随机动力系统的 Morse 分解

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摘要： 众所周知, Morse 分解对于研究动力系统的不变集的内部结构是非常有效的. 最近, 对于随机动力系统已经建立了 Morse 分解的结果, 从而可以用于研究随机不变集的内部结构, 例如随机吸引子的内部结构. 在这片注记中, 综述了随机动力系统的 Morse 分解定理.

关键词： 随机动力系统; Morse 分解; 随机吸引子
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Morse decomposition for random dynamical systems

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Abstract: It is well-known that Morse decomposition is very useful to study the inner structure of invariant sets for given dynamical systems. Recently, Morse decomposition is established for random dynamical systems, which can be used to investigate the inner structure of random invariant sets, e.g. random attractors. In this note, we review Morse decomposition theorem for random dynamical systems.

Key words: Random dynamical system; Morse decomposition; random attractor

0 Introduction

One important aspect of the qualitative analysis of differential equations and dynamical systems is the study of asymptotic, long-term behavior of solutions/orbits. Hence much of dynamical systems involves the study of the existence and structure of invariant sets. The classical Morse decomposition theorem, due to [1], states that any invariant compact set can be decomposed into finite disjoint invariant compact sets the connecting orbits between them. This theorem completely describe the dynamics on the invariant set, so it is very helpful for us understand the inner structure of invariant sets. See [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] etc for studies and reviews on attractors, Morse decomposition, and Lyapunov functions.

Random dynamical systems, see [12] for a comprehensive introduction, arise in the modeling of many phenomena in physics, biology, economics, climatology, etc and the random effects

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often reflect intrinsic properties of these phenomena rather than just to compensate for the defects in deterministic models. The history of study of random dynamical systems goes back to Ulam and von Neumann [13] and it has flourished since the 1980s due to the discovery that the solutions of stochastic ordinary differential equations yield a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. Recently, some authors studied the Morse decomposition and Lyapunov second method for random dynamical systems, see [14, 15, 16, 17, 18, 19, 20, 21] for details. In this note, we review Morse decomposition theorem for random dynamical systems.

1 Random dynamical systems

In this section, we will give some preliminary definitions and propositions for the later use. Firstly we give the definition of continuous random dynamical systems (cf Arnold [12]).

Definition 1.1. A continuous *random dynamical system (RDS)*, shortly denoted by φ, consists of two ingredients:
(i) A model of the noise, namely a metric dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t\omega$ is a measurable flow which leaves $\mathbb{P}$ invariant, i.e. $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{T}$.
(ii) A model of the system perturbed by noise, namely a cocycle φ over θ, i.e. a measurable mapping $\varphi : \mathbb{T} \times \Omega \times X \to X, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, such that $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and the family $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : X \to X$ of random self-mappings of X satisfies the cocycle property:

$$\varphi(0, \omega) = \mathrm{id}_X, \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \quad \text{for all} \quad t, s \in \mathbb{T}, \omega \in \Omega. \tag{1.1}$$

Remark 1.2. The time for the base flow (θ_t) is always assumed to be two-sided, even if φ is defined for nonnegative time only. Furthermore, the maps $\varphi(t, \omega) : X \to X$ are not assumed to be invertible a priori. The cocycle property implies that for two sided time ($\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$) $\varphi(t, \omega)$ is automatically invertible $\mathbb{P}$-a.s. for every $t \in \mathbb{T}$. In fact, in this case $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega)$ for every $t \in \mathbb{T}$.

We now give the definition of random set, which is a basic concept for RDS.

Definition 1.3. Let X be a metric space with a metric d_X. A set-valued map $\omega \mapsto D(\omega)$ taking values in the closed/compact subsets of X is said to be a *random closed/compact set* if the mapping $\omega \mapsto \mathrm{dist}_X(x, D(\omega))$ is measurable for any $x \in X$, where $\mathrm{dist}_X(x, B) := \inf_{y\in B} d_X(x, y)$. A set-valued map $\omega \mapsto U(\omega)$ taking values in the open subsets of X is said to be a *random open set* if $\omega \mapsto U^c(\omega)$ is a random closed set, where U^c denotes the complement of U.

Definition 1.4. A random set D is said to be *forward invariant* under the RDS φ if $\varphi(t,\omega)D(\omega) \subset D(\theta_t\omega)$ for all $t \geq 0$ almost surely; It is said to be *invariant* if $\varphi(t,\omega)D(\omega) = D(\theta_t\omega)$ for all $t \geq 0$ almost surely.

Now we enumerate some basic results about random sets in the following propositions, for details the reader can refer to Arnold [12], Castaing and Valadier [22], Chueshuv [23], Crauel [24], Hu and Papageorgiou [25].

