Enlargement of the Transformation Group of General Relativity: Spherically Symmetric Solutions

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Abstract

A theory has been presented previously in which the geometrical structure of a real four-dimensional space time manifold is expressed by a real orthonormal tetrad, and the group of diffeomorphisms is enlarged. Field equations were obtained from a variational principle which is invariant under the larger group. In this paper a suitable Lagrangian for a field with sources is presented and spherically symmetric solutions for both the free field and the field with sources are given. A stellar model and an external, free-field model are developed. The resulting models are compared to the internal and external Schwarzschild models. The theory implies that the external stress-energy tensor has non-compact support and hence may give the geometrical foundation for dark matter.

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1 Introduction

Let $\mathcal{X}^4$ be a 4-dimensional space with metric $g_{\mu\nu}$ and with orthonormal tetrad $h^i{}_\mu$. Thus $g_{\mu\nu} = \eta_{ij}\, h^i{}_\mu h^j{}_\nu$ where $\eta_{ij} = \mathrm{diag}\{-1, 1, 1, 1\}$. Let $\tilde{V}^\alpha$ be a vector density of weight $+1$. Then a conservation law of the form $\tilde{V}^\alpha_{;\alpha} = 0$ is invariant under all transformations satisfying

$$x^\nu{}_{,\overline{\alpha}}\big(x^{\overline{\alpha}}{}_{,\nu,\mu} - x^{\overline{\alpha}}{}_{,\mu,\nu}\big) = 0 \quad . \tag{1}$$

This property defines the group of conservative transformations of which the group of diffeomorphisms is a proper subgroup. Although we may view the space as a Riemannian manifold, the space is more general than a manifold [1]. In the Riemannian manifold interpretation we regard $x^{\bar{\mu}}$ as anholonomic when $x^{\bar{\alpha}}{}_{,\mu}$ is non-diffeomorphic [4]. The geometrical content of the theory is determined by the vector $C_\alpha \equiv h_i{}^\nu\big(h^i{}_{\alpha,\nu} - h^i{}_{\nu,\alpha}\big) = \gamma^\mu{}_{\alpha\mu}$, where the Ricci rotation coefficient is given by $\gamma^i{}_{\mu\nu} = h^i{}_{\mu;\nu}$ [1,2]. Pandres calls this the curvature vector. He shows that C_α is invariant under transformations from x^μ to $x^{\overline{\mu}}$ if and only if the transformation is conservative and thus satisfies (1). A suitable scalar Lagrangian for the free field is given by

$$\mathcal{L}_f = -\frac{1}{16\pi}\int C^\alpha C_\alpha\, h\, d^4x \tag{2}$$

where $h = \sqrt{-g}$ is the determinant of the tetrad. Using $h^i_\mu = h^I_\mu \Lambda^i_I$, we have extended the field variables [3] to include the tetrad $h^I{}_\mu$ and 4 internal vectors Λ^i_I, with internal space variable x^I. We assume that the metric on the x^I space is also $\eta_{IJ} = \mathrm{diag}\big(-1, 1, 1, 1\big)$. The definition of the Ricci rotation coefficient is also extended using the Λ^i_I to

$$\Upsilon^\alpha{}_{\mu\nu} \equiv h_I{}^\alpha h^I{}_{\mu;\nu} + h_i{}^\alpha h^I{}_\mu \Lambda^i_{I,\nu} \tag{3}$$

and the definition of C_α is also extended to $C_\alpha \equiv \Upsilon^\mu{}_{\alpha\mu}$. Using the extended Ricci rotation coefficients, one finds that

$$C^\alpha C_\alpha = R + \Upsilon^{\alpha\beta\nu}\gamma_{\alpha\nu\beta} - 2C^\alpha_{;\alpha} - \eta^{ij} h_j{}^\nu h_I{}^\alpha (\Lambda^I_{i,\alpha,\nu} - \Lambda^I_{i,\nu,\alpha}) \quad , \tag{4}$$

where R is the usual Ricci scalar curvature. Thus, when the physical space is interpreted as a manifold, the free field exhibits terms suggestive of non-gravitational interactions [2,4]. Setting the variations of $\mathcal{L}_f$ with respect to $h^I{}_\mu$ and Λ^i_I equal to zero along with the assumption that we may always choose Λ^i_I to correspond to a complex Lorentz transformation (since $h^i{}_\mu = h^I{}_\mu \Lambda^i_I$), yields the field equations

$$C_\mu = 0 \quad . \tag{5}$$

Henceforth in this paper we will assume that we are working with a solution of the field equations for which the curl of the Λ^I_i is zero. In this case, an identity for the

Einstein tensor is

$$
\begin{aligned}
G_{\mu\nu} =& C_{\mu;\nu} - C_\alpha \Upsilon^\alpha_{\ \mu\nu} - g_{\mu\nu} C^\alpha_{\ ;\alpha} - \frac{1}{2} g_{\mu\nu} C^\alpha C_\alpha \\
& + \Upsilon_{\mu\ \ \nu;\alpha}^{\ \alpha} + \Upsilon^\alpha_{\ \sigma\nu} \Upsilon^\sigma_{\ \mu\alpha} + \tfrac{1}{2} g_{\mu\nu} \Upsilon^{\alpha\beta\sigma} \Upsilon_{\alpha\sigma\beta} \quad .
\end{aligned}
$$

This expression is not manifestly symmetric in μ and ν, but the left-hand side is symmetric in its lower indices and hence the right-hand side must be as well. Thus we use a symmetrized expression to ensure this. Define for general $K_{\mu\nu}$, the symmetrized tensor by $K_{(\mu\nu)} = \frac{1}{2}(K_{\mu\nu} + K_{\nu\mu})$. Using (5) we see that the field equations may be also expressed in the form

$$
G_{\mu\nu} = \Upsilon_{(\mu\ \ \nu);\alpha}^{\ \ \alpha} + \Upsilon^\alpha_{\ \sigma(\nu} \Upsilon^\sigma_{\ \mu)\alpha} + \tfrac{1}{2} g_{\mu\nu} \Upsilon^{\alpha\beta\sigma} \Upsilon_{\alpha\sigma\beta} \quad \equiv 8\pi \big(T_f\big)_{\mu\nu} \tag{6}
$$

with free field stress energy tensor $\mathbf{T}_f$. The terms of $\mathbf{T}_f$ suggest that, when interpreted in Riemannian geometry, this new geometry produces a stress energy tensor with additional terms that could be the stress energy tensor for dark matter or dark energy [4].

