On critical cases of Sobolev's inequalities for Heisenberg groups

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Abstract: This paper deal with the problem of Sobolev imbedding in the critical cases. We prove some Trudinger-type inequalities on the whole Heisenberg group, extending to this context the Euclidean results by T. Ozawa. The procedue depend on optimal growth rate of Gagliardo-Nirenberg inequalities. We obtain the sharp constant for the Trudinger-type inequalities on the whole Heisenberg groups when the function is radial. Using these inequalities, the estimate of heat kernel and the Reisz transform, we obtain some Morrey's inequality and the Brezis-Gallouet-Wainger inequality.

Keywords: Heisenberg group;Sobolev's inequality;Brezis-Gallouet-Wainger inequality

1 Introduction

It is well known that the Sobolev space $H^{s,p}(\mathbb{R}^n)$ is continuously imbedding into $L^\infty(\mathbb{R}^n)$ provided $s>n/p$. The case when $s=n/p$ is called the critical exponent, which implies $H^{n/p,p}(\mathbb{R}^n)$ is not imbedded into $L^\infty(\mathbb{R}^n)$, but into $L^q(\mathbb{R}^n)$ for all q with $p\le q<\infty$. Recently, Ozawa ([1]) gave a precise investigation for this imbedding and obtained the following optimal growth rate as $q\to\infty$. Indeed, for every p with $1<p<\infty$, it holds

$$\|u\|_{L^q(\mathbb{R}^n)}\le C_{n,p}q^{1-1/p}\|u\|_{L^p(\mathbb{R}^n)}^{p/q}\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \qquad (1.1)$$

for all $u\in H^{n/p,p}(\mathbb{R}^n)$ and for all q with $p\le q<\infty$, where $C_{n,p}$ is a constant depending only on n and p, but not on q. Furthermore, as corollaries of (1.1), they obtain other inequalities which characterize the critical imbedding in the Sobolev space. That is, the estimate (1.1) yields the Trudinger type inequality and the Brezis-Gallouet-Wainger type inequality.

The aim of this note is to prove analogous inequality (1.1) on the Heisenberg group H_n, where the Laplace operator Δ is replaced by the Kohn's sublaplace Δ_H on H_n. We refer to Section 2 for a more detailed account on this terminology and background results of analysis on Heisenberg groups. To this end we have:

Theorem 1.1. Let $1<p<\infty$. Then there exists a constant $C_{Q,p}$ depending only on Q and p such that

$$\|u\|_{L^q(H_n)}\le C_{Q,p}q^{1-p}\|u\|_{L^p(H_n)}^{p/q}\|(-\Delta_H)^{\frac{Q}{2p}}u\|_{L^p(H_n)}^{1-p/q}, \qquad (1.2)$$

where $Q=2n+2$ is the homogenous dimension of H_n.

The Theorem 1.1 implies the following result.

Theorem 1.2. Let $1<p<\infty$. There exist positive constant α and $C>0$ such that for all

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$f \in H^{Q/p,p}(H_n)$ with $\|(-\Delta_H)^{\frac{Q}{2p}} u\|_{L^p(H_n)} \le 1$,

$$\int_{H_n} \left(\exp(\alpha |f(\xi)|^{p'}) - \sum_{0\le j<p-1;\, j\in\mathbb{N}} \frac{1}{j!} (\alpha |f(\xi)|^{p'})^j \right) d\xi \le C \|u\|^p_{L^p(H_n)},$$

where $p' = (p-1)/p$ is the Holder conjugate of p.

In the special case the optimal bound of α for which Trudinger's inequality holds is obtained. In fact, it is proved by W. Cohn and G. Lu ([2]) that there exist constants C such that for any domain $\Omega \subset H_n$, $|\Omega| < \infty$ and $f \in W_0^{1,Q}(\Omega)$, the horizontal Sobolev space on H_n, the following inequality holds:

$$\sup_{f\in W_0^{1,Q}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_Q \left(\frac{f(\xi)}{\|\nabla f\|_{L^Q(H_n)}} \right)^{Q'} \right) < C,$$

where $Q' = (Q-1)/Q$ is the Holder conjugate of Q and

$$\alpha_Q = Q \left(\frac{2\pi^n \Gamma(1/2)\Gamma((Q-1)/2)}{\Gamma(Q/2)\Gamma(n)} \right)^{Q'-1}. \qquad (1.3)$$