Proposition 1.5. Let X be a Polish space, then the following assertions hold.
(i) D is a random closed set in X if and only if the set $\{\omega : D(\omega)\bigcap U \neq \emptyset\}$ is measurable for any open set $U \subset X$.
(ii) if D is a random closed set, then so is the closure of D^c.
(iii) if D is a random open set, then the closure cl$D$ of D is a random closed set; if D is a random closed set, then int$D$, the interior of D, is a random open set.
(iv) D is a random compact set in X if and only if $D(\omega)$ is compact for every $\omega \in \Omega$ and the set $\{\omega : D(\omega)\bigcap C \neq \emptyset\}$ is measurable for any closed set $C \subset X$.
(v) if $\{D_n, n \in \mathbb{N}\}$ is a sequence of random closed sets with non-void intersection and there exists $n_0 \in \mathbb{N}$ such that D_{n_0} is a random compact set, then $\bigcap_{n\in\mathbb{N}} D_n$ is a random compact set in X.
(vi) if $f : \Omega \times X \to X$ is a function such that $f(\omega,\cdot)$ is continuous for all ω and $f(\cdot,x)$ is measurable for all x, then $\omega \mapsto f(\omega, D(\omega))$ is a random compact set provided that D is a random compact set.
(vii) if D is a random closed set, then graph$(D) := \{(\omega,x)|x \in D(\omega)\}$ is a measurable subset of $\mathcal{F} \times \mathcal{B}(X)$; conversely, given $D : \Omega \to 2^X$, taking values in the closed subsets of X, if graph$(D) \in \mathcal{F} \times \mathcal{B}(X)$, then D is an $\mathcal{F}^u$-measurable (in particular, $\mathcal{F}^{\mathbb{P}}$-measurable, with $\mathcal{F}^{\mathbb{P}}$ being the completion of the σ-algebra $\mathcal{F}$ with respect to the measure $\mathbb{P}$) random closed set, i.e. the mapping $\omega \mapsto \text{dist}_X(x, D(\omega))$ is $\mathcal{F}^u$-measurable for any $x \in X$;
(viii) if D is an $\mathcal{F}^{\mathbb{P}}$-measurable random closed set, then there exists a $\mathcal{F}$-measurable random closed set $\tilde{D}$ such that $D = \tilde{D}$ almost surely.
(ix) (*Measurable Selection Theorem*). Let a multifunction $\omega \mapsto D(\omega)$ take values in the subspace of closed non-void subsets of X. Then D is a random closed set if and only if there exists a sequence $\{v_n : n \in \mathbb{N}\}$ of measurable maps $v_n : \Omega \to X$ such that

$$v_n(\omega) \in D(\omega) \quad and \quad D(\omega) = \overline{\{v_n(\omega), n \in \mathbb{N}\}} \quad for\ all \quad \omega \in \Omega.$$

In particular if D is a random closed set, then there exists a measurable selection, i.e. a measurable map $v : \Omega \to X$ such that $v(\omega) \in D(\omega)$ for all $\omega \in \Omega$.
(x) (*Projection Theorem*). Let X be a Polish space and $M \subset \Omega \times X$ be a set which is measurable with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}(X)$. Then the set

$$\Pi_\Omega M = \{\omega \in \Omega : (\omega, x) \in M \text{ for some } x \in X\}$$

is universally measurable, i.e. belongs to $\mathcal{F}^u$, where Π_Ω stands for the canonical projection of $\Omega \times X$ to Ω. In particular, it is measurable with respect to the $\mathbb{P}$-completion $\bar{\mathcal{F}}^{\mathbb{P}}$ of $\mathcal{F}$.

Definition 1.6. Assume that D is a random set, then the *omega-limit set of* D, Ω_D, is defined to be

$$\Omega_D(\omega) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \varphi(s, \theta_{-s}\omega)D(\theta_{-s}\omega)}.$$

Definition 1.7. For given two random sets D and A, we say A *(pull-back) attracts* D if

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)|A(\omega)) = 0$$

holds almost surely, where $d(A|B)$ stands for the Hausdorff semi-metric between two sets A, B, i.e. $d(A|B) := \sup_{x\in A}\inf_{y\in B} d_X(x, y)$; and we say A *attracts* D *in probability* or *weakly attracts* D if

$$\mathbb{P} - \lim_{t\to\infty} d(\varphi(t, \omega)D(\omega)|A(\theta_t\omega)) = 0.$$

2 Morse decomposition for random dynamical systems

We now introduce "backward orbit" for random semiflow:

Definition 2.1. (i) For fixed ω and x, a mapping $\sigma.(\omega) : \mathbb{R}^- \to X$ is called a *backward orbit of* φ *through* x *driven by* ω if it satisfies the cocycle property:

$$\sigma_0(\omega) = x,\ \ \sigma_{t+s}(\omega) = \varphi(s, \theta_t\omega) \circ \sigma_t(\omega) \text{ for } \forall t \leq 0, s \geq 0, t + s \leq 0.$$

(ii) Let $\mathcal{M}$ denote the set of all X-valued random variables and $x \in \mathcal{M}$. A mapping $\sigma : \mathbb{R}^- \to \mathcal{M}$ is called a *backward orbit of* φ *through* x if for all $\omega \in \Omega$, the following cocycle property holds:

$$\sigma_0(\omega) = x(\omega),\ \ \sigma_{t+s}(\omega) = \varphi(s, \theta_t\omega) \circ \sigma_t(\omega) \text{ for } \forall t \leq 0, s \geq 0, t + s \leq 0.$$

Also we can introduce "entire orbit" for random semiflow:

Definition 2.2. (i) For fixed ω and x, a mapping $\sigma.(\omega) : \mathbb{R} \to X$ is called an *entire orbit of* φ *through* x *driven by* ω if it satisfies the cocycle property:

$$\sigma_0(\omega) = x,\ \ \sigma_{t+s}(\omega) = \varphi(s, \theta_t\omega) \circ \sigma_t(\omega) \text{ for } \forall t \in \mathbb{R}, s \geq 0.$$

(ii) Let $x \in \mathcal{M}$. A mapping $\sigma : \mathbb{R} \to \mathcal{M}$ is called an *entire orbit of* φ *through* x if for all $\omega \in \Omega$, the following cocycle property holds:

$$\sigma_0(\omega) = x(\omega),\ \ \sigma_{t+s}(\omega) = \varphi(s, \theta_t\omega) \circ \sigma_t(\omega) \text{ for } \forall t \in \mathbb{R}, s \geq 0.$$

By the cocycle property of σ, it is clear that for arbitrary $t \geq 0$ and $\omega \in \Omega$ we have

$$\sigma_t(\omega) = \varphi(t, \omega) \circ \sigma_0(\omega).$$

That is, when an entire orbit σ of φ is restricted to $\mathbb{R}^+$ (called *forward orbit*), then it coincides with the orbit of φ, which is the same as the deterministic case.

