一类肿瘤生长自由边界问题的解的正则性

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摘要：本文研究一类描述固体型肿瘤生长的自由边界问题。由于自由边界和表面张力的作用，该问题是一个含有非局部项的非线性问题。本文证明自由边界函数是时空变元的解析函数，即使初值只具有低正则性。

关键词：自由边界问题，肿瘤模型，正则性

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Regularity of solutions of a free boundary problem modeling the growth of solid tumors

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Abstract: In this paper we study a free boundary problem modeling the growth of solid tumors. Due to the free boundary and surface tension effects, this problem is a nonlinear problem involving non-local terms. We prove that the free boundary is real analytic in temporal and spatial variables, even if the given initial data admit less regularity.

Key words: Free boundary problem; tumor model; regularity.

0 Introduction

In this paper we study the regularity of solutions of the following free boundary problem

$$\begin{cases} \Delta\sigma = \tau_1\sigma + \beta & \text{in } \Omega(t),\ t>0, \\ \Delta\beta = \tau_2\beta & \text{in } \Omega(t),\ t>0, \\ \Delta p = -\eta(\sigma - \tilde{\sigma} - \iota\beta) & \text{in } \Omega(t),\ t>0, \\ \sigma = \bar{\sigma}, \beta = \bar{\beta} & \text{on } \Gamma(t),\ t>0, \\ p = \gamma\kappa & \text{on } \Gamma(t),\ t>0, \\ V = -\partial_{\hat{\nu}} p & \text{on } \Gamma(t),\ t>0, \\ \Gamma(0) = \Gamma_0 & \text{for } t=0. \end{cases} \quad (1)$$

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Here $\sigma = \sigma(t, x)$, $\beta = \beta(t, x)$ and $p = p(t, x)$ are unknown functions defined on the time-space manifold $\cup_{t\geq 0}(\{t\} \times \Omega(t))$, where $\Omega(t)$ is an *a priori* unknown time-dependent domain in $\mathbb{R}^n$ whose boundary, which we denote by $\Gamma(t)$, is free and has to be determined together with σ, β and p. The given initial data Γ_0 is a smooth closed hypersurface in $\mathbb{R}^n$ and encloses a bounded domain Ω_0 such that $\Omega(0) = \Omega_0$. In this model, Δ represents the Laplacian in the x-variable, V, κ and $\hat{\nu}$ denote the normal velocity, the mean curvature and the outward unit normal field, respectively, of the free boundary $\Gamma(t)$, and $\tau_1, \tau_2, \eta, \iota, \tilde{\sigma}, \bar{\sigma}, \bar{\beta}$ and γ are positive constants. The sign of κ is fixed on by the convention that $\kappa \geq 0$ at points where $\Gamma(t)$ is convex with respect to $\Omega(t)$.

This problem is a classical mathematical model describing the growth of solid tumors cultivated in laboratory [1, 2, 3]. In this model $\Omega(t)$ stands for the domain occupied by the tumor at time t, σ represents the nutrient concentration, β represents the inhibitor concentration, p denotes the internal pressure. The tumor region is regarded as a porous medium, so that Darcy's law and the law of conservation of mass yield the third equation in (1). The conditions $\sigma = \bar{\sigma}, \beta = \bar{\beta}$ on $\Gamma(t)$ mean that the tumor receives constant nutrient and inhibitor supply from the tumor surface, and the relation $p = \gamma\kappa$ reflects the cell-to-cell adhesiveness of the tumor. Finally, the equation $V = -\partial_{\hat{\nu}} p$ follows from the Stefan condition on $\Gamma(t)$.

The problem (1) has been well studied in the past a few years. More precisely, Cui and Escher considered the inhibitor-free case (i.e. $\beta = 0$) of this problem with general nonlinear terms, and studied existence of non-radial stationary solutions [4] and the stability of radial stationary solutions under small non-radial perturbation [5]. Then the work [6] extended the analysis given in [5] to the inhibitor-present case (i.e. $\beta \neq 0$). If the stationary diffusion equations $\Delta\sigma = \tau_1\sigma + \beta$ and $\Delta\beta = \tau_2\beta$ in (1) are replaced by their non-stationary versions, the resulting problem and its certain forms are also studied, cf. [7, 8, 9, 10, 11, 12, 13, 14].

In this paper our interest is to investigate the regularity of solutions of (1). We shall show that the free boundary is real analytic in time and space variables, even if the given initial data admit less regularity. This result is far from evident, by the fact that the system (1) is a nonlinear problem involving non-local terms. Our analysis relies on the employment of the functional analytic method and the theory of maximal regularity [15, 16, 17, 18], and some techniques developed in [19, 20].

To give a precise statement of our main result, we first introduce some notations. Given $m \in \mathbb{N}$, $\alpha \in (0, 1)$ and a bounded domain Ω in $\mathbb{R}^n$, we denote by $h^{m+\alpha}(\bar{\Omega})$ the so-called little Hölder space on Ω of index $m+\alpha$, i.e., the closure of $C^\infty(\bar{\Omega})$ in the usual Hölder space $C^{m+\alpha}(\bar{\Omega})$. Hereafter we shall fix $\alpha \in (0, 1)$. We use the notation C^ω to denote real analytic dependence. Assume that Γ_0 is a compact hypersurface in $\mathbb{R}^3$ of class $h^{3+\alpha}$. Let Γ_* be a compact embedded analytic hypersurface in $\mathbb{R}^3$ near Γ_0, such that Γ_0 is a $h^{3+\alpha}$-perturbation of Γ_* in the following sense: There exists a $h^{3+\alpha}$-function ρ_0 defined on Γ_*, with a sufficiently small C^1-norm, such that Γ_0 is the image of the mapping $x \mapsto x + \rho_0(x)\mathbf{n}(x)$, $x \in \Gamma_*$, where $\mathbf{n}$ denotes the outward unit normal field on Γ_*. Let Ω_0 and Ω_* be the bounded domain enclosed by Γ_0 and Γ_*, respectively. Ω_* will be used as the reference domain. In this paper, we identify a function $u : [0,T] \to C(\Gamma_*)$ with the corresponding function on $\Gamma_* \times [0,T]$ defined by $u(t,x) = u(t)(x)$ for $t \in [0,T]$ and $x \in \Gamma_*$. Similarly we identify a function $v : [0,T] \to C(\bar{\Omega}_*)$ with the corresponding function on $\bar{\Omega}_* \times [0,T]$ defined by $v(t,x) = v(t)(x)$ for $t \in [0,T]$ and $x \in \bar{\Omega}_*$. (σ,β,p,Γ) is called a solution of (1) (strict solution, in the sense of Lunardi [18]) if it satisfies:

(i) There exist $T > 0$ and $\rho \in C([0,T), h^{3+\alpha}(\Gamma_*)) \cap C^1([0,T), h^{\alpha}(\Gamma_*))$ such that the boundary $\Gamma(t)$ of $\Omega(t)$ is the image of the mapping $x \mapsto x + \rho(t,x)\nu(x)$, $x \in \Gamma_*$ for each $t \in [0,T)$.