Following [1], we have the following result:

Theorem 1.3. For any $\alpha \in (0, \alpha_Q)$ there exists a constant $C > 0$ such that for all radial function $f \in C_0^\infty(H_n)$,

$$\int_{H_n} \exp\left(\alpha \left(\frac{|f(\xi)|}{\|\nabla f\|_{L^Q(H_n)}} \right)^{Q'} \right) - \sum_{j=0}^{Q-2} \frac{1}{j!} \left(\alpha \left(\frac{|f(\xi)|}{\|\nabla f\|_{L^Q(H_n)}} \right)^{Q'} \right)^j d\xi \le C \frac{\|f\|^Q_{L^Q(H_n)}}{\|\nabla f\|^Q_{L^Q(H_n)}}.$$

The inequality above is false when $\alpha \ge \alpha_Q$.

Finally, we obtain the following Brezis-Gallouet-Wainger inequality on the Heisenberg group.

Theorem 1.4. Let $1 < p < \infty$, $1 < q < \infty$ and $m > \max\{Q/q, 1\}$. Then there exits a constant C >0 such that for all $f \in H^{Q/p,p}(H_n) \cap H^{m,q}(H_n)$ with $\|f\|_{H^{Q/p,p}(H_n)} \le 1$,

$$\|f\|_{L^\infty} \le C(1 + \log(1 + \|(-\Delta_H)^{\frac{m}{2}} u\|_{L^q(H_n)}))^{\frac{1}{p'}}.$$

2 Notations and preliminaries

Let $H_n = (\mathbb{R}^{2n} \times \mathbb{R}, \circ)$ be the $(2n+1)$-dimensional Heisenberg group whose group structure is given by

$$(x,t) \circ (x',t') = (x + x', t + t' + 2\sum_{j=1}^{n} (x_{2j} x'_{2j-1} - x'_{2j-1} x_{2j})).$$

The vector fields

$$X_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2x_{2j} \frac{\partial}{\partial t}, \qquad X_{2j} = \frac{\partial}{\partial x_{2j}} - 2x_{2j-1} \frac{\partial}{\partial t}$$

($j=1,\cdots,n$) are left invariant and generate the Lie algebra of H_n. The Kohn's sub-Laplace on H_n is

$$\Delta_H=\sum_{j=1}^{2n}X_j^2=\sum_{j=1}^{2n}\frac{\partial^2}{\partial x_j^2}+4|x|^2\frac{\partial^2}{\partial t^2}+4\sum_{k=1}^{n}(x_{2j}\frac{\partial}{\partial x_{2j-1}}-x_{2j-1}\frac{\partial}{\partial x_{2j}})\frac{\partial}{\partial t}$$

and the subgradient is the $(2n)$-dimensional vector given by

$$\nabla_H=(X_1,\cdots,X_{2n})=\nabla_x+2\Lambda x\frac{\partial}{\partial t},$$

where $\nabla_x=(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{2n}})$ and Λ is a skew symmetric and orthogonal matrix given by

$$\Lambda=\mathrm{diag}(J_1,\cdots,J_n),\ J_1=\cdots=J_n=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The Sobolev space $H^{s,p}(H_n)$ on the Heisenberg group H_n is the completion of $C_0^\infty(H_n)$ under the norm $\|(I-\Delta_H)^{\frac{s}{p}}u\|_{L^p(H_n)}$ ([3]).

For each real number $\lambda>0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_\lambda(x,t)=(\lambda x,\lambda^2 t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q=2n+2$

is the homogenous dimension. For simplicity, we use the notation

$\lambda(x,t)=(\lambda x,\lambda^2 t)$.