We can give an alternative definition of (forward, backward) invariant sets for random semiflow:

Definition 2.3. (i) A random set D is called *forward invariant* if $D = D^+_\varphi$ almost surely, where

$$D^+_\varphi(\omega) := \{x|\varphi(t,\omega)x \in D(\theta_t\omega) \text{ for all } t \geq 0\};$$

(ii) A random set D is called *backward invariant* if $D = D^-_\varphi$ almost surely, where

$$D^-_\varphi(\omega) := \{x|\exists \text{ a backward orbit } \sigma \text{ in } D \text{ through } x, \text{ i.e. } \sigma_t(\omega) \in D(\theta_t\omega), \forall t \leq 0\};$$

(iii) A random set D is called *invariant* if $D = D_\varphi$ almost surely, where

$$D_\varphi(\omega) := \{x|\exists \text{ an entire orbit } \sigma \text{ in } D \text{ through } x, \text{ i.e. } \sigma_t(\omega) \in D(\theta_t\omega), \forall t \in \mathbb{R}\}.$$

Remark 2.4. (i) Clearly a random set D is invariant if and only if it is both forward invariant and backward invariant.
(ii) The forward invariant set and invariant set defined in Definition 2.3 coincide with that of Definition 1.4. The equivalence of forward invariance is obvious. Invariance in Definition 2.3 clearly implies invariance in Definition 1.4. If D is invariant in the sense of Definition 1.4, for $x \in D(\omega)$ and any $k \in \mathbb{N}$, by the fact $\varphi(k, \theta_{-k}\omega)D(\theta_{-k}\omega) = D(\omega)$, there exists $x_k \in D(\theta_{-k}\omega)$ such that $\varphi(k, \theta_{-k}\omega)x_k = x$. Then $\sigma_t(\omega) := \varphi(k+t, \theta_{-k}\omega)x_k \in D(\theta_t\omega)$ by the invariance of D in the sense of Definition 1.4, for $t \in [-k, 0]$. Similarly, by $\varphi(1, \theta_{-(k+1)}\omega)D(\theta_{-(k+1)}\omega) = D(\theta_{-k}\omega)$, there exists $x_{k+1} \in D(\theta_{-(k+1)}\omega)$ such that $\varphi(1, \theta_{-(k+1)}\omega)x_{k+1} = x_k$. Inductively in this way, we obtain a backward orbit through x, hence D is invariant in the sense of Definition 2.3.
(iii) If D_1 and D_2 are forward invariant, then clearly $D_1 \cup D_2$ and $D_1 \cap D_2$ are forward invariant; If D_1 and D_2 are invariant, then clearly $D_1 \cup D_2$ is invariant, while $D_1 \cap D_2$ is not necessarily invariant (since $D_1 \cap D_2$ is not necessarily backward invariant), which differs from random flow case, see page 35 in [12].

It is known that given an invariant random set D and $x \in D(\omega)$, there exists a backward orbit lying on D through x. A natural question is, for any random variable $x \in D$, does there exist a backward orbit lying on D through x? The answer is yes, see the following lemma.

Lemma 2.5. *Assume that D is an invariant random closed set, then for any random variable on D there exists a backward orbit lying on D through this random variable.*

Proof. We know that, for given $k \in \mathbb{Z}^-$ and for each ω, there exists an $\tilde{x}_k(\omega) \in D(\theta_k\omega)$ from which we obtain a backward orbit from time k to time 0 (present time). Hence we need only to show that we can select appropriate $\tilde{x}_k$ such that the mapping $\omega \mapsto \tilde{x}_k(\omega)$ is measurable. In other words, we need to show $\sigma_k \in \mathcal{M}$, which implies $\sigma_s \in \mathcal{M}$, $\forall k \leq s \leq 0$. For arbitrary $t > 0$, denote $\varphi^{-1}(t,\omega)x$ the preimage of x under φ. Consider the semiflow Θ corresponding to φ, given by $\Theta_t(\omega, x) := (\theta_t\omega, \varphi(t,\omega)x)$, which is an $\mathcal{F} \times \mathcal{B}(X)$-measurable mapping from $\Omega \times X$ to itself for fixed $t \geq 0$. The preimage of (ω, x) under Θ_t is $\Theta_t^{-1}(\omega, x) := (\theta_{-t}\omega, \varphi^{-1}(t,\omega)x)$. Since D is a random closed set, we have

$$\text{graph}(D) := \{(\omega, x) | x \in D(\omega)\} \in \mathcal{F} \times \mathcal{B}(X)$$

by Proposition 1.5 (vii). Hence we have $\Theta_t^{-1}(\text{graph}(D)) \in \mathcal{F} \times \mathcal{B}(X)$ by the measurability of Θ_t. It is cleat that

$$\text{graph}(D^t) = \Theta_t^{-1}(\text{graph}(D(\theta_t\omega))),$$

where

$$D^t(\omega) := \varphi^{-1}(t,\omega)D(\theta_t\omega).$$

Therefore D^t is an $\mathcal{F}^u$-measurable random closed set, by Proposition 1.5 (vii) again. By Proposition 1.5 (viii) we may assume that D^t is an $\mathcal{F}$-measurable random closed set. In particular, for given $k \in \mathbb{Z}^-$, $D^{-k} \cap D$ is a nonempty random closed set by Proposition 1.5 (vii) and (viii). By the measurable selection theorem (see Proposition 1.5 (ix)), we can choose a random variable $\tilde{x}_k \in D^{-k} \cap D$. This completes the proof. □

Throughout this section, we use S to denote the invariant random compact set we will decompose, say, S is a global random attractor. By Lemma 2.5, for any point (random variable) on S, there exists backward orbit lying on S through this point (random variable). Afterwards, when we say backward orbits, we refer those lying on S unless otherwise stated (since there may be backward orbit not lying on S but lying on the entire state space — X).