(ii) There exists $\Theta \in C([0,T], h^{3+\alpha}(\Omega_*,\mathbb{R}^n)) \cap C^1([0,T), h^{\alpha}(\Omega_*,\mathbb{R}^n))$ such that $\Theta(t,\cdot) \in \mathrm{Diff}^{3+\alpha}(\Omega_*,\Omega(t))$ for each $t \in [0,T)$, and by writing $u(t,x) := \sigma(t,\Theta(t,x))$, $w(t,x) := \beta(t,\Theta(t,x))$ and $v(t,x) := p(t,\Theta(t,x))$, there holds $(u(t,\cdot), w(t,\cdot), v(t,\cdot)) \in h^{3+\alpha}(\bar{\Omega}_*) \times h^{3+\alpha}(\bar{\Omega}_*) \times h^{1+\alpha}(\bar{\Omega}_*)$ for each $t \in [0,T)$.

(iii) (σ,β,p,Γ) satisfies (1.1) pointwise.

Then our main result is formulated below:

Theorem 1. *Let Γ_0 be a compact hypersurface in $\mathbb{R}^n$ of class $h^{3+\alpha}$. Then the problem (1) has a unique solution (σ,β,p,Γ) on some time interval $[0,t^+)$ with $t^+ > 0$. Moreover, the time-space manifold $\bigcup_{t\in(0,t^+)}(\{t\} \times \Gamma(t))$ is real analytic, and $(\sigma(t,\cdot),\beta(t,\cdot),p(t,\cdot)) \in C^{\omega}(\bar{\Omega}(t)) \times C^{\omega}(\bar{\Omega}(t)) \times C^{\omega}(\bar{\Omega}(t))$ for each $t \in (0,t^+)$.*

The remainder of this paper is organized as follows. In the next section, we give local well-posedness of the problem (1). Section 2 aims at introducing a parameter-dependent mapping. In the last section, we give the proof of Theorem 1.

1 Local well-posedness

In this section, we establish local well-posedness of (1).

Let Γ_* be the compact embedded analytic hypersurface in $\mathbb{R}^n$ near Γ_0 introduced before. Recall that it encloses the reference domain Ω_*. Let $a_0 > 0$ and define

$$\mathcal{U} := \{\rho \in h^{2+\alpha}(\Gamma_*);\ \|\rho\|_{C^1(\Gamma_*)} < a_0\}.$$

For each $\rho \in \mathcal{U}$, we introduce a mapping

$$\theta_\rho : \Gamma_* \to \mathbb{R}^n, \quad \theta_\rho(x) := x + \rho(x)\nu(x),$$

where ν stands for the outward unit normal field of Γ_*. For each $\rho \in \mathcal{U}$, define an embedded hypersurface Γ_ρ in $\mathbb{R}^n$ by $\Gamma_\rho := \mathrm{im}(\theta_\rho) = \{\theta_\rho(x);\ x \in \Gamma_*\}$. It is not difficult to see that the operator θ_ρ is near the identity and $\theta_\rho \in \mathrm{Diff}^{2+\alpha}(\Gamma_*, \Gamma_\rho)$ provided $a_0 > 0$ is small enough which is assumed to be satisfied later on. Noticing that Γ_* is of class C^ω, one can easily find that

$$[\rho \mapsto \theta_\rho] \in C^\omega(h^{m+\alpha}(\Gamma_*) \cap \mathcal{U},\, (h^{m+\alpha}(\Gamma_*))^n), \;\; m \in \mathbb{N} \tag{2}$$

for sufficiently small $a_0 > 0$. We denote by Ω_ρ the bounded domain enclosed by Γ_ρ. Let $\pi \in \mathcal{L}(h^{m+\alpha}(\Gamma_*), h^{m+\alpha}(\bar{\Omega}_*))$ denote the right inverse of the trace operator $\mathrm{tr}(u) = u|_{\Gamma_*}$ which can be defined as follows: Given $\varphi \in h^{m+\alpha}(\Gamma_*)$, define $\pi(\varphi) := u$, where $u \in h^{m+\alpha}(\bar{\Omega}_*)$ is the solution of the elliptic boundary value problem

$$\Delta u = 0 \;\text{ in } \Omega_*, \quad u = \varphi \;\text{ on } \Gamma_*.$$

It is obvious that $\mathrm{tr}(\pi(\varphi)) = \varphi$ for $\varphi \in h^{m+\alpha}(\Gamma_*)$. Given $\rho \in \mathcal{U}$, define an operator

$$\Theta_\rho : \Omega_* \to \Omega_\rho, \quad \Theta_\rho := \mathrm{Id}_{\Omega_*} + \pi(\theta_\rho - \mathrm{Id}_{\Gamma_*}).$$

It can be verified that Θ_ρ is near the identity and $\Theta_\rho \in \mathrm{Diff}^{2+\alpha}(\Omega_*, \Omega_\rho)$ for sufficiently small a_0. Moreover, it follows from (2) that

$$[\rho \mapsto \Theta_\rho] \in C^\omega(h^{m+\alpha}(\Gamma_*) \cap \mathcal{U},\, (h^{m+\alpha}(\bar{\Omega}_*))^n), \;\; m \in \mathbb{N}, \tag{3}$$

provided $a_0 > 0$ is small enough. It is obvious that $\Theta_\rho|_{\Gamma_*} = \theta_\rho$.

The corresponding pull-back and push-forward operators induced by Θ_ρ will be denoted by Θ_ρ^* and Θ_*^ρ, respectively, i.e.,

$$\Theta_\rho^* u := u \circ \Theta_\rho \;\text{ for } u \in C(\bar{\Omega}_\rho), \qquad \Theta_*^\rho v := v \circ \Theta_\rho^{-1} \;\text{ for } v \in C(\bar{\Omega}_*).$$

Given $\rho \in \mathcal{U}$, we define the transformed differential operators $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ by

$$\mathcal{A}(\rho)u := \Theta_\rho^* \Delta(\Theta_*^\rho u), \quad \mathcal{B}(\rho)u := \theta_\rho^* \langle \Upsilon_1 \nabla(\Theta_*^\rho u),\, \hat{\nu}\rangle \qquad \text{for } u \in C^2(\bar{\Omega}_*),$$

where Υ_1 stands for the trace operator on Γ_ρ, $\hat{\nu}$ represents the outward normal field on Γ_ρ, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. It follows from lemma 2.2 in [20] that

$$\begin{aligned} &[\rho \mapsto \mathcal{A}(\rho)] \in C^\omega(h^{m+2+\alpha}(\Gamma_*) \cap \mathcal{U},\, \mathcal{L}(h^{m+2+\alpha}(\bar{\Omega}_*), h^{m+\alpha}(\bar{\Omega}_*))), \\ &[\rho \mapsto \mathcal{B}(\rho)] \in C^\omega(h^{m+2+\alpha}(\Gamma_*) \cap \mathcal{U},\, \mathcal{L}(h^{m+2+\alpha}(\bar{\Omega}_*), h^{m+1+\alpha}(\Gamma_*))) \end{aligned} \tag{4}$$