The anisotropic dilation introduces a homogenous norm $\rho(x,t)=(|x|^4+t^2)^{\frac{1}{4}}$. For simplicity, we always write it ρ. A function f is said to be radial if $f(x,t)=f(\rho)$.If f is a radial function, we have, using the polar coordinates related to ρ (see e.g. [2])

$$\int_{H_n}f(x,t)dxdt=c_0\int_0^\infty f(\rho)\rho^{Q-1}d\rho \tag{2.1}$$

and

$$\int_{H_n}|\nabla_H f(x,t)|^Q\,dxdt=\int_{H_n}|f'(\rho)|^Q\frac{|x|^Q}{\rho^Q}dxdt=c_Q\int_0^\infty(f'(\rho))^Q\rho^{Q-1}d\rho, \tag{2.2}$$

where for $\beta>-2n$,

$$c_\beta=\frac{\omega_{2n-1}\Gamma(1/2)\Gamma((Q-2+\beta)/2)}{\Gamma((Q+\beta)/4)}$$

and ω_{2n-1} is the volume of unite sphere in $\mathbb{R}^{2n}$.

We call a curve $\gamma:[a,b]\to H_n$ a horizontal curve connecting two points $\xi,\eta\in H_n$ if $\gamma(a)=\xi$, $\gamma(b)=\eta$ and $\dot\gamma(s)\in\mathrm{span}\{X_1,\cdots,X_{2n}\}$ for all s. The Carnot-Caratheodory distance between ξ,η is defined as

$$d_{cc}(\xi,\eta)=\inf_\gamma\int_a^b\sqrt{\langle\dot\gamma(s),\dot\gamma(s)\rangle}ds$$

where the infimum is taken over all horizontal curves γ connecting ξ and η. It is known that any two points ξ,η on H_n can be joined by a horizontal curve of finite length and then d_{cc} is a metric on H_n. An important feature of this distance function is that the distance is left-invariant. For simplicity, we always write $d_{cc}(\xi)=d_{cc}(\xi,e)$, where $e=(0,0)$ the origin of is H_n. Given any $\xi=(x,t)\in H_n$, set $x^*=\dfrac{z}{d_{cc}(x,t)}$, $t^*=\dfrac{t}{d_{cc}(x,t)^2}$ and $\xi^*=(x^*,t^*)$. The polar coordinates on H_n associated with d_{cc} is the following (cf. [4]):

$$\int_{H_n} f(x,t)dxdt=\int_0^\infty\int_\Sigma f(\lambda(x^*,t^*))\lambda^{Q-1}d\sigma d\lambda,\ f\in L^1(H_n),$$

where $\Sigma=\{\xi\in H_n; d_{cc}(\xi)=1\}$ is the Heisenberg unit sphere.

Let P_h denote the heat kernel (that is, the integral kernel of $e^{h\Delta_H}$) on H_n. For convenience, we set $P_h(x,t)=P_h((x,t);e)$. It is well known that P_h has the form (cf. [5, 6, 7])

$$P_h(x,t)=\frac{1}{2(4\pi h)^{n+1}}\int_{\mathbb{R}}\exp\left(\frac{i\lambda t-|x|^2\lambda\coth\lambda}{4h}\right)\left(\frac{\lambda}{\sinh\lambda}\right)^n d\lambda. \tag{2.3}$$

The following global estimates for P_h can be found in [8]: for every $\varepsilon>0$, there exist constants $C_{Q,\varepsilon}>0$ such that

$$P_h(x,t)\le C_{Q,\varepsilon}\frac{1}{h^{n+1}}e^{-\frac{d_{cc}^2(x,t)}{(4+\varepsilon)h}}. \tag{2.4}$$

3 The proofs

Following [1], we prove the following useful lemma.

Lemma 3.1. There exists $C_{p,Q}$ depends only on p and Q such that for any q with $1\le p\le q<\infty$ and $f\in C_0^\infty(H_n)$,

$$\|u\|_{L^q(H_n)}\le C_{p,Q}q^{p'}\|(-\Delta_H)^{\frac{Q}{2}(\frac{1}{p}-\frac{1}{q})}u\|_{L^p(H_n)},$$

where $p'=(p-1)/p$.