One feature of the extended theory with field variables $h^I_{\ \mu}$ and Λ^i_I is that the internal fields associated with Λ^i_I may be specified after finding a tetrad $h^I_{\ \alpha}$ which satisfies the condition $h_I^{\ \nu}\big(h^I_{\ \mu,\nu} - h^I_{\ \nu,\mu}\big) = 0$. The tetrad $h^I_{\ \alpha}$ determines the gravitational field via $g_{\mu\nu}$ and changes in Λ^i_I have no effect on the corresponding Riemannian manifold [3]. If Λ^i_I is a constant field such that $\eta_{ij} = \eta_{IJ}\Lambda^I_i \Lambda^J_j = \mathrm{diag}\big(-1,1,1,1\big)$, then the field equations are satisfied, however Λ^i_I may be non-constant. There exist nonconstant (non-diffeomorphic) values of Λ^i_I that satisfy the conservative condition, $\Lambda^J_k\big(\Lambda^k_{I,J} - \Lambda^k_{J,I}\big) = 0$. With Λ^i_I that satisfy this condition, the field equations, $C_\mu = 0$ remain satisfied.

The motion of a free particle in the inertial coordinate system is given by

$$
\frac{d^2x^i}{ds^2} = 0\,, \tag{7}
$$

where $-ds^2 = \eta_{ij} dx^i dx^j$. This equation when transformed to internal coordinates, x^I is

$$
\frac{d^2x^I}{ds^2} = -\Lambda^I_i \Lambda^i_{J,K} \frac{dx^J}{ds}\frac{dx^K}{ds}\ , \tag{8}
$$

where the right hand side of this equation is zero when there are no internal forces. We interpret the Λ^i_I as the internal fields that via $\Lambda^i_{I,J}$ correspond to electroweak and strong interactions. In the manifold view, with coordinates x^α equation (7) becomes

$$
\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\Upsilon^\alpha_{\ \mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\ . \tag{9}
$$

From (3) one sees that this equation of motion depends on $\Lambda^i_{I,\nu}$.

In the presence of sources the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_s = \int \left(-\frac{1}{16\pi} C^\alpha C_\alpha + L_s \right) h \, d^4x \tag{10}$$

where $L_s = L_s(x^\alpha)$ is the appropriate Lagrangian density function for the source. In this case C_α is nonzero and variation of (10) with respect to the tetrad results in

$$\int \left[-\frac{1}{16\pi} \left(C_{(\mu;\nu)} - C_\alpha \Upsilon^\alpha_{\ (\mu\nu)} - \frac{1}{2} g_{\mu\nu} C^\alpha C_\alpha + - g_{\mu\nu} C^\alpha_{\ ;\alpha} \right) + \frac{1}{2} (T_s)_{\mu\nu} \right] h \, h^{i\nu} \delta h_i^{\ \mu} \, d^4x = 0$$

Here, $(T_s)_{\mu\nu}$ is the usual stress-energy tensor of the source the standard theory [5]. Thus

$$C_{(\mu;\nu)} - C_\alpha \Upsilon^\alpha_{\ (\mu\nu)} - \frac{1}{2} g_{\mu\nu} C^\alpha C_\alpha + - g_{\mu\nu} C^\alpha_{\ ;\alpha} = 8\pi (T_s)_{\mu\nu} \tag{11}$$

and also we have the following identity for the Einstein tensor,

$$G_{\mu\nu} = \left(\Upsilon_{(\mu\ \ \nu);\alpha}^{\ \ \alpha} + \Upsilon^\alpha_{\ \sigma(\nu} \Upsilon^\sigma_{\ \ \mu)\alpha} + \tfrac{1}{2} g_{\mu\nu} \Upsilon^{\alpha\beta\sigma} \Upsilon_{\alpha\sigma\beta} \right) + 8\pi (T_s)_{\mu\nu} \tag{12}$$

or

$$G_{\mu\nu} = 8\pi \big(T_f \big)_{\mu\nu} + 8\pi \big(T_s \big)_{\mu\nu} \quad . \tag{13}$$

We call $\mathbf{T}_f$ the free field stress energy and $\mathbf{T}_s$ the stress energy for the source.

2 Spherically symmetric solutions.

A. Free Field Case. We now exhibit spherically symmetric solutions of the field equations for a free field (5). Let $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. If $f(r)$ is a positive differentiable function of r, then the tetrad field given by

$$h^i_{\ \mu} = \delta^i_0 \delta^0_\mu \sqrt{f(r)} + \frac{1}{\sqrt[4]{f(r)}} (\delta^i_1 \delta^1_\mu + \delta^i_2 \delta^2_\mu + \delta^i_3 \delta^3_\mu) \tag{14}$$

yields $C_\mu = 0$ and hence is a solution of the field equations (5). The line element (metric) in spherical coordinates is given by

$$ds^2 = -f(r) dt^2 + \frac{1}{\sqrt{f(r)}} dr^2 + \frac{r^2}{\sqrt{f(r)}} d\theta^2 + \frac{r^2 \sin^2\theta}{\sqrt{f(r)}} d\phi^2 \quad . \tag{15}$$

Now change the radial coordinate $r \to \overline{r}$ so that $\overline{r}^2 = \frac{r^2}{\sqrt{f(r)}}$ and $f(r) = e^{2\Phi(\overline{r})}$. Since these are differentiable functions, this change of coordinates $(t, r, \theta, \phi) \to (t, \overline{r}, \theta, \phi)$ is a diffeomorphism and hence the field equations remain satisfied. The mapping $r \to \overline{r}$

is the simply the inverse of the function $r = r(\overline{r}) = \overline{r}e^{\frac{1}{2}\Phi(\overline{r})}$. After this change in the radial coordinate r, we will now rename $\overline{r}$ as simply r. The tetrad in spherical coordinates may be expressed by

$$h^i{}_\mu = \begin{bmatrix} e^\Phi & 0 & 0 & 0 \\ 0 & \left(1+\frac{1}{2}r\Phi'\right)\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \left(1+\frac{1}{2}r\Phi'\right)\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \left(1+\frac{1}{2}r\Phi'\right)\cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{16}$$

where the upper index refers to the row and the prime indicates differentiation with respect to r. One finds that $C_\mu = 0$ for this tetrad. The new metric is