for $m \in \mathbb{N}$ and sufficiently small $a_0 > 0$, where $\mathcal{L}(Z_1, Z_0)$ denotes the Banach space of all linear continuous mappings from the Banach space Z_1 to the Banach space Z_0. We introduce the transformed mean curvature operator $\mathcal{N}$ by $\mathcal{N}(\rho) := \theta_\rho^* \kappa$, where κ is the mean curvature of the hypersurface Γ_ρ. From Lemma 3.1 in [20] we know that there exist

$$(\mathcal{P}_1, \mathcal{P}_2) \in C^\omega(h^{m+1+\alpha}(\Gamma_*) \cap \mathcal{U},\, \mathcal{L}(h^{m+2+\alpha}(\Gamma_*), h^{m+\alpha}(\Gamma_*)) \times h^{m+\alpha}(\Gamma_*))$$

such that

$$\begin{aligned}&\mathcal{N}\in C^\omega(h^{m+2+\alpha}(\Gamma_*)\cap\mathcal{U},\,h^{m+\alpha}(\Gamma_*)),\quad m\in\mathbb{N},\\&\mathcal{N}(\rho)=\mathcal{P}_1(\rho)\rho+\mathcal{P}_2(\rho)\qquad\text{for}\ \ \rho\in h^{m+2+\alpha}(\Gamma_*)\cap\mathcal{U}.\end{aligned}\tag{5}$$

With these preparations we can transform the free boundary problem (1) to a new system defined on the fixed reference domain Ω_*. For this, we define $\mathcal{O}:=h^{3+\alpha}(\Gamma_*)\cap\mathcal{U}$. Let $T>0$ be given and consider a function $\rho\in C([0,T],\mathcal{O})\cap C^1([0,T],h^\alpha(\Gamma_*))$. Denote $\Gamma(t):=\Gamma_{\rho(t)}$ and $\Omega(t):=\Omega_{\rho(t)}$ for $t\in[0,T]$. Using these notations and denoting $u(t):=\Theta^*_\rho\sigma(t,\cdot)$, $w(t):=\Theta^*_\rho\beta(t,\cdot)$ and $v(t):=\Theta^*_\rho p(t,\cdot)$, we see that the problem (1) is converted into the following problem:

$$\begin{cases}\mathcal{A}(\rho)u=\tau_1u+w & \text{in }\Omega_*,\ t>0,\\ \mathcal{A}(\rho)w=\tau_2w & \text{in }\Omega_*,\ t>0,\\ \mathcal{A}(\rho)v=-\eta(u-\tilde\sigma-\iota w) & \text{in }\Omega_*,\ t>0,\\ u=\bar\sigma,\ w=\bar\beta & \text{on }\Gamma_*,\ t>0,\\ v=\gamma\mathcal{N}(\rho) & \text{on }\Gamma_*,\ t>0,\\ \partial_t\rho=-\mathcal{B}(\rho)v & \text{on }\Gamma_*,\ t>0,\\ \rho(0)=\rho_0 & \text{for }t=0,\end{cases}\tag{6}$$

where ρ_0 is the function introduced before to define the initial data Γ_0.

In the following, we fuse the system (6) into an evolution equation containing the unknown ρ merely. To do so, let $\rho\in\mathcal{O}$ be given and firstly consider the elliptic boundary value problem

$$\mathcal{A}(\rho)w=\tau_2w\ \text{ in }\Omega_*,\quad w=\bar\beta\ \text{ on }\Gamma_*.\tag{7}$$

By the theory of elliptic PDEs and the perturbation theory for operators we know that (7) has a unique solution $w\in h^{3+\alpha}(\bar\Omega_*)$ depending on ρ, which we denote by $w=\mathcal{Q}_1(\rho)$. By defining

$$\mathbb{K}:\mathcal{O}\times h^{3+\alpha}(\bar\Omega_*)\to h^{1+\alpha}(\bar\Omega_*)\times h^{3+\alpha}(\Gamma_*),\quad \mathbb{K}(\rho,w):=(\mathcal{A}(\rho)w-\tau_2w,\Upsilon_0w-\bar\beta)$$

and by using the implicit function theorem and the analytic dependence of $\mathbb{K}(\rho,w)$ on (ρ,w), we can prove

$$[\rho\mapsto\mathcal{Q}_1(\rho)]\in C^\omega(\mathcal{O},\,h^{3+\alpha}(\bar\Omega_*))\cap C^\omega(\mathcal{U},\,h^{2+\alpha}(\bar\Omega_*)).\tag{8}$$

Next, we consider the problem

$$\mathcal{A}(\rho)u=\tau_1u+\mathcal{Q}_1(\rho))\ \text{ in }\Omega_*,\quad u=\bar\sigma\ \text{ on }\Gamma_*,\tag{9}$$

where we have replaced w with $\mathcal{Q}_1(\rho)$. Following the same step we see that the problem (9) has a unique solution $u=\mathcal{Q}(\rho)$ satisfying

$$[\rho\mapsto\mathcal{Q}(\rho)]\in C^\omega(\mathcal{O},\,h^{3+\alpha}(\bar\Omega_*))\cap C^\omega(\mathcal{U},\,h^{2+\alpha}(\bar\Omega_*)).\tag{10}$$

Then we consider the following elliptic boundary value problem

$$\mathcal{A}(\rho)v = -\eta(\mathcal{Q}(\rho) - \tilde{\sigma} - \iota\mathcal{Q}_1(\rho)) \text{ in } \Omega_*, \quad v = \gamma\mathcal{N}(\rho) \text{ on } \Gamma_*, \tag{11}$$

where we have replaced u, w with $\mathcal{Q}(\rho)$ and $\mathcal{Q}_1(\rho)$. Given $\rho \in \mathcal{O}$, we introduce two operators $\mathcal{T}_1(\rho)$ and $\mathcal{T}_2(\rho)$ by defining $v_1 = \mathcal{T}_1(\rho)h_1$ and $v_2 = \mathcal{T}_2(\rho)h_2$ to be solutions of the problems

$$\begin{cases} \mathcal{A}(\rho)v_1 = 0 & \text{in } \Omega_*, \\ v_1 = h_1 & \text{on } \Gamma_* \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{A}(\rho)v_2 = h_2 & \text{in } \Omega_*, \\ v_2 = 0 & \text{on } \Gamma_*. \end{cases}$$

We use the theory of elliptic PDEs to get a unique solution of (11), which is written as

$$v = \gamma\mathcal{T}_1(\rho)\mathcal{N}(\rho) - \mathcal{T}_2(\rho)\eta(\mathcal{Q}(\rho) - \tilde{\sigma} - \iota\mathcal{Q}_1(\rho)).$$