Proof. Step 1. Consider the potential I_λ of order $\lambda\in(0,Q)$, defined by

$$(I_\lambda f)(\xi)=\int_{H_n} d_{cc}(\eta^{-1}\circ\xi)^{\lambda-Q}f(\eta)d\eta=(K_\lambda * f)(\xi).$$

Following [9], we decompose $K_\lambda=K_{\lambda,s}^{(0)}+K_{\lambda,s}^{(1)}$ for $s>0$, where

$$K_{\lambda,s}^{(0)}(\eta)=\begin{cases}K_\lambda(\eta), & d_{cc}(\eta)<s;\\ 0, & d_{cc}(\eta)\ge s.\end{cases}$$

We denote by $I_\lambda=I_\lambda^{(0)}(s)+I_\lambda^{(1)}(s)$ the corresponding decomposition of the potential. By polar coordinates and Young's inequality, for any p with $1\le p<\infty$,

$$\|I_\lambda^{(0)}(s)f\|_{L^p(H_n)}\le\|K_{\lambda,s}^{(0)}\|_{L^1(H_n)}\|f\|_{L^p(H_n)}=\frac{|\Sigma|}{\lambda}s^\lambda\|f\|_{L^p(H_n)}, \tag{3.1}$$

$$\| I_\lambda^{(1)}(s) f\|_{L^\infty(H_n)} \le \| K_{\lambda,s}^{(1)}\|_{L^{p'}(H_n)} \| f\|_{L^p(H_n)} = \left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{1}{p'}} s^{\lambda - \frac{Q}{p}} \| f\|_{L^p(H_n)}, \qquad (3.2)$$

where $|\Sigma| = \int_\Sigma d\sigma$, $1/q = 1/p - \lambda/Q$ and we set $\left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{1}{p'}} = 1$ when $p=1$ and $p' = \infty$.

Define, for $t>0$ and $k=0,1$,

$$J_\lambda^{(k)}(t) = I_\lambda^{(k)}\left(\left(\frac{2}{t}\left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{1}{p'}}\right)^{\frac{q}{Q}}\right).$$

Then I_λ has the decomposition $I_\lambda = J_\lambda^{(0)}(t) + J_\lambda^{(1)}(t)$. By (3.1) and (3.2), we have

$$\| J^{(0)}(t) f\|_{L^p(H_n)} \le \frac{|\Sigma|}{\lambda}\left(\frac{2}{t}\left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{1}{p'}}\right)^{\frac{q}{p}-1} \| f\|_{L^p(H_n)},.$$

$$\| I_\lambda^{(1)}(s) f\|_{L^\infty(H_n)} \le \frac{t}{2}\| f\|_{L^p(H_n)}.$$

Therefore, by Chebyshev's inequality, we obtain

$$|\{\xi \in H_n; |(I_\lambda f)(\xi)| > t\| f\|_{L^p(H_n)}\}|$$

$$\le \left|\{\xi \in H_n; |(J_\lambda^{(0)} f)(\xi)| > \frac{t}{2}\| f\|_{L^p(H_n)}\}\right| + \left|\{\xi \in H_n; |(J_\lambda^{(1)} f)(\xi)| > \frac{t}{2}\| f\|_{L^p(H_n)}\}\right|$$

$$= \left|\{\xi \in H_n; |(J_\lambda^{(0)} f)(\xi)| > \frac{t}{2}\| f\|_{L^p(H_n)}\}\right|$$

$$\le \left(\frac{|\Sigma|}{\lambda}\left(\frac{2}{t}\left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{1}{p'}}\right)^{\frac{q}{p}-1} \| f\|_{L^p(H_n)}\right)^p \left(\frac{t}{2}\| f\|_{L^p(H_n)}\right)^{-p}$$

$$= 2^q \left(\frac{|\Sigma|}{\lambda}\right)^p \left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{q-p}{p'}} t^{-q}$$

and hence

$$\sup_{t>0} t |\{\xi \in H_n; |(I_\lambda f)(\xi)| > t\}|^{\frac{1}{q}} = \sup_{t>0} t\| f\|_{L^p(H_n)} |\{\xi \in H_n; |(I_\lambda f)(\xi)| > t\| f\|_{L^p(H_n)}\}|^{\frac{1}{q}}$$

$$\le 2\left(\frac{|\Sigma|}{\lambda}\right)^{\frac{p}{q}} \left(\frac{|\Sigma| q}{Qp'}\right)^{\frac{q-p}{p'q}} \| f\|_{L^p(H_n)} = 2\left(\frac{|\Sigma| q}{Q}\right)^{1-\frac{1}{p}+\frac{1}{q}} \frac{(p-1)^{(p-1)(1/p-1/q)}}{p^{1-1/p-(2p-1)q}(q-p)^{p/q}} \| f\|_{L^p(H_n)}$$

whenever $1/q = 1/p - \lambda/Q$, $1 \le p < q < \infty$, $0 < \lambda < Q$.

