all almost all t_0 with $0 < t_0 < t \leq T$ and also for $t_0 = 0$(see for example[10]).

Duo to the properties of A_ε (see section 2 for further details), by using a Galerkin method, we can prove, like in the case of the classical Navier-Stokes equations (see[10]), that (2.1)-(2.4) admit a weak Leray-Hopf global solution $u(t)$ in $[0,\infty)$, for any $u_0 \in H_\varepsilon$. We also notice that $u(t) \in V_\varepsilon$ for $t \in \mathcal{F}^T_u$,where $\mathcal{F}^T_u \subset [0,T]$ is a measurable set of full measure. In 1987 for 3D Navier-Stokes equation Foias and Temam [18] introduced the set J_ε consisting of all weak Leray-hopf solutions existing in $(-\infty, +\infty)$ and bounded in $L^\infty((-\infty,+\infty); H_\varepsilon)$. This set is not empty since it contains $\mathcal{A}_\varepsilon$. Foias and Temam [18] also showed that this set is compact in H^{weak}_ε and that,for any weak Leray-Hopf solution $u(t)$ in $(0,+\infty), u(t) \ \rightarrow \ J_\varepsilon$ in H^{weak}_ε as $t \ \rightarrow \ +\infty$. We next show that $J_\varepsilon = \mathcal{A}_\varepsilon$,for $0 < \varepsilon \leq \varepsilon_1$,where $\varepsilon_1 > 0$ is small enough.We use the same arguments as in [13,Theorem 3.12,Chapter 3],[3],[4].

Theorem 6.1. Assume that the conditions (6.1) and (5.1) hold, and that $f \in L^2(\Omega_\varepsilon)^3$ satisfies (5.1). Then, there exist a positive constant $\varepsilon_1 \leq \varepsilon_0$ and, for any $r > 0$,for any $\varepsilon \in (0, \varepsilon_1]$, a time $T(\varepsilon,r)\geq 0$ such that, for any weak Leray-Hopf solution $u(t)$ of (2.1)-(2.4),with $\|u(0)\|_{L^2}\leq r$, there is a positive time t_1, $0<t_1\leq T(\varepsilon,r)$ so that $u(t)\in\mathcal{B}_\varepsilon$ for $t\geq t_1$. In particular, $J_\varepsilon=\mathcal{A}_\varepsilon$.

Proof. Let $u(t)$ be weak Leray-Hopf solution of (2.1)-(2.4), Arguing as in section 5 (5.4), we deduce from (6.4) and (2.5) that, for $t\geq 0$, for $\varepsilon\in(0,\varepsilon_1]$,

$$\frac{1}{t}\int_0^t\|u(s)\|_{H^1}^2ds\leq\frac{2}{tc_0}\|u_0\|_{L^2}^2+\frac{2c_1}{c_0}(\|M_\varepsilon f\|_{L^2}^2+\varepsilon^2\|(I-M_\varepsilon)f\|_{L^2}^2),$$

Using the conditions (5.1), we obtain that

$$\frac{1}{t}\int_0^t\|u(s)\|_{H^1}^2ds\leq\frac{3c_1}{c_0}(K_1^2+\varepsilon K_0^2),\quad\text{for all } t\geq T(\varepsilon,r),\tag{6.5}$$

where

$$T(\varepsilon,r)=\frac{2r^2}{c_1(K_1^2+\varepsilon K_0^2)}.$$

The estimate (6.5) and the condition (6.1) imply that,

$$\frac{1}{t}\int_0^t\|u(s)\|_{H^1}^2ds\leq\frac{3c_1}{c_0}(K_1^2+\varepsilon K_0^2)<k_1^2,\quad\text{for all } t\geq T(\varepsilon,r),$$

We can choose $\varepsilon_1>0$, with $\varepsilon_1\leq\varepsilon_0$ such that $\frac{3c_1}{c_0}(K_1^2+\varepsilon_1K_0^2)<k_0^2\varepsilon_1^{-1}$. Thus, we have, for $\varepsilon\in(0,\varepsilon_1]$,

$$\frac{1}{t}\int_0^t\|u(s)\|_{H^1}^2ds<\min(k_0^2\varepsilon^{-1},k_1^2),\quad\text{for all } t\geq T(\varepsilon,r).$$

Therefore, there exists a subset $\mathcal{F}_0\subset[0,T(\varepsilon,r)]$ of positive measure such that

$$\|u(t)\|_{H^1}^2\leq\min(k_0^2\varepsilon^{-1},k_1^2),\quad\text{for all } t\geq T(\varepsilon,r).$$

Since $\mathcal{F}_u^{T(\varepsilon,r)}$ is a set of full measure in $[0,T(\varepsilon,r)]$, we have that $\mathcal{F}_u^{T(\varepsilon,r)}\cap\mathcal{F}_0\neq\emptyset$. Therefore, there exists $t_1\in\mathcal{F}_u^{T(\varepsilon,r)}\cap\mathcal{F}_0\subset[0,T(\varepsilon,r)]$ such that

$$\|u(t_1)\|_{H^1}\leq k_0\varepsilon^{-\frac{1}{2}},\quad\|M_\varepsilon u(t_1)\|_{L^2}\leq k_1.$$

We deduce now from Theorem 5.1 and from uniqueness of strong solution of (2.1)-(2.4) and that $u(t)\in\mathcal{B}_\varepsilon$, for $t\geq t_1$. As a direct consequence, we obtain the equality $J_\varepsilon=\mathcal{A}_\varepsilon$. This completes the proof of Theorem 6.1.

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