Definition 2.6. An invariant random compact set $A \subset S$ is called a *(local) attractor* if there exists a random open neighborhood U of A relative to S such that $\Omega_U(\omega) = A(\omega)$. (without loss of generality, we can assume that U is forward invariant.) The basin of attraction of A is defined by

$$B(A)(\omega) := \{x \in S(\omega) | \varphi(t,\omega)x \in U(\theta_t\omega) \text{ for some } t \geq 0\}$$

and the *repeller R corresponding to A* is defined by

$$R(\omega) = S(\omega) \backslash B(A)(\omega).$$

(A, R) is called an *attractor-repeller pair*.

Note that since S is a random compact set, by Lemma 3.2 in [17] (the proof of Lemma 3.2 is also applicable here), $B(A)$ is independent of the choice of U.

Lemma 2.7. *Assume that (A, R) is an attractor-repeller pair in S, then A, $B(A)$, and R are invariant random sets.*

Proof. (i) The invariance of A follows immediately from its definition.
(ii) The forward invariance of $B(A)$ follows directly from the definition of $B(A)$ and the forward invariance of U. For arbitrary $x \in B(A)(\omega)$, if for any backward orbit σ through x, there exists some $t_0 < 0$ such that $\sigma_{t_0} \in R(\theta_{t_0}\omega)$. By the definition of R, we know that any point in R can not enter into $B(A)$ in positive time, so we obtain that R is forward invariant. Therefore,

$$\varphi(-t_0, \theta_{t_0}\omega)\sigma_{t_0}(\omega) = \sigma_0(\omega) = x \in R(\omega),$$

a contradiction. That is, $B(A)$ is backward invariant.
(iii) By (ii) we only need to show the backward invariance of R. For arbitrary $x \in R(\omega)$, if for any backward orbit σ through x, there exists some $t_0 < 0$ such that $\sigma_{t_0} \in B(A)(\theta_{t_0}\omega)$. Then by the forward invariance of $B(A)$, we have

$$\varphi(-t_0, \theta_{t_0}\omega)\sigma_{t_0}(\omega) = \sigma_0(\omega) = x \in B(A)(\omega),$$

a contradiction. Hence R is backward invariant. □

Remark 2.8. Denote $B(R)(\omega) := S(\omega) \setminus A(\omega)$ for each ω.

1. Similar to the proof of Lemma 2.7 (ii), we obtain that if $D \subset S$ is forward invariant, then $D^c := S \setminus D$ is backward invariant set. Furthermore, D^c is *strongly backward invariant* in the sense that *any* backward orbit through the point (hence the random variable, note that the backward orbit through a random variable is a choice of a backward orbit through a point) on S lies on D^c.

2. Observe that different from the random flow case, the complementary of a backward invariant set need not be forward invariant. Particularly, $B(R)$ is not necessarily forward invariant since the forward orbit through the point in $B(R)$ may enter A.

3. Since A is forward invariant, $B(R)$ is strongly backward invariant. Similarly, the random set $S \setminus (A \cup R)$ is strongly backward invariant, but not necessarily forward invariant. Note that the forward orbit through the point in $S \setminus (A \cup R)$ can enter A, but never enter R.

4. Note that if a random set $D \subset S$ is strongly backward invariant in the above sense, then D^c is forward invariant. That is, the reason that the complementary of a backward invariant set is not necessarily forward invariant lies in that the set is not strongly backward invariant.

Definition 2.9. Assume that x is a random variable in S, and σ is an entire orbit through x. Then the *omega-limit set* Ω_σ and the *alpha-limit set* Ω^*_σ of σ are defined to be

$$\Omega_\sigma(\omega) := \bigcap_{T\geq 0}\overline{\bigcup_{t\geq T}\{\sigma_t(\theta_{-t}\omega)\}}$$

and

$$\Omega^*_\sigma(\omega) := \bigcap_{T\geq 0}\overline{\bigcup_{t\geq T}\{\sigma_{-t}(\theta_t\omega)\}},$$

respectively.

It is clear that

$$\Omega_\sigma(\omega) := \bigcap_{T\geq 0}\overline{\bigcup_{t\geq T}\{\varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega)\}},$$

i.e. the omega-limit set of σ only depends on the random variable x, so Ω_σ can also be denoted by Ω_x; while the alpha-limit set depends on the entire orbit σ. Clearly a point $y \in \Omega_\sigma(\omega)$ (respectively $y \in \Omega^*_\sigma(\omega)$) if and only if there exist sequences $t_n \to +\infty$ (respectively $t_n \to -\infty$) and $y_n = \sigma_{t_n}(\theta_{-t_n}\omega)$ such that $y_n \to y$ as $n \to +\infty$.

Lemma 2.10. *Assume that x is a random variable in S, and σ is an entire orbit through x. Then Ω_σ and Ω^*_σ are invariant random compact sets.*

Proof. The random variable x can be regarded as a random set consisting of just a single point, so Ω_σ is an invariant random compact set.