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1+\frac{1}{2}r\Phi'(r)\right)^2 dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad . \tag{17}$$

After a long, but straightforward calculation, one finds that the Einstein tensor equals a diagonal tensor which is in general nonzero: $G_{\mu\nu} = 8\pi\left(T_f\right)_{\mu\nu}$. The non-zero components are (with Φ representing $\Phi(r)$)

$$G_{tt} = 8\pi\left(T_f\right)_{tt} = \frac{e^{2\Phi}\left(\frac{1}{8}(r\Phi')^3 + \frac{3}{4}(r\Phi')^2 + 2r\Phi' + r^2\Phi''\right)}{r^2\left(1+\frac{1}{2}r\Phi'\right)^3} \quad , \tag{18}$$

$$G_{rr} = 8\pi\left(T_f\right)_{rr} = \frac{r\Phi' - \frac{1}{4}(r\Phi')^2}{r^2} \tag{19}$$

and

$$\frac{G_{\theta\theta}}{r^2} = \frac{G_{\phi\phi}}{r^2\sin^2\theta} = \frac{8\pi T_{\theta\theta}}{r^2} = \frac{8\pi T_{\phi\phi}}{r^2\sin^2\theta} = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2\left(1+\frac{1}{2}r\Phi'\right)^3} \quad . \tag{20}$$

One difference between this and the Schwarzschild metric [6] is that there is only one unknown function ($\Phi(r)$) instead of two (the standard $\Lambda(r)$ and $\Phi(r)$ functions).

We will first work on the G_{tt} term. One finds that

$$e^{-2\Phi(r)}G_{tt} = \frac{2}{r^2}\cdot\frac{d}{dr}\left(\frac{r}{2} - \frac{r}{2(1+\frac{1}{2}r\Phi')^2}\right) \equiv \frac{2}{r^2}w'(r) \equiv 8\pi\rho_f \quad , \tag{21}$$

where $w(r) \equiv \frac{r}{2} - \frac{r}{2(1+\frac{1}{2}r\Phi')^2}$. Hence

$$\Phi'(r) = \frac{2}{r}\left[\left(1-\frac{2w(r)}{r}\right)^{-\frac{1}{2}} - 1\right] \quad . \tag{22}$$

Thus

$$g_{rr} = \left(1+\frac{1}{2}r\Phi'\right)^2 = \left(1-\frac{2w(r)}{r}\right)^{-1} \quad , \tag{23}$$

and

$$g_{tt} = -e^{2\Phi(r)} \quad , \text{ where } \Phi(r) = \int \frac{2}{r}\left[\left(1 - \frac{2w(r)}{r}\right)^{-\frac{1}{2}} - 1\right] dr \tag{24}$$

(this defines $\Phi(r)$ up to a constant). The function $w(r)$ is related to the mass inside a ball of radius r for the free field and ρ_f represents the density of the free field in the manifold interpretation.

Let p_R represent the radial pressure of the free field. Then one finds [6] that the radial pressure of the free field is given by

$$8\pi p_R = \frac{G_{rr}}{\left(1 + \frac{1}{2} r\Phi'\right)^2} = \frac{r\Phi' - \frac{1}{4}(r\Phi')^2}{r^2\left(1 + \frac{1}{2} r\Phi'\right)^2} \tag{25}$$

and from (22) one finds that

$$8\pi p_R = \frac{4r\sqrt{1 - \frac{2w(r)}{r}} - 4r + 6w(r)}{r^3} \quad . \tag{26}$$

Let the tangential pressure of the free field be denoted by p_T. We also find that $8\pi p_T = \dfrac{G_{\theta\theta}}{r^2} = \dfrac{G_{\phi\phi}}{r^2 \sin^2\theta}$ and thus,

$$8\pi p_T = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2} r\Phi' + \frac{1}{2} r^2 \Phi''}{r^2\left(1 + \frac{1}{2} r\Phi'\right)^3} \quad . \tag{27}$$

Using (22), the tangential pressure may be expressed in terms of $w(r)$ and r by

$$8\pi p_T = \frac{8r - 9w(r) - 8r\sqrt{1 - \frac{2w(r)}{r}} + rw'(r)}{r^3} \quad . \tag{28}$$

Since $p_R \neq p_T$ there are shear stresses and we see that $\left(T_f\right)_{\mu\nu}$ does not model a perfect fluid.

B. General spherical tetrads. With $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, the general tetrad in spherical coordinates may be expressed by

$$h^i{}_\mu = \begin{bmatrix} e^{\Phi(r)} & 0 & 0 & 0 \\ 0 & e^{\Lambda(r)} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{\Lambda(r)} \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{\Lambda(r)} \cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{29}$$

where the upper index refers to the row. The curvature vector for this tetrad field is given by

$$C_\mu = \frac{e^\Lambda}{r}\left[\, 0,\ 2 - e^{-\Lambda}\left(r\Phi' + 2\right),\ 0,\ 0 \right] \tag{30}$$

where components are in the order $[t, r, \theta, \phi]$ and the prime denotes the derivative with respect to r. The tetrad (29) leads to the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad . \tag{31}$$

Comparison of metrics (17) and (31) implies that for the metric of (17), $(r\Phi'+2) = 2e^\Lambda$ which then implies that C_μ in equation (30) would be identically zero. From (30) we see that the general spherically symmetric tetrad field does not generally yield $C_\mu = 0$, hence we consider whether there exists a spherically symmetric solution of the field equations which flow from (10). The metric (31) leads to a diagonal Einstein tensor with nonzero elements:

$$G^t_{\ t} = \frac{1}{r^2}\big(-2re^{-2\Lambda}\Lambda' + e^{-2\Lambda} - 1\big) = -\frac{2}{r^2}\frac{d}{dr}\left[\tfrac{1}{2}r(1-e^{-2\Lambda})\right] \quad , \tag{32}$$

$$G^r_{\ r} = \frac{1}{r^2}\big(2re^{-2\Lambda}\Phi' + e^{-2\Lambda} - 1\big) \tag{33}$$

and

$$G^\theta_{\ \theta} = G^\phi_{\ \phi} = \frac{e^{-2\Lambda}}{r}\big(r\Phi'' + r(\Phi')^2 - r\Phi'\Lambda' + \Phi' - \Lambda'\big) \quad . \tag{34}$$