Arguing as above one can prove that

$$\begin{aligned} &[\rho \mapsto \mathcal{T}_1(\rho)] \in C^\omega(h^{m+2+\alpha}(\Gamma_*) \cap \mathcal{U},\, \mathcal{L}(h^{m+1+\alpha}(\Gamma_*), h^{m+1+\alpha}(\bar{\Omega}_*))), \\ &[\rho \mapsto \mathcal{T}_2(\rho)] \in C^\omega(h^{m+2+\alpha}(\Gamma_*) \cap \mathcal{U},\, \mathcal{L}(h^{m+\alpha}(\bar{\Omega}_*), h^{m+2+\alpha}(\bar{\Omega}_*))), \quad m \in \mathbb{N}. \end{aligned} \tag{12}$$

We introduce two mappings

$$\begin{aligned} &R(\rho) := \gamma\mathcal{B}(\rho)\mathcal{T}_1(\rho)\mathcal{P}_2(\rho) - \mathcal{B}(\rho)\mathcal{T}_2(\rho)\eta(\mathcal{Q}(\rho) - \tilde{\sigma} - \iota\mathcal{Q}_1(\rho)), \\ &\Phi(\rho) := \gamma\mathcal{B}(\rho)\mathcal{T}_1(\rho)\mathcal{P}_1(\rho) \qquad \text{for } \rho \in \mathcal{U}. \end{aligned} \tag{13}$$

It follows from (2)–(5), (8), (10), (12), (13) and the fact that the composition of analytic mappings is also analytic that

$$[\rho \mapsto \Phi(\rho)] \in C^\omega(\mathcal{U},\, \mathcal{L}(h^{3+\alpha}(\Gamma_*), h^\alpha(\Gamma_*))), \quad [\rho \mapsto R(\rho)] \in C^\omega(\mathcal{U},\, h^\alpha(\Gamma_*)). \tag{14}$$

By writing

$$\Psi(\rho) := \Phi(\rho)\rho + R(\rho) \quad \text{for } \rho \in \mathcal{O}, \tag{15}$$

we see that the system (6) is fused into the following evolution equation for the unknown ρ:

$$\frac{d\rho}{dt} + \Psi(\rho) = 0, \quad \rho(0) = \rho_0. \tag{16}$$

Summarizing the above deductions we get:

Lemma 1. *The problem* (16) *is equivalent to the problem* (6)*. Moreover, the nonlinear mapping* Ψ *introduced in* (15) *satisfies*

$$[\rho \mapsto \Psi(\rho)] \in C^\omega(\mathcal{O},\, h^\alpha(\Gamma_*)). \tag{17}$$

We shall treat the problem (16) as a fully nonlinear evolution equation and establish the maximal regularity for the linearization in the sense of Da Prato and Grisvard [17]. Due to this point, given $T > 0$, set $I := [0, T]$ and

$$E_\Upsilon := h^{3+\alpha}(\Gamma_*), \;\; \mathbb{E}_0(I) := C([0,T], h^\alpha(\Gamma_*)), \;\; \mathbb{E}_1(I) := C([0,T], h^{3+\alpha}(\Gamma_*)) \cap C^1([0,T], h^\alpha(\Gamma_*)).$$

Write Υ for the (temporal) trace operator in $\mathbb{E}_1(I)$, i.e.

$$\Upsilon:\ \mathbb{E}_1(I) \to E_\Upsilon, \quad u \mapsto u(0).$$

Let X_0 and X be Banach spaces such that X_0 is continuously injected and dense in X. Denote by $\mathcal{H}(X_0, X)$ the subset of all $A \in \mathcal{L}(X_0, X)$ such that $-A$, considered as an unbounded operator on X, generates a strongly continuous analytic semigroup on X. Write $\Psi'(\rho)$ for the Fréchet derivative of Ψ at ρ. Let $\mathcal{L}_{is}(X, Y)$ represent the set of all bounded isomorphisms from the Banach space X into the Banach space Y. We have the following result:

Lemma 2. *Let $\rho \in \mathcal{O}$ be given. Then $(\mathbb{E}_1(I), \mathbb{E}_0(I))$ is a pair of maximal regularity for $\Psi'(\rho)$, that is,*

$$(\frac{d}{dt} + \Psi'(\rho), \Upsilon) \in \mathcal{L}_{is}(\mathbb{E}_1(I),\ \mathbb{E}_0(I) \times E_\Upsilon), \quad \rho \in \mathcal{O}.$$

Proof. Given $\rho \in \mathcal{O}$, it follows from (15) that

$$\Psi'(\rho)\xi = \Phi(\rho)\xi + [\Phi'(\rho)\xi]\rho - R'(\rho)\xi \quad \text{for } \xi \in h^{3+\alpha}(\Gamma_*).$$

Noticing $\gamma > 0$, arguing similarly as in the proof of Theorem 4.1 in [20] we can prove

$$\Phi(\rho) \in \mathcal{H}(h^{3+\alpha}(\Gamma_*), h^\alpha(\Gamma_*)), \quad \rho \in \mathcal{O}.$$

On the other hand, from (14) we know that

$$[\Phi'(\rho)\cdot\,]\rho \in \mathcal{L}(\mathcal{U}, h^\alpha(\Gamma_*)), \ \ R'(\rho) \in \mathcal{L}(\mathcal{U}, h^\alpha(\Gamma_*)), \quad \rho \in \mathcal{O}.$$

It follows from the well-known perturbation result of generators (cf. Section 2.4 in [18]) that

$$\Psi'(\rho) \in \mathcal{H}(h^{3+\alpha}(\Gamma_*), h^\alpha(\Gamma_*)), \ \ \rho \in \mathcal{O}.$$

Combining this with the fact that little Hölder spaces are stable under continuous interpolation method $(\cdot,\cdot)^0_{\theta,\infty}, \theta \in (0,1)$ of Da Prato and Grisvard (cf. [16, 17, 18]), we get the assertion. □

Then we obtain the following local well-posedness of (16) (see also Theorem 2.7 in [16]):

Theorem 2. *Given $\rho_0 \in \mathcal{O}$, there exist $t^+ := t^+(\rho_0) > 0$ and a unique maximal solution*

$$\rho := \rho(\cdot, \rho_0) \in C([0, t^+), \mathcal{O}) \cap C^1([0, t^+), h^\alpha(\Gamma_*))$$

of the problem (16). The map $(t, \rho_0) \mapsto \rho(t, \rho_0)$ defines a local smooth semiflow on $\mathcal{O}$.

2 Transformation

In this section we introduce a parameter-dependent transformation and study its smoothing.