Step 2. Fix $p \in (1,\infty)$. Define $\lambda = Q(1/p - 1/q)$, $1 < p < q < \infty$. Then

$0<\lambda<Q$ and $1<p<Q/\lambda$.

Set $(1/p_0,1/q_0)=(1,1-1/p+1/q)$ and $(1/p_1,1/q_1)=(1/p-1/q+1/(q+1),1/(q+1))$.

By (3.3), $I_{Q(1/p-1/q)}$ satisfies the following weak types $(1/p_0,1/q_0)$ and $(1/p_1,1/q_1)$ simultaneously:

$$\sup_{t>0} t\,|\{\xi\in H_n;|(I_\lambda f)(\xi)|>t\}|^{\frac{1}{q_0}}\le 2\left(\frac{|\Sigma|q_0}{Q(q_0-1)}\right)^{\frac{1}{q_0}}\|f\|_{L^1(H_n)};$$

$$\sup_{t>0} t\,|\{\xi\in H_n;|(I_\lambda f)(\xi)|>t\}|^{\frac{1}{q_1}}\le 2\left(\frac{|\Sigma|q_1}{Q}\right)^{1-\frac{1}{p_1}+\frac{1}{q_1}}\frac{(p_1-1)^{(p_1-1)(1/p_1-1/q_1)}}{p_1^{1-1/p_1-(2p_1-1)q_1}(q_1-p_1)^{p_1/q_1}}\|f\|_{L^{p_1}(H_n)}.$$

Set $\theta=\theta(q)=(1-1/p)/(1-1/p+1/q-1/(q+1))$. Then

$1<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, $1/p=(1-\theta)/p_0+\theta/p_1$,

$1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$.

Therefore, by Marcinkiewicz interpolation theorem,

$$I_{Q(1/p-1/q)}\le 4\left(q+\frac{pq}{q-p}\right)^{1/q}M_0(q)^{1-\theta}M_1(q)^{\theta}\|f\|_{L^{p_1}(H_n)},$$

where $M_0(q)=\left(\frac{|\Sigma|q_0}{Q(q_0-1)}\right)^{\frac{1}{q_0}}$

$$M_1(q)=\left(\frac{|\Sigma|q_1}{Q}\right)^{1-\frac{1}{p_1}+\frac{1}{q_1}}\frac{(p_1-1)^{(p_1-1)(1/p_1-1/q_1)}}{p_1^{1-1/p_1-(2p_1-1)q_1}(q_1-p_1)^{p_1/q_1}}.$$

Notice that

$$\lim_{q\to\infty}\theta(q)=1,\quad \lim_{q\to\infty}M_0(q)=\left(\frac{|\Sigma|p}{Q}\right)^{1-\frac{1}{p}},\lim_{q\to\infty}M_0(q)=\left(\frac{|\Sigma|(p-1)}{Qp}\right)^{1-\frac{1}{p}}.$$

We have, for any q with $p<q<\infty$,

$$\|I_{Q(1/p-1/q)}f\|_{L^q(H_n)}\le C_{Q,p}q^{1-1/p}\|f\|_{L^p(H_n)}\qquad(3.4)$$

with some positive constant $C_{Q,p}$ depending only on Q and p.