For arbitrary $y \in \Omega^*_\sigma(\omega)$, there exist sequences $t_n \to +\infty$ and $y_n = \sigma_{-t_n}(\theta_{t_n}\omega)$ such that $y_n \to y$ as $n \to +\infty$. For $s > 0$, we have

$$\begin{aligned}
\varphi(s,\omega)y &= \lim_{n\to+\infty}\varphi(s,\omega)\circ\sigma_{-t_n}(\theta_{t_n}\omega)\\
&= \lim_{n\to+\infty}\varphi(s,\omega)\circ\sigma_{-t_n}(\theta_{t_n-s}\circ\theta_s\omega)\\
&= \lim_{n\to+\infty}\sigma_{s-t_n}(\theta_{t_n-s}\circ\theta_s\omega)\\
&= \lim_{n\to+\infty}\sigma_{-\tau_n}(\theta_{\tau_n}\circ\theta_s\omega)\quad (\text{let } t_n - s = \tau_n)\\
&\in \Omega^*_\sigma(\theta_s\omega),
\end{aligned}$$

where the 1st equality holds by the continuity property of φ with respect to x, the 3rd equality holds by the cocycle property of σ, and the last inclusion relation holds by the definition of Ω^*_σ. This verifies the forward invariance of Ω^*_σ.

For arbitrary $y \in \Omega^*_\sigma(\theta_s\omega)$ with $s > 0$, there exist sequences $t_n \to +\infty$ and $y_n = \sigma_{-t_n}(\theta_{t_n}\circ\theta_s\omega)$ such that $y_n \to y$ as $n \to +\infty$. Then we have

$$y = \lim_{n\to+\infty}\sigma_{-t_n}(\theta_{t_n}\circ\theta_s\omega)$$

$$\begin{aligned}
&= \lim_{n\to+\infty} \sigma_{-(\tau_n-s)}(\theta_{\tau_n-s}\circ\theta_s\omega) \quad (\text{let } t_n+s=\tau_n)\\
&= \lim_{n\to+\infty} \varphi(s,\omega)\sigma_{-\tau_n}(\theta_{\tau_n-s}\circ\theta_s\omega)\\
&= \lim_{n\to+\infty} \varphi(s,\omega)\sigma_{-\tau_n}(\theta_{\tau_n}\omega)\\
&= \varphi(s,\omega)\lim_{n\to+\infty} \sigma_{-\tau_n}(\theta_{\tau_n}\omega)\\
&= \varphi(s,\omega)x,
\end{aligned}$$

where the last two equalities hold by the pre-compactness of $\{\sigma_{-\tau_n}(\theta_{\tau_n}\omega)|n\in\mathbb{N}\}$, and by taking a subsequence we assume that the subsequence converges to $x\in\Omega^*_\sigma$. This verifies $\Omega^*_\sigma(\theta_s\omega)\subset\varphi(s,\omega)\Omega^*_\sigma(\omega)$.

Therefore, we have showed that $\varphi(s,\omega)\Omega^*_\sigma(\omega)=\Omega^*_\sigma(\theta_s\omega)$, hence completed the proof. □

Lemma 2.11. *Assume that x is a random variable with σ being an entire orbit through x, and A is a random attractor with R being the corresponding repeller. Then we have:*
(i) if $x\in R$ almost surely, then $\Omega_\sigma\subset R$ and $\Omega^*_\sigma\subset R$ almost surely;
(ii) if $x\in B(A)\backslash A$ almost surely, then $\Omega_\sigma\subset A$ and $\Omega^*_\sigma\subset R$ almost surely;
(iii) if $x\in A$ almost surely, then $\Omega_\sigma\subset A$ almost surely; if $\Omega^*_\sigma\subset A$ almost surely, then σ lies on A almost surely, i.e. for arbitrary $t\in\mathbb{R}$, we have $\sigma_t\subset A$ almost surely;
(iv) if $x\in B(A)$ almost surely, then $\Omega_\sigma\subset A$ almost surely; if $x\in B(R):=S\backslash A$ almost surely, then $\Omega^*_\sigma\subset R$ almost surely.

Proof. (i) By the forward invariance of R, the former is obvious. By the forward invariance of $B(A)$ we obtain that all backward orbits through x must lie on R, so by the definition of Ω^*_σ we have $\Omega^*_\sigma\subset R$ almost surely.

(ii) The former follows directly from Lemma 4.3 in [20] (it is clear that Lemma 4.3 also holds for random semiflow). Assume that U is a forward invariant random open neighborhood of A relative to S such that $\Omega_U=A$ and let $V=S\backslash U$. For arbitrary random variable $y\in V$, let σ^y be a backward orbit through y. Then by the forward invariance of U, σ^y lies on V. Hence we have $\Omega^*_{\sigma^y}\subset V$ almost surely. If $\Omega^*_{\sigma^y}\not\subset R$ with positive probability, letting $R_1:=R\cup\Omega^*_{\sigma^y}$, then R_1 is an invariant random compact set by Lemma 2.10 and Remark 2.4 (iii). Then we can choose a random variable z such that $z\in R_1$ almost surely and $z\in R_1\backslash R$ with positive probability. On one hand, $\Omega_z\subset R_1$ almost surely by the invariance of R_1, which implies

$$\mathbb{P}-\lim_{t\to\infty} d(\varphi(t,\omega)z(\omega)|R_1(\theta_t\omega))=0.$$

On the other hand $z\in B(A)$ with positive probability, which implies that

$$\lim_{t\to\infty} d(\varphi(t,\cdot)z(\omega)|A(\theta_t\cdot))=0$$

with positive probability, a contradiction to the fact that $R_1\cap A=\emptyset$ almost surely. Therefore, for arbitrary random variable $y\in V$, we have $\Omega^*_{\sigma^y}\subset R$ almost surely.