Using $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we now decompose the stress-energy tensor using (13). From $8\pi\big(T_f\big)_{\mu\nu} = \Upsilon_{\mu\ \ \nu;\alpha}^{\ \alpha} + \Upsilon^\alpha_{\ \sigma\nu}\Upsilon^\sigma_{\ \mu\alpha} + \frac{1}{2}g_{\mu\nu}\Upsilon^{\alpha\beta\sigma}\Upsilon_{\alpha\sigma\beta}$, one finds that $\mathbf{T}_f$ is diagonal with elements

$$8\pi\big(T_f\big)_{tt} = \frac{e^{2\Phi}e^{-2\Lambda}\big(r^2\Phi'' + \frac{1}{2}(r\Phi')^2 - r^2\Phi'\Lambda' + 2r\Phi' + 2e^\Lambda - e^{2\Lambda} - 1\big)}{r^2} \quad , \tag{35}$$

$$8\pi\big(T_f\big)_{rr} = \frac{1}{r^2}\left(-\frac{1}{2}(r\Phi')^2 + e^{2\Lambda} - 1\right) \qquad \text{and} \tag{36}$$

$$\frac{8\pi\big(T_f\big)_{\theta\theta}}{r} = \frac{8\pi\big(T_f\big)_{\phi\phi}}{r\sin^2\theta} = e^{-2\Lambda}\big(\frac{1}{2}r(\Phi')^2 - \Phi' + \Lambda' + e^\Lambda\Phi'\big) \ . \tag{37}$$

As indicated by (12) and (13), $\mathbf{T}_s$ is determined by variation of the L_s term in the Lagrangian (10).

3 Models for the Interior of a Star.

We will use the general spherical tetrad and the field equations which are derived from the Lagrangian (10) with $L_s = \rho(r)$, representing the density as a function of r. It is well known that this Lagrangian with appropriate thermodynamic conditions lead to the usual perfect fluid stress-energy tensor [7,8]. With a tetrad that corresponds to a stationary basis (velocity of the observer is zero if $h^0_{\ \mu} = 0$ for $\mu = 1, 2$ and 3), one finds [6]

$$T^\mu_\nu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \quad . \tag{38}$$

Using the tetrad field of (29), we require that the radial and tangential pressures of the corresponding source stress-energy tensor (11) be equal, leading to the following differential equation with primes denoting derivatives with respect to r:

$$r^2\Phi'' - \left(r^2\Lambda' + re^\Lambda\right)\Phi' = 2 - 2e^{2\Lambda} + 2r\Lambda' \tag{39}$$

After multiplying by an integrating factor and integrating, (39) implies that

$$\left(r\Phi' + 2\right)e^{-\Lambda} = 2 - \kappa r e^{\int(r^{-1}e^\Lambda)} \tag{40}$$

where κ is arbitrary. We assume that C_α has compact support and is a smooth function and hence integration of the $C^\alpha_{;\alpha}$ term over the region of support results in a value of zero and hence does not affect the overall mass of the source. If one drops this term (or incorporates it into the $\rho(r)$ function), then one finds that

$$8\pi\rho = \tfrac{1}{2}\left(\kappa e^{\int(r^{-1}e^\Lambda)}\right)^2 \tag{41}$$

and

$$8\pi p = \frac{\kappa e^{\int(r^{-1}e^\Lambda)}}{r} - \tfrac{1}{2}\left(\kappa e^{\int(r^{-1}e^\Lambda)}\right)^2 \quad . \tag{42}$$

We also note that for this internal solution that the curvature vector in the order t, r, θ, ϕ is given by

$$C_\mu = \left[0,\ \kappa e^\Lambda e^{\int(r^{-1}e^\Lambda)},\ 0,\ 0\right] \quad . \tag{43}$$

This gives $C^\mu C_\mu = \kappa^2 e^{2\int(r^{-1}e^\Lambda)}$ and as in the free field case, the field equations imply that the Lagrangian has a value of zero.

Let R represent the radius of the star and let $M = m(R)$ denote the mass-energy inside the star. The determining factor for the radius of the star is that the pressure of the gas drops to zero at the surface, i.e., $p(R) = 0$. This implies that either κ is identically zero or that $\kappa e^{\int(r^{-1}e^\Lambda)} = \frac{2}{r}$ when $r = R$. At this point we will specify the arbitrary constant in $\int(r^{-1}e^\Lambda)$ by expressing it as a definite integral, $\int_R^r(r^{-1}e^\Lambda)dr$ and hence $\kappa = \frac{2}{R}e^{-\int_R^R(r^{-1}e^\Lambda)} = \frac{2}{R}$. Hence we have

$$8\pi\rho = \frac{2}{R^2}e^{2\int_R^r(r^{-1}e^\Lambda)} \tag{44}$$

and

$$8\pi p = \frac{2}{Rr}e^{\int_R^r(r^{-1}e^\Lambda)} - \frac{2}{R^2}e^{2\int_R^r(r^{-1}e^\Lambda)} \tag{45}$$

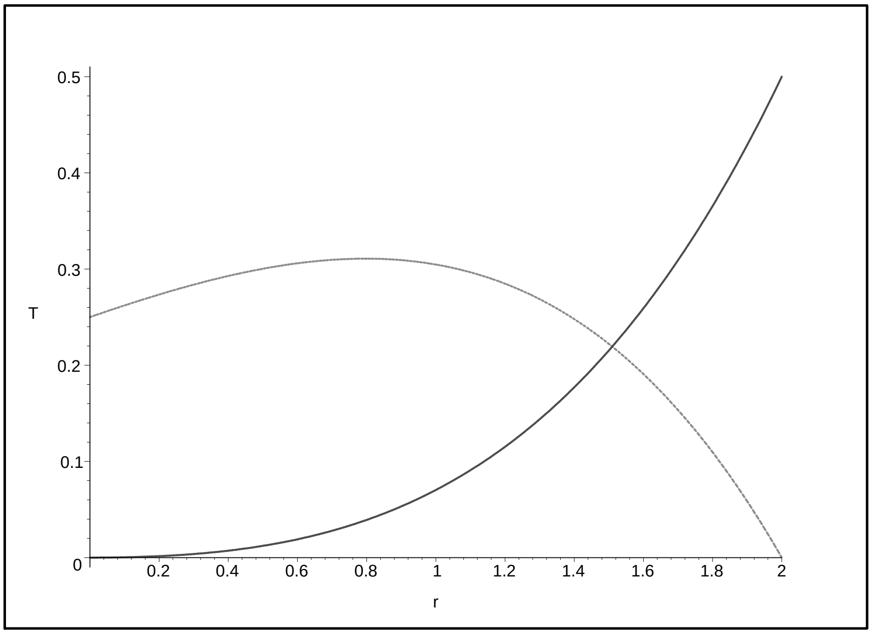

Figure 1: Stress Energy Tensor Components: Solid: Density, Dashed: Pressure

Using (44), we have $\rho = \frac{1}{4\pi R^2} e^{2\int_R^r (r^{-1}e^{\Lambda})}$ and hence