Step 3. By the following identity

$$(-\Delta_H)^{-\mu}=\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}e^{t\Delta_H}dt,$$

we have, using the estimates for P_h (see (2.2)),

$$|(-\Delta_H)^{Q/2(1/p-1/q)}f|(\eta)=\frac{1}{\Gamma(Q/2(1/p-1/q))}\left|\int_0^\infty h^{Q/2(1/p-1/q)-1}P_h*fdh\right|$$

$$\le\frac{C_{Q,\varepsilon}}{\Gamma(Q/2(1/p-1/q))}\left|\int_0^\infty h^{Q/2(1/p-1/q)-n-2}\int_{H_n}e^{-\frac{d_{cc}^2(\xi)}{(4+\varepsilon)h}}|f(\eta^{-1}\circ\xi)|d\xi dh\right|$$

$$=\frac{C_{Q,\varepsilon}}{\Gamma(Q/2(1/p-1/q))}\left|\int_{H_n}\int_0^{\infty} h^{Q/2(1/p-1/q)-n-2} e^{-\frac{d_{cc}^2(\xi)}{(4+\varepsilon)h}} \,|\, f(\eta^{-1}\circ\xi)\,|\, dh d\xi\right|$$

$$\leq C'_{Q,\varepsilon}(I_{Q(1/p-1/q)}|f|)(\xi).$$

Therefore,

$$\|(-\Delta_H)^{Q(1/p-1/q)} f\|_{L^q(H_n)} \leq C_{Q,\varepsilon} \| I_{Q(1/p-1/q)}|f|\|_{L^q(H_n)} \leq C_{Q,\varepsilon,p} q^{1-1/p} \| f\|_{L^p(H_n)}.$$

The desired result follows by choosing $\varepsilon\in(0,1/4)$.

Proof of Theorem 1.1 It has been proved by Xu C.-J. ([9], pp. 112) that there exists a positive constant $C_{Q,p}$, depending only on Q and p, such that the following Gagliardo - Niremberg inequalities holds for all $f\in C_0^{\infty}(H_n)$:

$$\|(-\Delta_H)^{\frac{\sigma(1-\theta)}{2}} f\|_{L^p(H_n)} \leq C_{Q,p}\|(-\Delta_H)^{\frac{\sigma}{2}} f\|_{L^p(H_n)}^{1-\theta}\| f\|_{L^p(H_n)}^{\theta}, \quad \theta\in(0,1). \tag{3.5}$$

The desired result follows from Lemma 3.1 and inequality (3.5).

Proof of Theorem 1.2 Recalling the results in Theorem 1.1, we can see that the proof is completely analogous to the context of $\mathbb{R}^n$ ([1]). These complete the proof of Theorem 1.2.

Proof of Theorem 1.3. Notice that if f is radial, then by Eq.(2.1)-(2.2),

$$\int_{H_n} f(x,t)dxdt = c_0\int_0^{\infty} f(\rho)\rho^{Q-1}d\rho = \frac{c_0}{\omega_{Q-1}}\int_{\mathbb{R}^Q} f(|x|)dx$$

and

$$\int_{H_n} |\nabla_H f(x,t)|^Q\, dxdt = c_Q\int_0^{\infty}(f'(\rho))^Q\rho^{Q-1}d\rho = \frac{c_Q}{\omega_{Q-1}}\int_{\mathbb{R}^Q}|\nabla f(|x|)|^Q\, dx,$$

Where ω_{Q-1} is the volume of unite sphere in $\mathbb{R}^Q$ (see [2]). The proof is just the same as in [10]. These complete the proof of Theorem 1.3.

Proof of Theorem 1.4. **Step 1**. We assume $0<m-Q/q<1$ and $m>1$. By Morrey's inequality (see e.g. [11])

$$|f(\xi)-f(\eta)|\leq C\|\nabla_H f\|_{L^{\frac{qQ}{q(1-m)+Q}}(H_n)} d_{cc}(\eta^{-1}\circ\xi)^{m-Q/q}.$$

Since the Reisz transform on the Heisenberg group is bounded (cf. [12]), we have

$$|f(\xi)-f(\eta)|\leq C\|(-\Delta_H)^{\frac{1}{2}} f\|_{L^{\frac{qQ}{q(1-m)+Q}}(H_n)} d_{cc}(\eta^{-1}\circ\xi)^{m-Q/q}.$$

By Sobolev inequality (cf. [5])

$$\|(-\Delta_H)^{\frac{1}{2}} f\|_{L^{\frac{qQ}{q(1-m)+Q}}(H_n)} \leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)},$$

we obtain the following Morrey's inequality

$$|f(\xi)-f(\eta)|\leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)} d_{cc}(\eta^{-1}\circ\xi)^{m-Q/q}.$$