Let $U_n := \varphi(n, \theta_{-n}\omega)U(\theta_{-n}\omega)$, $n \in \mathbb{N}$, then we have $U_{n+1} \subset U_n$ by the forward invariance of U. Moreover, each U_n is a forward invariant random open neighborhood of A relative to S and $\Omega_{U_n} = A$. Clearly we have

$$A(\omega) = \lim_{n\to\infty} U_n(\omega).$$

Letting $V_n = S \backslash U_n$, for arbitrary random variable $x \in V_n$, we have $\Omega^*_{\sigma^x} \subset R$ almost surely by the above argument. Since n is arbitrary, for arbitrary random variable in $S\backslash A$ with a backward orbit σ, we have $\Omega^*_\sigma \subset R$ almost surely. This completes the proof of (ii).

(iii) The former is trivial. Since A is forward invariant, any backward orbit through a random variable in $S\backslash A$ must lie on $S\backslash A$. If there exists some $t_0 \in \mathbb{R}$ such that $\mathbb{P}\{\omega|\sigma_{t_0}(\omega) \not\subset A(\theta_{t_0}\omega)\} = \delta > 0$, then for all $s \le t_0$ we have $\mathbb{P}\{\omega|\sigma_s(\omega) \not\subset A(\theta_s\omega)\} \ge \delta$, i.e. $\mathbb{P}\{\omega|\sigma_s(\omega) \subset S(\theta_s\omega)\backslash A(\theta_s\omega)\} \ge \delta$. Then by the proof of (ii) it follows that $\Omega^*_\sigma \subset R$ with positive probability, a contradiction to the fact that $\Omega^*_\sigma \subset A$ almost surely.

(iv) The former follows directly from (ii) and (iii), while the later has been proved during the proof of (ii). □

Remark 2.12. Similar to the proof of Lemma 2.11 (iii), if $\Omega_\sigma \subset R$ a.s., then $\sigma_t \in R$ a.s. for all $t \in \mathbb{R}$.

Definition 2.13. Assume that (A_i, R_i) are attractor-repeller pairs of φ on the invariant random compact set S with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = S \text{ and } S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset.$$

Then the family $D = \{M_i\}_{i=1}^n$ of invariant random compact sets, defined by

$$M_i = A_i \cap R_{i-1},\ 1 \le i \le n,$$

is called a *Morse decomposition of* S, and each M_i is called *Morse set*. If D is a Morse decomposition, M_D is defined to be $\bigcup_{i=1}^n M_i$.

Remark 2.14. (i) For $i \ne j$, say $i < j$, $M_i \cap M_j = A_i \cap R_{i-1} \cap A_j \cap R_{j-1} \subset A_i \cap R_{j-1} \subset A_i \cap R_i = \emptyset$. (ii) Each Morse set M_i is invariant, which is trivial if φ is a random flow, but requires explanation in the case of random semiflow. Clearly each M_i is forward invariant. For any point $x \in M_i(\omega) = A_i(\omega) \cap R_{i-1}(\omega)$, there exists a backward orbit σ in A_i by the backward invariance of A_i. By the forward invariance of $R^c_{i-1} = B(A_{i-1})$, any backward orbit through a point in R_{i-1} must lie on R_{i-1}. Hence we have σ lying on M_i, i.e. M_i is backward invariant.

Lemma 2.15. *Assume that A is an attractor in S with a forward invariant neighborhood $N \subset S$ and the basin of attraction $B(A) \subset S$. Then for arbitrary random closed set $K \subset B(A)$, there exists a measurable mapping $\omega \mapsto T_K(\omega) \ge 0$ such that, for $\mathbb{P}$-almost all $\omega \in \Omega$, we have*

$$\varphi(t,\omega)K(\omega) \subset N(\theta_t\omega), \qquad \forall t \ge T_K(\omega). \tag{2.1}$$

Proof. For given $K \subset B(A)$ and $\forall x \in K(\omega)$, there exists a $\tau(\omega, x)$ such that

$$\varphi(\tau(\omega, x), \omega)x \in \text{int}N(\theta_{\tau(\omega,x)}\omega)$$

by the definition of basin of attraction. So there exists an open neighborhood $U(\omega, x)$ of x such that

$$\varphi(\tau(\omega, x), \omega)U(\omega, x) \subset \text{int}N(\theta_{\tau(\omega,x)}\omega).$$

By the compactness of $K(\omega)$, there exists a finite collection of such neighborhoods suffice to cover $K(\omega)$. So there exists a $\tau(\omega, K)$ such that $\varphi(\tau(\omega, K), \omega)K(\omega) \subset \text{int}N(\theta_{\tau(\omega,K)}\omega)$.