$$e^{\Lambda} = \frac{r\rho'}{2\rho} \quad . \tag{46}$$

Note that this implies that $\rho' > 0$. From (40) we find $\Phi' = \frac{\rho'}{\rho} - \frac{2}{r} - \frac{2\sqrt{\pi} r\rho'}{\sqrt{\rho}}$ and hence

$$e^{\Phi} = \frac{C\rho}{r^2} e^{-2\sqrt{\pi}\int_R^r \frac{r\rho'}{\sqrt{\rho}}} \quad , \tag{47}$$

where C is an arbitrary constant which may be chose so as to make $g_{\mu\nu}$ continuous at the surface of the star. Thus the model is determined by specifying $\rho(r)$ and the positive constant of integration for integration in (47) with the additional condition that $\rho' > 0$.

Example. As a reasonable model, suppose that $\rho(r) = \frac{1}{4\pi R^2} \cdot \frac{r^2(r+2)^2}{64}$. If the value of R is set to be $R = 2$ (note, in geometrized units the radius of the sun is $R = 2.333$), then (46) and (47) give $e^{\Lambda} = \dfrac{2(r+1)}{r+2}$ and $e^{\Phi} = C(r+2)^2 e^{-\frac{1}{12}r^3 - \frac{1}{8}r^2 + \frac{7}{6}}$. The curvature vector is $C_\mu = \left[0 \quad \frac{r(r+1)}{4}, 0, 0\right]$. One find that $-G^t_{\ t} = \frac{3r^2+9r+8}{4r(r+1)^3}$. Note that $\lim_{r\to 0^+} -G^t_{\ t} = \infty$ and at $r = 2$, the value of $-G^t_{\ t}$ is $\frac{19}{108}$. We also find $-(T_f)^t_{\ t} = \dfrac{r^8 + 7r^7 + 19r^6 - 23r^5 - 224r^4 - 428r^3 - 240r^2 + 192r + 256}{128r(r+1)^3}$, which has an infinite value at $r = 0$ and approaches a negative value $(-\frac{89}{108})$ at $r = 2$. If one investigates the pressures, one finds that $8\pi p = \frac{1}{128}(2-r)(r+2)(r^2+4r+8)$, which

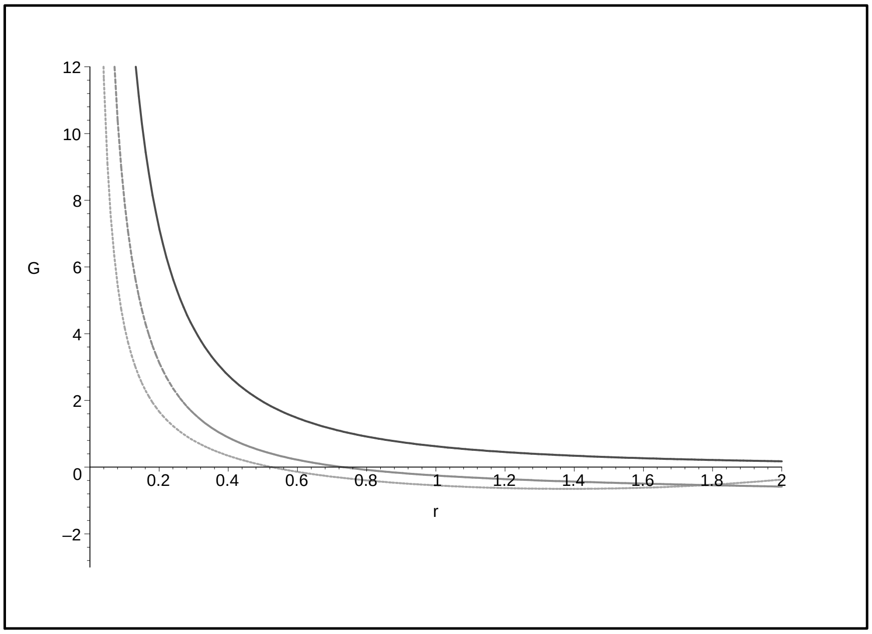

Figure 2: Einstein Tensor Components: Solid: Density, Dashed: Radial Pressure, Dotted: Tangential Pressure

is positive for $0 \leq r < 2$. The total pressure associated with G^r_r term is given by $G^r_r = \dfrac{-r^5 - 6r^4 - 13r^3 - 10r^2 + 6r + 32}{8r(r+1)^3}$, which has an infinite positive value at $r = 0$ and becomes a small negative value (≈ -0.5) as r approaches 2. Similar features occur for $G^\theta_\theta = \dfrac{r^8 + 7r^7 + 19r^6 - 3r^5 - 112r^4 - 200r^3 - 72r^2 + 48r + 32}{64r(r+1)^3}$. For $\mathbf{T}_f$ one finds $(T_f)^r_r = \dfrac{-r^7 - 6r^6 - 13r^5 + 4r^4 + 44r^3 + 32r^2 + 32r + 128}{128r(r+1)^2}$ and also $(T_f)^\theta_\theta = \dfrac{r^8 + 7r^7 + 19r^6 + r^5 - 80r^4 - 116r^3 + 80r^2 + 160r + 64}{128r(r+1)^3}$. Both of these approach $+\infty$ as r approaches 0. The $(T_f)^r_r$ component drops to a small negative value as r approaches 2 (≈ -0.2). The $(T_f)^\theta_\theta$ component is strictly positive on $0 \leq r \leq 2$.