Recall that Γ_* is a compact embedded analytic hypersurface in $\mathbb{R}^n$. We denote by $\mathcal{V}^\omega(\Gamma_*)$ the vector space of all real analytic vector fields on Γ_*. For each point $x \in \Gamma_*$, let $T_x\Gamma_*$ stand for the tangent space of Γ_* at x. We rely on the following result of Escher and Prokert [19]:

Lemma 3. *([19]) There exist an integer $N \in \mathbb{N}$ and a mapping*

$$\Pi \in C^{\omega}(\mathbb{R}^N \times \mathbb{R} \times \Gamma_*, \Gamma_*) \tag{18}$$

satisfying the following properties:

$$\Pi(\mu, t, \cdot) \in \mathrm{Diff}^{\omega}(\Gamma_*) \quad \textit{for } (\mu, t) \in \mathbb{R}^N \times \mathbb{R}, \tag{19}$$

$$\left\{V_\mu(x) := \frac{\partial}{\partial t}\Pi(\mu, t, x)|_{t=0};\, \mu \in \mathbb{R}^N\right\} = T_x\Gamma_* \quad \textit{for } x \in \Gamma_*, \tag{20}$$

$$[\mu \mapsto V_\mu(\cdot)] \in \mathrm{Hom}(\mathbb{R}^N, \mathcal{V}^{\omega}(\Gamma_*)). \tag{21}$$

In the following, we will convert the problem (16) into a new problem with the help of the mapping Π. For this, like in Section 1, given $(\mu, t) \in \mathbb{R}^N \times \mathbb{R}$, we denote by $\Pi(\mu, t, \cdot)^*$ and $\Pi(\mu, t, \cdot)_*$ the pull-back and push-forward operators induced by the mapping $\Pi(\mu, t, \cdot)$, i.e.,

$$\Pi(\mu, t, \cdot)^* u := u \circ \Pi(\mu, t, \cdot), \quad \Pi(\mu, t, \cdot)_* v := v \circ \Pi^{-1}(\mu, t, \cdot) \quad \text{for } u, v \in C(\Gamma_*).$$

Given $(\mu, x) \in \mathbb{R}^N \times \Gamma_*$, from the proof of Lemma 3.1 in [19] we know that $\Pi(\mu, \cdot, x)$ is the unique global solution to the initial value problem

$$z'(t) = V_\mu(z), \quad z(0) = x. \tag{22}$$

Given $(\mu, t) \in \mathbb{R}^N \times \mathbb{R}$, define an operator $S_\mu(t):\ h^{3i+\alpha}(\Gamma_*) \to h^{3i+\alpha}(\Gamma_*), i = 0, 1$, by

$$S_\mu(t)v := \Pi(\mu, t, \cdot)^* v \quad \text{ for } v \in h^{3i+\alpha}(\Gamma_*), i = 0, 1.$$

Our next lemma shows that $[t \to S_\mu(t)]$ is a strongly continuous group on $h^{3i+\alpha}(\Gamma_*), i = 0, 1$. Then the infinitesimal generator of $\{S_\mu(t);\, t \in \mathbb{R}\}$ on $h^{\alpha}(\Gamma_*)$ will be denoted by D_μ. It follows from [21] that D_μ is a closed operator on $h^{\alpha}(\Gamma_*)$. Thus its domain $\mathrm{dom}(D_\mu)$, endowed with the graph norm of D_μ, is a well-defined Banach space.

Lemma 4. *Let $S_\mu(t)$ and D_μ be defined as above. Then*

(i) Given $\mu \in \mathbb{R}^N$, $[t \to S_\mu(t)]$ is a strongly continuous group on $h^{3i+\alpha}(\Gamma_*), i = 0, 1$.

(ii) $h^{1+\alpha}(\Gamma_*) \hookrightarrow \mathrm{dom}(D_\mu)$ and $D_\mu u(x) = T_x u V_\mu(x)$ for $x \in \Gamma_*$ and $u \in h^{3+\alpha}(\Gamma_*)$.

(iii) $[(\mu, u) \to D_\mu u] \in \mathcal{L}^2(\mathbb{R}^N \times h^{3+\alpha}(\Gamma_*), h^{\alpha}(\Gamma_*))$.

Proof. (i) Observe the fact that $\Pi(\mu, \cdot, \cdot)$ is a flow on Γ_*, so that the group properties of $S_\mu(t)$ follow readily. It remains to show that $S_\mu(t)$ is strongly continuous on $h^{3i+\alpha}(\Gamma_*), i = 0, 1$. This can be easily verified with the help of the density of $C^\infty(\Gamma_*)$ in $h^{3i+\alpha}(\Gamma_*), i = 0, 1$.

(ii) A elementary calculation shows that

$$\frac{d}{dt}u(\Pi(\mu, t, x))|_{t=0} = T_x u \frac{\partial}{\partial t}\Pi(\mu, t, x)|_{t=0} = T_x u V_\mu(x), \quad x \in \Gamma_*, u \in h^{1+\alpha}(\Gamma_*),$$

where $T_x u$ denotes the derivative of u in the tangent space $T_x\Gamma_*$. If $u \in h^{1+\alpha}(\Gamma_*)$ then

$$\|\frac{S_\mu(t)u - u}{t}\|_{h^\alpha(\Gamma_*)} = \|\frac{u\circ\Pi(\mu,t,\cdot) - u\circ\Pi(\mu,0,\cdot)}{t}\|_{h^\alpha(\Gamma_*)} \le C\|Du\|_{h^\alpha(\Gamma_*)},$$

so that $\|T_x u V_\mu(x)\|_{h^\alpha(\Gamma_*)} \le C\|Du\|_{h^\alpha(\Gamma_*)}$. Thus we have

$$D_\mu u(x) = T_x u V_\mu(x) \quad \text{for } x \in \Gamma_*, u \in h^{1+\alpha}(\Gamma_*), \tag{23}$$

$$h^{3+\alpha}(\Gamma_*) \hookrightarrow h^{1+\alpha}(\Gamma_*) \hookrightarrow \mathrm{dom}(D_\mu) \hookrightarrow h^\alpha(\Gamma_*). \tag{24}$$

(iii) From the inclusion (24) we know that $[w \to D_\mu w] \in \mathcal{L}(h^{3+\alpha}(\Gamma_*), h^\alpha(\Gamma_*))$ for any $\mu \in \mathbb{R}^N$. Combining this with (21) and (23) we get (iii). $\square$

Let $\rho_0 \in \mathcal{O}$ be given and let $\rho = \rho(\cdot,\rho_0)$ be the unique maximal solution to the problem (16), guaranteed by Theorem 2. By subdividing the interval $[0,t^+)$ we may assume without loss of generality that $t^+ \le 1$. Hereafter we consider the solution on the interval $I := [0,T]$ with fixed $T \in (0,t^+)$. Let $B_{\mathbb{R}^N}(0,r_0)$ denote the ball of radius r_0 centered at the origin of $\mathbb{R}^N$, where $r_0 > 0$ is small enough and is assumed to be satisfied later on. Since $\mathrm{dist}(\rho[0,T],\partial\mathcal{O}) > 0$, we know that there exist an open neighborhood $\tilde{\mathcal{O}} \subset \mathcal{O}$ of $\rho[0,T]$ and $r_0 > 0$ such that $S_\mu(t)\tilde{\mathcal{O}} \subset \mathcal{O}$ and $S_\mu(t)(\rho[0,T]) \subset \tilde{\mathcal{O}}$ for all $\mu \in B_{\mathbb{R}^N}(0,r_0), t \in I$. It follows from the compactness of I and the fact that $\tilde{\mathcal{O}}$ is open in $h^{3+\alpha}(\Gamma_*)$ that $\mathbb{O}(I) := C([0,T],\tilde{\mathcal{O}}) \cap C^1([0,T],h^\alpha(\Gamma_*))$ is an open subset of $\mathbb{E}_1(I)$. Choose $\varepsilon_0 > 0$ sufficiently small such that $\lambda t \in [0,t^+)$ for $t \in I$ and $\lambda \in (1-\varepsilon_0, 1+\varepsilon_0)$. Given $(\lambda,\mu) \in (1-\varepsilon_0,1+\varepsilon_0) \times B_{\mathbb{R}^N}(0,r_0)$, define

$$\rho_{\lambda,\mu}(t) := S_\mu(t)\rho(\lambda t) = \Pi(\mu,t,\cdot)^*\rho(\lambda t) = \rho(\lambda t, \Pi(\mu,t,\cdot)), \qquad t \in I. \tag{25}$$