Let

$$T_K(\omega) := \inf\{t \geq 0 | \varphi(t, \omega)K(\omega) \subset N(\theta_t\omega)\}.$$

By the above argument, $T_K(\omega)$ is finite. By the measurable selection theorem, we have

$$d(K(\omega)|N(\omega)) = \sup_n \inf_m d(x_n(\omega), y_m(\omega)),$$

where x_n, y_m are two sequences of random variables such that $K(\omega) = \overline{\{x_n(\omega), n \in \mathbb{N}\}}$ and $N(\omega) = \overline{\{y_m(\omega), m \in \mathbb{N}\}}$, respectively. Hence the mapping $\omega \mapsto d(K(\omega)|N(\omega))$ is measurable. For arbitrary $a \geq 0$, by the forward invariance of N, we have

$$\{\omega | T_K(\omega) \geq a\} = \bigcap_{s<a, s\in\mathbb{Q}} \{\omega | d(\varphi(s, \omega)K(\omega)|N(\theta_s\omega)) > 0\}.$$

So the mapping $\omega \mapsto T_K(\omega)$ is measurable. This completes the proof. □

Lemma 2.16. *Assume that A is a random attractor in S and $B(A) \subset S$ is the corresponding basin of attraction. Then for any random closed set $D \subset B(A)$, A pull-back attracts D.*

Proof. Assume that N is a forward invariant neighborhood of A, then by Lemma 2.15 we know that for any random closed set $D \subset B(A)$ there exists a measurable $T_D \geq 0$ such that, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\varphi(t, \omega)D(\omega) \subset N(\theta_t\omega), \qquad \forall t \geq T_D(\omega).$$

Hence for given non-random $\epsilon > 0$ and given non-random $k \in \mathbb{N}$, there exists non-random $k_\epsilon > 0$ such that $\mathbb{P}\{\omega | T_D(\theta_{-k}\omega) \leq k_\epsilon\} \geq 1 - \epsilon$. So we have

$$\begin{aligned}
&\mathbb{P}\{\omega | \varphi(t, \theta_{-t} \circ \theta_{-k}\omega)D(\theta_{-t} \circ \theta_{-k}\omega) \subset N(\theta_{-k}\omega), t \geq k_\epsilon\} \\
=&\mathbb{P}\{\omega | \varphi(t, \theta_{-k}\omega)D(\theta_{-k}\omega) \subset N(\theta_t \circ \theta_{-k}\omega), t \geq k_\epsilon\} \\
\geq&1 - \epsilon
\end{aligned}$$

by the measure preserving property of θ_t. Hence

$$\mathbb{P}\{\omega | \varphi(k, \theta_{-k}\omega) \circ \varphi(t, \theta_{-t} \circ \theta_{-k}\omega)D(\theta_{-t} \circ \theta_{-k}\omega)$$

$$\subset \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega), t \geq k_\epsilon\} \geq 1-\epsilon.$$

Therefore,

$$\mathbb{P}\{\omega | \overline{\bigcup_{t\geq k_\epsilon} \varphi(t+k,\theta_{-t-k}\omega)D(\theta_{-t-k}\omega)} \subset \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega)\} \geq 1-\epsilon.$$

By the definition of omega-limit set, we obtain that

$$\mathbb{P}\{\omega | \Omega_D(\omega) \subset \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega)\} \geq 1-\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\Omega_D \subset \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega)$ $\mathbb{P}$-a.s. Thus

$$\Omega_D(\omega) \subset \bigcap_{k\in\mathbb{N}} \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega) = \Omega_N(\omega) = A(\omega)$$

holds for $\mathbb{P}$-almost all $\omega \in \Omega$. Here $\bigcap_{k\in\mathbb{N}} \varphi(k,\theta_{-k}\omega)N(\theta_{-k}\omega) = \Omega_N(\omega)$ holds because N is forward invariant, which follows that

$$\varphi(t,\theta_{-t}\omega)N(\theta_{-t}\omega) \subset \varphi(s,\theta_{-s}\omega)N(\theta_{-s}\omega), \quad \forall t > s.$$

Note that Ω_D pull-back attracts D, so A also does. This completes the proof of the lemma. □

Lemma 2.17. *Assume that $A_1, A_2 \subset S$ are two random attractors with basins of attraction $B(A_1)$, $B(A_2)$, respectively. Assume that D is a random compact set satisfying $D \subset B(A_1) \cup B(A_2)$ almost surely. Then $A_1 \cup A_2$ pull-back attracts D.*

Proof. Choose a random compact set $D_1 \subset B(A_1)$ almost surely and choose $D_2 \subset B(A_2)$ almost surely such that $D \subset D_1 \cup D_2$ almost surely. Therefore, by (ii) and Lemma 2.16, we obtain for $\mathbb{P}$-almost all ω

$$\Omega_D(\omega) \subset \Omega_{D_1\cup D_2}(\omega) = \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega) \subset A_1(\omega) \cup A_2(\omega).$$

By the definition of omega-limit set, it is clear that Ω_D pull-back attracts D, so $A_1 \cup A_2$ pull-back attracts D. This completes the proof. □

Remark 2.18. It is obvious that the result of Lemma 2.17 holds for finite case, i.e. if the random compact set $D \subset \bigcup_{i=1}^n B(A_i)$ almost surely, then $\bigcup_{i=1}^n A_i$ pull-back attracts D.

Theorem 2.19. *Assume that $D = \{M_i\}_{i=1}^n$ is a Morse decomposition of S, determined by attractor-repeller pairs (A_i, R_i), $i = 1, \ldots, n$. Then M_D determines the limiting behavior of φ on S. Moreover, there are no "cycles" between the Morse sets. More precisely, we have:*

(i) *For any random variable x in S, there exists an entire orbit σ through x such that $\Omega_\sigma \subset M_D$ and $\Omega^*_\sigma \subset M_D$ almost surely.*

(ii) *If σ is an entire orbit through the random variable x satisfying that $\Omega_\sigma \subset M_p$ almost surely and $\Omega^*_\sigma \subset M_q$ almost surely for some $1 \le p, q \le n$, then $p \le q$; Moreover, $p = q$ if and only if σ lies on M_p.*