4 External Solution.

In order for the external solution to agree with the weak-field solution as $r \to \infty$, we will identify $w(r) = \frac{1}{2}M$ for $r \geq R$, the radius of the star. Thus from (21) we have

$\frac{1}{2}M = \dfrac{r}{2} - \dfrac{r}{2(1+\frac{1}{2}r\Phi')^2}$ and hence

$$\Phi(r) = \int \left[\frac{2}{r\sqrt{1-\frac{M}{r}}} - \frac{2}{r} \right] dr \tag{48}$$

which can be easily integrated to find $\Phi(r) = 4\ln(1+\sqrt{1-\frac{M}{r}}) + \frac{1}{2}\ln C$ for some arbitrary $C > 0$. Thus

$$g_{tt} = -e^{2\Phi(r)} = -C\left(1+\sqrt{1-\frac{M}{r}}\right)^8 \quad . \tag{49}$$

The arbitrary constant C is determined by the usual weak field approximation [6] which is $g_{tt} \approx -1 + \dfrac{2M}{r}$. This implies that $C = \frac{1}{256}$. Hence

$$g_{tt} = -\frac{1}{256}\left(1+\sqrt{1-\frac{M}{r}}\right)^8 \quad . \tag{50}$$

We thus obtain the following line element:

$$ds^2 = -\frac{1}{256}\left(1+\sqrt{1-\frac{M}{r}}\right)^8 dt^2 + \left(1-\frac{M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad . \tag{51}$$

Expanding g_{tt} in powers of $\frac{M}{r}$, we find that asymptotically, to second order,

$$g_{tt} \approx -1 + \frac{2M}{r} - \frac{5M^2}{4r^2} \quad , \; r >> M \quad . \tag{52}$$

Using (18-20), the Einstein field equations for the external solution are

$$\begin{aligned}
G_{tt} &= 8\pi T_{tt} = 0 \\
G_{rr} = 8\pi T_{rr} &= \frac{M\left(3\sqrt{1-\frac{M}{r}}-1\right)}{r^3\left(1-\frac{M}{r}\right)\left(1+\sqrt{1-\frac{M}{r}}\right)} \\
\frac{G_{\theta\theta}}{r^2} = \frac{G_{\phi\phi}}{r^2\sin^2\theta} = \frac{8\pi T_{\theta\theta}}{r^2} = \frac{8\pi T_{\phi\phi}}{r^2\sin^2\phi} &= \frac{-M\left(9\sqrt{1-\frac{M}{r}}-7\right)}{2r^2\left(1+\sqrt{1-\frac{M}{r}}\right)} \quad .
\end{aligned} \tag{53}$$

Or

$$\begin{aligned}
8\pi\rho &= 0 \\
8\pi p_R &= \frac{M\left(3\sqrt{1-\frac{M}{r}}-1\right)}{r^3\left(1+\sqrt{1-\frac{M}{r}}\right)} \\
8\pi p_T &= \frac{-M\left(9\sqrt{1-\frac{M}{r}}-7\right)}{2r^2\left(1+\sqrt{1-\frac{M}{r}}\right)} \quad .
\end{aligned} \tag{54}$$

Asymptotically we have

$$\begin{aligned} 8\pi p_R &\approx \frac{M}{r^3}\Big(1-\frac{M}{2r}\Big) \\ 8\pi p_T &\approx -\frac{M}{2r^3}\Big(1-\frac{2M}{r}\Big) \end{aligned} \quad , \quad r >> M \quad . \tag{55}$$

From ${T^\mu}_{\nu;\mu} = 0$, with ${T^\mu}_\nu = \mathrm{diag}\big(0, p_R(r), p_T(r), p_T(r)\,\big)$ and $\frac{d\Phi}{dr} = \frac{2}{r\sqrt{1-\frac{M}{r}}} - \frac{2}{r}$, one finds (22)

$$\frac{dp_R}{dr} = -2p_R(r)\Bigg(\frac{1}{r\sqrt{1-\frac{M}{r}}} - \frac{1}{r}\Bigg) + \frac{2}{r}\Big(p_T(r) - p_R(r)\Big) \quad . \tag{56}$$

We see that this equation asymptotically gives

$$\frac{dp_R}{dr} \approx -p_R(r)\bigg(\frac{M}{r^2}\bigg) + \frac{2}{r}\Big(p_T(r) - p_R(r)\Big) \quad . \tag{57}$$

We note that at $r = R$, there may be a discontinuity in the metric. One constant of integration in the internal solution is available to make the g_{tt} term continuous. The discontinuity in the g_{rr} may be smoothed out using appropriation transition functions or perhaps explained in terms of surface tension.

Noncompact External Solutions. If the density outside (for $r \geq R$) is small but nonzero, a realistic function that agrees with the weak field solution must have the property, $\lim_{r\to\infty} w(r) = \frac{1}{2}M$. These noncompact solutions may realistically model the exterior of a star or the distribution of matter in a star cluster. One particularly simple model is given by

$$w(r) = \frac{M}{2} - \frac{M^2}{8r} \quad . \tag{58}$$

With this choice, $1 - \frac{2w(r)}{r} = 1 - \frac{M}{r} + \frac{M^2}{4r^2} = (1 - \frac{M}{2r})^2$. Thus, using (17), (21), (26) and (28) we have

$$ds^2 = -\Big(1-\frac{M}{2r}\Big)^4 dt^2 + \Big(1-\frac{M}{2r}\Big)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad , \tag{59}$$

and

$$\begin{aligned} 8\pi\rho &= \frac{M^2}{4r^4} \\ 8\pi p_r &= \frac{M}{r^3}\Big(1-\frac{3M}{r}\Big) \\ 8\pi p_T &= -\frac{M}{2r^3}\Big(1-\frac{5M}{2r}\Big) \end{aligned} \tag{60}$$

These equations are exact.