We calculate that

$$\begin{aligned}\frac{d}{dt}(\rho_{\lambda,\mu}(t)) &= D_\mu S_\mu(t)\rho(\lambda t) + \lambda S_\mu(t)\frac{d\rho}{dt}(\lambda t) = D_\mu S_\mu(t)\rho(\lambda t) - \lambda S_\mu(t)\Psi(\rho(\lambda t))\\ &= D_\mu \rho_{\lambda,\mu}(t) - \lambda S_\mu(t)\Psi(\Pi(\mu,t,\cdot)_*\Pi(\mu,t,\cdot)^*\rho(\lambda t))\\ &= D_\mu \rho_{\lambda,\mu}(t) - \lambda\Pi(\mu,t,\cdot)^*\Psi(\Pi(\mu,t,\cdot)_*\rho_{\lambda,\mu}(t)),\end{aligned}$$

where we used (16), (25) and the fact that $S_\mu(t)$ and D_μ commute on $h^{3+\alpha}(\Gamma_*)$. By writing

$$\mathbb{F}(\mu,v)(t) := \Pi(\mu,t,\cdot)^*\Psi(\Pi(\mu,t,\cdot)_* v(t)), \tag{26}$$

we see that $\rho_{\lambda,\mu}(t)$ solves the following problem:

$$\frac{d}{dt}h + \lambda\mathbb{F}(\mu,h) - D_\mu h = 0, \quad h(0) = \rho_0. \tag{27}$$

Moreover, we have the following result:

Lemma 5. *Let $\mathbb{F}(\mu,v)$ be defined as in (26). There hold*

(i) $[(\mu,w) \mapsto D_\mu w] \in \mathcal{L}^2(B_{\mathbb{R}^N}(0,r_0) \times \mathbb{E}_1(I), \mathbb{E}_0(I))$.

(ii) $[(\mu,v) \mapsto \mathbb{F}(\mu,v)] \in C^\omega(B_{\mathbb{R}^N}(0,r_0) \times \mathbb{O}(I), \mathbb{E}_0(I))$.

To give the proof of Lemma 5, we need some preparations. Given $(\mu,\xi)\in B_{\mathbb{R}^N}(0,r_0)\times\mathcal{U}$ and $t=1$, define

$$\rho := \Pi(\mu,1,\cdot)_*\xi = \xi\circ\Pi^{-1}(\mu,1,\cdot). \tag{28}$$

It is obvious that $\rho\in\mathcal{U}$ for each fixed $\mu\in\mathbb{R}^N$. From the proof of Lemma 3.1 in [19] we know that $V_0=0$ for $\mu=0$ and $\Pi(\mu,\cdot,x)$ is the unique global solution of (22). It follows that

$$\Pi(0,1,\cdot)=\mathrm{Id}_{\Gamma_*}\quad\text{and}\quad \rho\equiv\xi\quad \text{for } \mu=0. \tag{29}$$

Given $(\mu,\xi)\in B_{\mathbb{R}^N}(0,r_0)\times\mathcal{U}$, we denote by $\theta_{\mu,\xi}$ the composition of the two mappings $\Pi(\mu,1,\cdot):\Gamma_*\to\Gamma_*$ and $\theta_\rho:\Gamma_*\to\Gamma_\rho$, that is,

$$\theta_{\mu,\xi}:\,\Gamma_*\to\Gamma_\rho,\quad \theta_{\mu,\xi}:=\theta_\rho(\cdot)\circ\Pi(\mu,1,\cdot)=\Pi(\mu,1,\cdot)+\xi(\cdot)(\nu\circ\Pi(\mu,1,\cdot)), \tag{30}$$

where as before ν stands for the unit outward normal field of Γ_*, θ_ρ is defined in the beginning of Section 2 and we have used the relation (28). Since Γ_* is of class C^ω and $\Pi\in C^\omega(\mathbb{R}^N\times\mathbb{R}\times\Gamma_*,\,\Gamma_*)$ (cf. (18)), we can prove

$$[\mu\mapsto\nu\circ\Pi(\mu,1,\cdot)]\in C^\omega(B_{\mathbb{R}^N}(0,r_0),\ (h^{m+\alpha}(\Gamma_*))^n),\quad m\in\mathbb{N}.$$

Thus for sufficiently small $r_0>0$ and $a_0>0$, we have $\theta_{\mu,\xi}\in\mathrm{Diff}^{2+\alpha}(\Gamma_*,\Gamma_\rho)$ and

$$[(\mu,\xi)\mapsto\theta_{\mu,\xi}]\in C^\omega\Big(B_{\mathbb{R}^N}(0,r_0)\times(h^{m+\alpha}(\Gamma_*)\cap\mathcal{U}),\ (h^{m+\alpha}(\Gamma_*))^n\Big),\quad m\in\mathbb{N}. \tag{31}$$

Given $(\mu,\xi)\in B_{\mathbb{R}^N}(0,r_0)\times\mathcal{U}$, define an operator

$$\Theta_{\mu,\xi}:\,\Omega_*\to\Omega_\rho,\quad \Theta_{\mu,\xi}:=\mathrm{Id}_{\Omega_*}+\pi(\theta_{\mu,\xi}-\mathrm{Id}_{\Gamma_*}).$$

From the above relation and the property of the operator π we know that

$$[(\mu,\xi)\mapsto\Theta_{\mu,\xi}]\in C^\omega\Big(B_{\mathbb{R}^N}(0,r_0)\times(h^{m+\alpha}(\Gamma_*)\cap\mathcal{U}),\ (h^{m+\alpha}(\bar\Omega_*))^n\Big),\quad m\in\mathbb{N}. \tag{32}$$