(iii) *If $\sigma_1, \ldots, \sigma_l$ are l entire orbits through the random varibales $x_1, \ldots, x_l$ respectively such that for some $1 \le j_0, \ldots, j_l \le n$, $\Omega_{\sigma_k} \subset M_{j_{k-1}}$ and $\Omega^*_{\sigma_k} \subset M_{j_k}$ for $k = 1, \ldots, l$, then $j_0 \le j_l$. Moreover, $j_0 < j_l$ if and only if σ_k does not lie on M_D with positive probability for some k, otherwise $j_0 = \cdots = j_l$.*

Proof. (i) Since

$$\emptyset = R^c_0 \subsetneq R^c_1 \subsetneq \cdots \subsetneq R^c_n = S,$$

let $\tilde{R}_i = R^c_i \backslash R^c_{i-1}$. Then $S = \bigcup_{i=1}^n \tilde{R}_i$ almost surely and $\tilde{R}_i = B(A_i)\backslash B(A_{i-1})$. Hence for arbitrary random variable in $\tilde{R}_i$, it is attracted by A_i but not by A_{i-1}. For arbitrary random variable x in S, choose n random variables $x_1, \ldots, x_n$ such that $x_i \in \tilde{R}_i$ almost surely and $x(\omega) = x_i(\omega)$ when $\omega \in \Omega_i$, where $\Omega_i := \{\omega | x(\omega) \in \tilde{R}_i(\omega)\}$, $i = 1, \ldots, n$. By the fact $x_i \in \tilde{R}_i = R^c_i \cap R_{i-1}$ we know that x_i is attracted by $A_i \cap R_{i-1} = M_i$ almost surely. Then by Lemma 2.17 we obtain for $\mathbb{P}$-almost all ω

$$\Omega_\sigma(\omega) = \Omega_x(\omega) \subset \bigcup_{i=1}^n \Omega_{x_i}(\omega) \subset \bigcup_{i=1}^n M_i(\omega) = M_D(\omega).$$

Since

$$S = A^c_0 \supsetneq A^c_1 \supsetneq \cdots \supsetneq A^c_n = \emptyset,$$

let $\tilde{A}_i = A^c_{i-1} \backslash A^c_i = B(R_{i-1}) \cap A_i$, $i = 1, \ldots, n$. Then $S = \bigcup_{i=1}^n \tilde{A}_i$ almost surely. By (iv) of Lemma 2.11, for given random variable $x \in \tilde{A}_i$, we have $\Omega^*_\sigma \subset R_{i-1}$ almost surely for *any* backward orbit σ through x. Since $x \in \tilde{A}_i \subset A_i$, by the invariance of A_i, *there exists* a backward orbit σ through x lying on A_i (we can not guarantee generally that any backward orbit through x must lie on A_i). Hence for this σ we have $\Omega^*_\sigma \subset A_i$ almost surely. Therefore, we have obtained that for any random variable $x \in \tilde{A}_i$, there exists a backward orbit σ through it such that $\Omega^*_\sigma \subset A_i \cap R_{i-1} = M_i$ almost surely. For arbitrary random variable $y \in S$, choose n random variables y_i, $i = 1, \ldots, n$ such that $y_i \in \tilde{A}_i$ almost surely and $y(\omega) = y_i(\omega)$ when $\omega \in \Omega_i$, where $\Omega_i := \{\omega | y(\omega) \in \tilde{A}_i(\omega)\}$. By above argument, for each i, there exists a backward orbit σ_i through y_i such that $\Omega^*_{\sigma_i} \subset M_i$ almost surely. "Attaching" the corresponding parts of these σ_i's together when y lies on $\tilde{A}_i$, we obtain a backward orbit σ through y. By the choice of σ, we have

$$\Omega^*_\sigma \subset \bigcup_{i=1}^n \Omega^*_{\sigma_i} \subset \bigcup_{i=1}^n M_i = M_D$$

almost surely as desired.

(ii) Since $\Omega_\sigma \subset M_p = A_p \cap R_{p-1}$ almost surely, we have $x \in A^c_{p-1}$ almost surely. By the fact that $\Omega^*_\sigma \subset M_q = A_q \cap R_{q-1}$ almost surely, we have σ lying on A_q almost surely by (iii) of

Lemma 2.11. In particular, $\sigma_0 = x \in A_q$ almost surely. Hence we have $x \in A^c_{p-1} \cap A_q$ almost surely. If $q < p$, then $A_q \subset A_{p-1}$, hence $A_q \cap A^c_{p-1} = \emptyset$ almost surely, a contradiction.

If σ lies on M_p, then we have $\Omega_\sigma, \Omega^*_\sigma \subset M_p$ almost surely by the fact that M_p is an invariant random compact set. That is, we must have $p = q$. Conversely, if $p = q$, the fact $\Omega_\sigma = \Omega_x \subset M_p = A_p \cap R_{p-1}$ implies $x \in R_{p-1}$ almost surely. It follows that σ lies on R_{p-1} since any backward orbit through a random variable in R_{p-1} must lie on it. $\Omega^*_\sigma \subset M_p = A_p \cap R_{p-1}$ implies that σ lies on A_p by (iii) of Lemma 2.11. So we have obtained that σ lies on $A_p \cap R_{p-1} = M_p$ almost surely.

(iii) follows from (ii) immediately. □

Remark 2.20. In (i) of Theorem 2.19, we obtain that, for given random variable x, there exists an entire orbit through x satisfying $\Omega_\sigma \subset M_D$ and $\Omega^*_\sigma \subset M_D$ almost surely. While in the deterministic case, any entire orbit has this property, see [1] for the flow case and [10] for the semiflow case. But their methods are not applicable here. We are not sure whether the similar result holds for random semiflow.

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