A second model is given by

$$w(r) = \frac{M\left(1 - \frac{7M}{4r}\right)}{2\left(1 - \frac{3M}{2r}\right)^2} \quad . \tag{61}$$

With this choice, one finds that the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{\left(1 - \frac{3M}{2r}\right)^2}{\left(1 - \frac{2M}{r}\right)^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad , \tag{62}$$

and

$$\begin{aligned} 8\pi\rho &= -\frac{5M^2\left(1 - \frac{21M}{10r}\right)}{4r^4\left(1 - \frac{3M}{2r}\right)^3} \\ 8\pi p_r &= \frac{M\left(1 - \frac{9M}{4r}\right)}{r^3\left(1 - \frac{3M}{2r}\right)^2} \\ 8\pi p_T &= -\frac{M\left(1 - \frac{M}{r}\right)\left(1 - \frac{3M}{r}\right)}{2r^3\left(1 - \frac{3M}{2r}\right)^3} \end{aligned} \tag{63}$$

5 Motion of a Test Particle.

We now investigate the motion of a test particle in the external field solution. An efficient procedure for doing this is one that extremalizes an appropriate Lagrangian. The motion of the particle will be an extremum of the Lagrangian $\mathcal{L} = \frac{1}{2}g_{\mu\nu}u^\mu u^\nu$, where the velocity $u^\alpha = \dfrac{dx^\alpha}{d\tau}$. For convenience, we will use the "dot" notation for the components of u^α, i.e. $u^\alpha = \langle \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi} \rangle$. We will first investigate the motion in the geometry determined by (51). From (51),

$$\mathcal{L} = -\frac{1}{512}\left(1 + \sqrt{1 - \frac{M}{r}}\right)^8 \dot{t}^2 + \frac{1}{2}\left(1 - \frac{M}{r}\right)^{-1}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\sin^2\theta\dot{\phi}^2 \quad . \tag{64}$$

We may arbitrarily restrict our motion to the equatorial plane with $\theta \equiv \frac{\pi}{2}$ and hence

$$\mathcal{L} = -\frac{1}{512}\left(1 + \sqrt{1 - \frac{M}{r}}\right)^8 \dot{t}^2 + \frac{1}{2}\left(1 - \frac{M}{r}\right)^{-1}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 \quad . \tag{65}$$

From the Euler-Lagrange equations we have

$$\frac{\partial\mathcal{L}}{\partial t} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0 \quad \rightarrow \quad \frac{1}{256}\left(1 + \sqrt{1 - \frac{M}{r}}\right)^8 \dot{t} = E \quad , \tag{66}$$

and hence

$$\dot{t} = 256E\left(1+\sqrt{1-\frac{M}{r}}\right)^{-8} \quad . \tag{67}$$

In these equations the arbitrary constant E is identified with the conserved energy of the test particle. The Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$ similarly leads to

$$\dot{\phi} = \frac{L}{r^2} \quad , \tag{68}$$

where the arbitrary constant, L, is identified with the conserved angular momentum. Finally we look at the r equation, $\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0$. This implies that

$$\frac{-M\left(1+\sqrt{1-\frac{M}{r}}\right)^7}{128r^2\sqrt{1-\frac{M}{r}}}\,\dot{t}^2 - \frac{M}{2r^2\left(1-\frac{M}{r}\right)^2}\,\dot{r}^2 + r\dot{\phi}^2 - \frac{d}{d\tau}\left[\left(1-\frac{M}{r}\right)^{-1}\dot{r}\right] = 0 \quad . \tag{69}$$

Using (67) and (68) and differentiating we have

$$\frac{-512ME^2}{r^2\sqrt{1-\frac{M}{r}}\left(1+\sqrt{1-\frac{M}{r}}\right)^9} + \frac{M}{2r^2\left(1-\frac{M}{r}\right)^2}\,\dot{r}^2 + \frac{L^2}{r^3} - \frac{1}{1-\frac{M}{r}}\,\ddot{r} = 0 \quad . \tag{70}$$

Thus

$$\ddot{r} = -\frac{512ME^2\sqrt{1-\frac{M}{r}}}{r^2\left(1+\sqrt{1-\frac{M}{r}}\right)^9} + \frac{M}{2r^2\left(1-\frac{M}{r}\right)}\dot{r}^2 + \frac{L^2\left(1-\frac{M}{r}\right)}{r^3} \quad . \tag{71}$$

We now impose a normalization on the velocity by requiring that $g_{\mu\nu}u^\mu u^\nu = -1$. Therefore $-\frac{1}{256}\left(1+\sqrt{1-\frac{M}{r}}\right)^8\dot{t}^2 + \left(1-\frac{M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$. Using (67) and (68) we may eliminate the $\dot{t}$ and $\dot{\phi}$ terms. This leads to $256E^2\left(1+\sqrt{1-\frac{M}{r}}\right)^{-8} = \left(1-\frac{M}{r}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} + 1$. Substituting this into (71), we arrive at

$$\begin{aligned}\ddot{r} = & -\frac{2M\sqrt{1-\frac{M}{r}}}{r^2\left(1+\sqrt{1-\frac{M}{r}}\right)} + \frac{L^2\sqrt{1-\frac{M}{r}}\left(3\sqrt{1-\frac{M}{r}}-2\right)}{r^3} \\ & - \frac{M\left(3\sqrt{1-\frac{M}{r}}-1\right)}{2r^2\left(1-\frac{M}{r}\right)\left(1+\sqrt{1-\frac{M}{r}}\right)}\dot{r}^2 \quad .\end{aligned} \tag{72}$$

A similar computation with the metric given by (59) results in

$$\ddot{r} = -\frac{M}{r^2}\left(1-\frac{M}{2r}\right) - \frac{M}{2r^2}\left(1-\frac{M}{2r}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^3}\left(1-\frac{M}{r}+\frac{M^2}{2r^2}\right) \quad . \tag{73}$$

With the metric given by (62) we similarly get

$$\ddot{r} = -\frac{M\big(1-\frac{2M}{r}\big)}{r^2\big(1-\frac{3M}{2r}\big)^2} - \frac{M\big(1-\frac{3M}{r}\big)}{2r^2\big(1-\frac{3M}{2r}\big)\big(1-\frac{2M}{r}\big)}\dot{r}^2 + \frac{L^2\big(1-\frac{2M}{r}\big)\big(1-\frac{3M}{r}\big)}{r^3\big(1-\frac{3M}{2r}\big)^2} \quad . \tag{74}$$

Kepler's Law. The angular velocity is given by $\omega = \dfrac{\dot{\phi}}{\dot{t}}$, and so when the orbit is circular ($\ddot{r} = \dot{r} = 0$) we see from (72) that

$$\omega^2 r^3 = \frac{M\Big(1+\sqrt{1-\frac{M}{r}}\Big)^7}{128\sqrt{1-\frac{M}{r}}} \quad , \tag{75}$$

which asymptotically gives

$$\omega^2 r^3 \approx M\Big(1-\frac{5M}{4r}\Big) \quad , \quad r >> M \quad . \tag{76}$$

For the motion under the metric (59) one gets

$$\omega^2 r^3 = M\Big(1-\frac{3M}{2r}+\frac{3M^2}{4r^2}-\frac{M^3}{8r^3}\Big) \quad . \tag{77}$$