Obvious there holds $\Theta_{\mu,\xi}|_{\Gamma_*}=\theta_{\mu,\xi}$. The corresponding pull-back and push-forward operators induced by $\Theta_{\mu,\xi}$ are respectively denoted by $\Theta^*_{\mu,\xi}$ and $\Theta_*^{\mu,\xi}$, i.e.,

$$\Theta^*_{\mu,\xi}u:=u\circ\Theta_{\mu,\xi}\ \text{ for } u\in C(\bar\Omega_\rho),\qquad \Theta_*^{\mu,\xi}v:=v\circ\Theta^{-1}_{\mu,\xi}\ \text{ for } v\in C(\bar\Omega_*).$$

By parallel argument like in Section 1, we can convert (1) into a new system by using $\Theta^*_{\mu,\xi}$. For this, given $(\mu,\xi)\in B_{\mathbb{R}^N}(0,r_0)\times\mathcal{U}$, we introduce the transformed differential operators

$$\mathcal{A}_\mu(\xi)u:=\Theta^*_{\mu,\xi}\Delta(\Theta_*^{\mu,\xi}u),\ \ \mathcal{B}_\mu(\xi)u:=\theta^*_{\mu,\xi}\langle\Upsilon_1\nabla(\Theta_*^{\mu,\xi}u),\,\hat\nu\rangle,\ \ \mathcal{N}_\mu(\xi):=\theta^*_{\mu,\xi}\kappa\quad\text{for } u\in C^2(\bar\Omega_*).$$

With the help of (31) and (32), arguing similarly as in the proof of Lemma 2.2 and Lemma 3.3 in [20] we can prove the analytic dependence of $\mathcal{A}_\mu(\xi)$, $\mathcal{B}_\mu(\xi)$ and $\mathcal{N}_\mu(\xi)$ on (μ,ξ) for small

$r_0 > 0$ and $a_0 > 0$. By denoting $\tilde{u}(t) := \Theta^*_{\mu,\xi}\sigma(t,\cdot)$, $\tilde{w}(t) := \Theta^*_{\mu,\xi}\beta(t,\cdot)$ and $\tilde{v}(t) := \Theta^*_{\mu,\xi}p(t,\cdot)$, we can transform (1) to the following equivalent problem

$$\begin{cases} \mathcal{A}_\mu(\xi)\tilde{u} = \tau_1\tilde{u} + \tilde{w} & \text{in } \Omega_*,\ t > 0, \\ \mathcal{A}_\mu(\xi)\tilde{w} = \tau_2\tilde{w} & \text{in } \Omega_*,\ t > 0, \\ \mathcal{A}_\mu(\xi)\tilde{v} = -\eta(\tilde{u} - \tilde{\sigma} - \iota\tilde{w}) & \text{in } \Omega_*,\ t > 0, \\ \tilde{u} = \bar{\sigma},\ \ \tilde{w} = \bar{\beta} & \text{on } \Gamma_*,\ t > 0, \\ \tilde{v} = \gamma\mathcal{N}_\mu(\xi) & \text{on } \Gamma_*,\ t > 0, \\ \partial_t\xi = -\mathcal{B}_\mu(\xi)\tilde{v} & \text{on } \Gamma_*,\ t > 0, \\ \xi(0) = \xi_0 & \text{for } t = 0, \end{cases} \tag{33}$$

where $\xi_0 = \rho_0$ by the fact that $\Pi(\mu, 0, \cdot) = \mathrm{Id}_{\Gamma_*}$ (cf. (29)). Moreover, by following the same reduction process like in Section 1 and using similar notations, we fuse (33) into the equation:

$$\frac{d\xi}{dt} + \Psi_\mu(\xi) = 0, \quad \xi(0) = \xi_0, \tag{34}$$

where

$$\begin{aligned} &\Psi_\mu(\xi) := \Phi_\mu(\xi)\xi + R_\mu(\xi) \quad \text{for } (\mu,\xi) \in B_{\mathbb{R}^N}(0,r_0) \times \mathcal{O}, \\ &[(\mu,\xi) \mapsto \Phi_\mu(\xi)] \in C^\omega(B_{\mathbb{R}^N}(0,r_0) \times \mathcal{U},\ \mathcal{L}(h^{3+\alpha}(\Gamma_*), h^\alpha(\Gamma_*))), \\ &[(\mu,\xi) \mapsto R_\mu(\xi)] \in C^\omega(B_{\mathbb{R}^N}(0,r_0) \times \mathcal{U},\ h^\alpha(\Gamma_*)). \end{aligned} \tag{35}$$

Comparing the above steps carefully with those in Section 1 we get

Lemma 6. *Let Ψ and Ψ_μ be defined by* (15) *and* (3.18)*, respectively. Then*

$$[(\mu,\xi) \mapsto \Psi_\mu(\xi)] \in C^\omega(B_{\mathbb{R}^N}(0,r_0) \times \mathcal{O},\ h^\alpha(\Gamma_*)), \tag{36}$$

$$\Psi_\mu(\xi) = \Pi(\mu,1,\cdot)^*\Psi(\Pi(\mu,1,\cdot)_*\xi) \quad \textit{for } (\mu,\xi) \in B_{\mathbb{R}^N}(0,r_0) \times \mathcal{O}. \tag{37}$$

Now we can give the proof of Lemma 5.

Proof of Lemma 5. The assertion (i) in Lemma 5 follows readily from Lemma 4 and the fact that I is compact. In the following, we prove (ii).

It is not difficult to prove that

$$\Pi(t\mu,1,\cdot) = \Pi(\mu,t,\cdot) \quad \text{for } (\mu,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{38}$$

Thus it follows from (25), (26), (37) and (38) that

$$\mathbb{F}(\mu,v)(t) = \Pi(\mu,t,\cdot)^*\Psi(\Pi(\mu,t,\cdot)_*v(t)) = \Pi(t\mu,1,\cdot)^*\Psi(\Pi(t\mu,1,\cdot)_*v(t)) = \Psi_{t\mu}(v(t)) \tag{39}$$

for $(\mu,t,v) \in B_{\mathbb{R}^N}(0,r_0) \times I \times \mathbb{O}(I)$. Observing that (36) is valid for time-independent functions $v \in \tilde{\mathcal{O}}$ and the fact that I is compact, by employing the perturbation argument like in the proof of Lemma 3.5 in [19], we can prove that for time-dependent functions $v \in \mathbb{O}(I)$, there holds

$$[(\mu,v) \mapsto \Psi_{t\mu}(v(t))] \in C^\omega(\mathbb{B}_{\mathbb{R}^N}(0,r) \times \mathbb{O}(I),\ \mathbb{E}_0(I)). \tag{40}$$

Combining (39) and (40) we complete the proof of the assertion (ii) in Lemma 5. □

3 Proof of Theorem 1

In this section we give the proof of Theorem 1.

Let $\rho_0 \in \tilde{\mathcal{O}}$ be given and let $\rho = \rho(\cdot, \rho_0)$ be the unique maximal solution of the problem (16). Recall that $I = [0, T]$ for $T \in (0, t^+)$ and $\rho_{\lambda,\mu}(t)$ is defined in (25).