Now let $0<\epsilon<e^{-p}$ and let $\tau\in H_n$ satisfying $d_{cc}(\tau)\leq 1$. Then,

$$|f(\xi)-f(\xi\circ(\epsilon\tau))|\leq C\epsilon^{\sigma}\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)},$$

Where $\sigma = m - Q/q \in (0,1)$. By Holder inequality and Theorem 1.1, there exists a constant $C_{Q,p}$ such that for any $r \geq p$,

$$\int_{d_{cc}(\tau)\leq 1} |f(\xi\circ(\epsilon\tau))|\,d\tau \leq \left(\int_{d_{cc}(\tau)\leq 1} d\tau\right)^{1-1/r} \left(\int_{d_{cc}(\tau)\leq 1} |f(\xi\circ(\epsilon\tau))|^r\,d\tau\right)^{1/r}$$

$$\leq \left(\int_{d_{cc}(\tau)\leq 1} d\tau\right)^{1-1/r} \epsilon^{-Q/r}\|f\|_{L^r(H_n)} \leq C_{Q,p}\epsilon^{-Q/r} r^{1-1/p}\|f\|_{H^{Q/p,p}(H_n)}$$

$$\leq C_{Q,p}\epsilon^{-Q/r} r^{1-1/p}.$$

Let $\epsilon = e^{-r}$. Then for any ϵ with $0<\epsilon<e^{-p}$ we have

$$\int_{d_{cc}(\tau)\leq 1} |f(\xi\circ(\epsilon\tau))|\,d\tau \leq C_{Q,p} e^Q r^{1-1/p} \leq C'_{Q,p}\left(-\log\epsilon\right)^{1-1/p}.$$

Following ([9]), we have

$$|f(\xi)| = \frac{1}{\int_{d_{cc}(\tau)\leq 1} d\tau}\int_{d_{cc}(\tau)\leq 1} |f(\xi)|\,d\tau$$

$$\leq C_Q \int_{d_{cc}(\tau)\leq 1} (|f(\xi) - f(\xi\circ(\epsilon\tau))| + |f(\xi\circ(\epsilon\tau))|)d\tau$$

$$\leq C_{Q,p}\epsilon^{\sigma}\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)} + C'_{Q,p}\left(-\log\epsilon\right)^{1-1/p}$$

The desired result follows by setting

$$\epsilon = \frac{1}{e^p + \|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)}^{1/\sigma}}.$$

Step 2. In the case $m - Q/q \geq 1$, we choose s satisfying $s - Q/q \in (0,1)$ and $s \geq 1$. Then $0 < s < m$. By step 1, we have

$$\|f\|_{L^\infty} \leq C(1+\log(1+\|(-\Delta_H)^{\frac{s}{2}} u\|_{L^q(H_n)}))^{\frac{1}{p'}}.$$

On the other hand, by Gagliardo-Niremberg inequalities on H_n ([9], pp. 112)

$$\|(-\Delta_H)^{\frac{s}{2}} f\|_{L^q(H_n)} \leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)}^{s/m}\|f\|_{L^q(H_n)}^{1-s/m}$$

$$\leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)}^{s/m}\|f\|_{H^{Q/p,p}(H_n)}^{1-s/m} \leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)} + C\|f\|_{H^{Q/p,p}(H_n)} \leq C\|(-\Delta_H)^{\frac{m}{2}} f\|_{L^q(H_n)} + C$$

The result now follows.

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Heisenberg 上临界情形的 Sobolev 不等式

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摘要：本文主要考虑在临界情形的 Sobolev 嵌入问题。我们证明了在整个 Heisenberg 群上存在一类 Trudinger 型不等式，这一点推广了由 T. Ozawa 所证明的欧式空间情形。过程依赖于对一类 Gagliardo-Nirenberg 不等式的最优估计。如果函数限制为仅仅依赖于范数的一元函数，我们得到了全空间 Heisenberg 群上 Trudinger 不等式的最优常数。利用上述不等式，相关热核估计以及 Reisz 变换的有界性，我们得到了一类 Morrey 不等式以及 Brezis-Gallouet-Wainger.

关键词：Heisenberg 群；Sobolev 不等式；Brezis-Gallouet-Wainger 不等式

中图分类号：O152.5；O178