We note that these results agree with Kepler's Law for large r. For the metric of (62) one gets exact agreement with Kepler's Law:

$$\omega^2 r^3 = M \quad . \tag{78}$$

Radial Motion. For pure radial motion ($L = 0$), (72) asymptotically yields

$$\ddot{r} \approx -\frac{M}{r^2}\Big(1-\frac{3M}{4r}\Big) - \frac{M}{2r^2}\Big(1+\frac{M}{2r}\Big)\dot{r}^2 \quad , \quad r >> M \quad . \tag{79}$$

From the metric (59), one finds similarly the pure radial motion to be exactly given by

$$\ddot{r} = -\frac{M}{r^2}\Big(1-\frac{M}{2r}\Big) - \frac{M}{2r^2}\Big(1-\frac{M}{2r}\Big)^{-1}\dot{r}^2 \quad . \tag{80}$$

From the metric (62), one gets

$$\ddot{r} \approx -\frac{M}{r^2}\Big(1+\frac{M}{r}\Big) - \frac{M}{2r^2}\Big(1+\frac{M}{2r}\Big)\dot{r}^2 \quad , r >> M \quad . \tag{81}$$

The magnitude of the $\dot{r}^2$ terms in (79-81) do not appear to be large enough to explain the Pioneer anomaly. The Pioneer spacecraft is traveling out of the solar system. A small acceleration toward the sun which cannot be explained by general relativity has been observed over a period of years [9]. When $\dot{r}$ is small, the second terms in (79-81) would correspond to small accelerations toward the sun. For Pioneer the magnitude of this term should be about 8.74×10^{-10} m s^{-2}, but (79-81) yields approximately 10^{-15} m s^{-2}.

Redshift. The difference between the values of g_{tt} in this model and the standard Schwarzschild solution would produce small differences in the predicted redshift. The redshift $z = \frac{\Delta\lambda}{\lambda} = |g_{tt}|^{-\frac{1}{2}} - 1$ for stationary objects. From (49) we find that

$$z = 16\left(1 + \sqrt{1 - \tfrac{M}{r}}\,\right)^{-4} - 1 \ \approx \frac{M}{r} \quad , \quad r >> M \quad , \tag{82}$$

and from (59) we find

$$z = \left(1 - \frac{M}{2r}\right)^{-2} - 1 \ \approx \frac{M}{r} \quad , \quad r >> M \quad . \tag{83}$$

Asymptotically, these results agree with the value found in the Schwarzschild geometry, i.e. $z \approx \frac{M}{r}$. At the distance of the earth from the sun, one finds the value given by (82) differs from the standard value by -6.25×10^{-17}, with a relative difference of -6.25×10^{-9}. From (83) we find the value of z differs from the standard value by -7.5×10^{-17} with a relative difference of -7.5×10^{-9}. For the metric of (62) one find exact agreement with the standard redshift result:

$$z = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} - 1 \ \approx \frac{M}{r} \quad , \quad r >> M \quad . \tag{84}$$

Precession of Perihelion. We now consider the precession of perihelion problem. We follow the procedure of Misner, Thorne and Wheeler [6], starting with conservation of 4-momentum which leads to

$$g^{tt}E^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 + \mu^2 = 0 \tag{85}$$

where μ is the rest mass of the particle. Again, we restrict the motion to the plane $\theta = \frac{\pi}{2}$. With $\tilde{E} = \frac{E}{\mu}$, $\tilde{L} = \frac{L}{\mu}$, and $\tau = \mu\lambda$, we find that

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{-g^{tt}\tilde{E}^2}{g_{rr}} - \frac{1}{g_{rr}}\left(1 + \frac{\tilde{L}^2}{r^2}\right) \tag{86}$$

Using $\frac{dr}{d\phi} = \frac{r^2}{\tilde{L}}\frac{dr}{d\tau}$, with $u \equiv \frac{M}{r}$ and $L^\dagger \equiv \frac{\tilde{L}}{M}$, one finds that

$$\left(L^\dagger\frac{du}{d\phi}\right)^2 \ = \ f(u) \ \equiv \ \left(\frac{-g^{tt}}{g_{rr}}\right)\tilde{E}^2 - \frac{1}{g_{rr}}\left(1 + (L^\dagger)^2u^2\right) \quad . \tag{87}$$

When the orbit is circular at $u = u_0 = \frac{M}{r_0}$, the effective potential, $f(u)$, has a minimum with both $f(u_0) = 0$ and $f'(u_0) = 0$. Hence $f(u) \approx \frac{1}{2} f''(u_0)(u - u_0)^2$. Via the chain rule, one has $2(L^\dagger)^2 \frac{du}{d\phi}\frac{d^2u}{d\phi^2} = f'(u)\frac{du}{d\phi}$. Thus, one finds that,

$$\frac{d^2}{d\phi^2}\Big(u - u_0\Big) - \frac{f''(u_0)}{2(L^\dagger)^2}\Big(u - u_0\Big) = 0 \quad . \tag{88}$$

When $f''(u_0) < 0$, the solution is periodic with

$$\text{Period} \ = \frac{2\pi}{\sqrt{-\frac{f''(u_0)}{2(L^\dagger)^2}}} \quad . \tag{89}$$

From the metric given in (51), we find that

$$-\frac{f''(u_0)}{2(L^\dagger)^2} \approx 1 - \frac{19}{4} u_0 \tag{90}$$

and thus the perihelion is shifted by

$$\Delta\phi \approx \left(\frac{19\pi}{4}\right)\frac{M}{r_0} \quad . \tag{91}$$

where r_0 is the radius of the near-circular orbit. If the metric (59) is used one finds

$$\Delta\phi \approx \left(\frac{7\pi}{2}\right)\frac{M}{r_0} \quad . \tag{92}$$

For the metric given in (62), one finds that

$$\Delta\phi \approx \frac{6\pi M}{r_0} \quad , \tag{93}$$

which is the standard result.

6 Conclusion.

The theory based on the conservative transformation group may provide a theoretical basis for a unified field theory and may also provide a theoretical basis for dark matter and the correct modification of general relativity. The Lagrangian for the field with sources may be used in a variety of applications, including quantization. The internal solution and its corresponding stellar model needs additional work to produce more realistic models. The external solutions, being non-compact, show promise for explaining dark matter.

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