Theorem 3. *There exists an open neighborhood* $\Lambda(\varepsilon) := (1-\varepsilon, 1+\varepsilon) \times (-\varepsilon, \varepsilon)^N \subset (1-\varepsilon_0, 1+\varepsilon_0) \times \mathbb{B}_{\mathbb{R}^N}(0, r_0)$ *of* $(1,0)$ *such that* $[(\lambda,\mu) \mapsto \rho_{\lambda,\mu}] \in C^\omega(\Lambda(\varepsilon), \mathbb{O}(I))$.

Proof. Recall that $\rho_{\lambda,\mu}$ is the solution of (27). Define

$$\mathbb{G}((\lambda,\mu), h) := \Big(\frac{d}{dt}h + \lambda\mathbb{F}(\mu, h) - D_\mu h,\ h(0) - \rho_0\Big).$$

It follows from Lemma 5 that

$$[((\lambda,\mu), h) \mapsto \mathbb{G}((\lambda,\mu), h)] \in C^\omega(\mathbb{R} \times B_{\mathbb{R}^N}(0, r_0) \times \mathbb{O}(I), \mathbb{E}_0(I) \times E_\Upsilon). \tag{41}$$

Given $h \in \mathbb{E}_1(I)$, we get from (29), (37) and (39) that

$$\partial_2\mathbb{G}((1,0),\rho)h = \frac{d}{d\varepsilon}\mathbb{G}((1,0), \rho + \varepsilon h)|_{\varepsilon=0} = \Big(\frac{d}{dt}h + \Psi'(\rho)h,\ h(0)\Big), \quad \rho \in \mathbb{O}(I). \tag{42}$$

From Lemma 2 we know that for fixed time-independent functions $\rho \in \tilde{\mathcal{O}} \subset \mathcal{O}$, there holds

$$\Big(\frac{d}{dt} + \Psi'(\rho), \Upsilon\Big) \in \mathcal{L}_{is}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\Upsilon).$$

Combining Lemma 2 with Remark III 3.4.2(c) in [15] we get that, given $(\phi, \varphi) \in \mathbb{E}_0(I) \times E_\Upsilon$, there is a unique solution $u \in \mathbb{E}_1(I)$ to the inhomogeneous evolution equation

$$\frac{d}{dt}u + \Psi'(\rho(t))u = \phi(t), \quad u(0) = \varphi,$$

that is, $(\frac{d}{dt} + \Psi'(\rho), \Upsilon)$ is surjective. Combining this with the open mapping theorem we get

$$\partial_2\mathbb{G}((1,0),\rho) = \Big(\frac{d}{dt} + \Psi'(\rho(t)), \Upsilon\Big) \in \mathcal{L}_{is}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\Upsilon), \quad \rho \in \mathbb{O}(I). \tag{43}$$

Since $\mathbb{G}((\lambda,\mu),h) = (0,0)$ holds if and only if h is a solution of (27), the implicit function theorem on Banach spaces and (41)–(43) yield that there is a neighborhood $\Lambda(\varepsilon) := (1-\varepsilon, 1+\varepsilon) \times (-\varepsilon,\varepsilon)^N \subset (1-\varepsilon_0, 1+\varepsilon_0) \times \mathbb{B}_{\mathbb{R}^N}(0, r_0)$ of $(1,0)$ such that

$$[(\lambda,\mu) \mapsto \rho_{\lambda,\mu}] \in C^\omega(\Lambda(\varepsilon), \mathbb{O}(I)). \quad \square$$

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. Local well-posedness of (1) is guaranteed by Theorem 2 and the equivalences between (1), (6) and (16). In the following we prove the analyticity.

Let $\Lambda(\varepsilon)$ be the neighborhood of $(1,0)$ in $\mathbb{R}\times\mathbb{R}^N$, ensured by Theorem 3. Pick a point $(t_0,x_0)\in(0,t^+)\times\Gamma_*$. Let $T\in(0,t^+)$ be given with $T>t_0$, and set $I=[0,T]$ as before. It follows from Lemma 3 that there exists a series of unit vectors $\mu_1,\mu_2,\cdots,\mu_n\in\mathbb{R}^N$ such that $(V_{\mu_1},V_{\mu_2},\cdots,V_{\mu_n})$ forms a basis of $T_{x_0}\Gamma_*$. Given a vector $a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, we write

$$\tilde{a}:=(a_1,a_2,\cdots,a_n,0,\cdots,0)\in\mathbb{R}^N,\quad \tilde{\mu}:=\sum_{k=1}^n a_k\mu_k\in\mathbb{R}^N.$$

For sufficiently small $\delta\in(0,\varepsilon)$, define a neighborhood $\tilde{\mathbb{A}}(\delta):=(1-\delta,1+\delta)\times(-\delta,\delta)^n\times\{0\}\subset\mathbb{R}^{N+1}$. We introduce a mapping

$$\Xi:\ \tilde{\mathbb{A}}(\delta)\to(0,t^+)\times\Gamma_*,\quad (\lambda,\tilde{a})\mapsto(\lambda t_0,\Pi(\tilde{\mu},t_0,x_0)).$$

By virtue of Lemma 3 we know that

$$\Xi\in C^\omega(\tilde{\mathbb{A}}(\delta),(0,t^+)\times\Gamma_*) \tag{44}$$

and

$$T_{(1,0)}\Xi(\eta,\tilde{b})=t_0\Big(\eta,\sum_{k=1}^n b_kV_{\mu_k}\Big)\in\mathbb{R}\times T_{x_0}\Gamma_*$$

for all $\eta\in\mathbb{R}$ and $\tilde{b}:=(b_1,b_2,\cdots,b_n,0,\cdots,0)\in\mathbb{R}^N$, which indicates that $T_{(1,0)}\Xi$ is bijective. Hence it follows from (44) and the inverse function theorem that Ξ is an analytic parametrization of an open neighborhood $O_{(t_0,x_0)}$ of (t_0,x_0) in $(0,t^+)\times\Gamma_*$, provided $\delta>0$ is chosen small enough. Noticing the obvious inclusion $\mathbb{O}(I)\subset C(I,C(\Gamma_*))$, we see that the evaluation mapping $\mathbb{O}(I)\to\mathbb{R}, f\mapsto f(t_0)(x_0)$ is well-defined and analytic, which combined with Theorem 3 yields

$$[(\lambda,\tilde{a})\mapsto\rho_{\lambda,\tilde{\mu}}(t_0)(x_0)]\in C^\omega(\tilde{\mathbb{A}}(\delta),\mathbb{R}). \tag{45}$$

On the other hand, it follows from the definition of Ξ and (24) that

$$\Xi^*\rho(\lambda,\tilde{a})=\rho(\lambda,\tilde{a})\circ\Xi=\rho(\lambda t_0,\Pi(\tilde{\mu},t_0,x_0))=\rho_{\lambda,\tilde{\mu}}(t_0)(x_0),\quad(\lambda,\tilde{a})\in\tilde{\mathbb{A}}(\delta). \tag{46}$$

Combining (44)–(46) we get $\rho\in C^\omega(O_{(t_0,x_0)},\mathbb{R})$. Since (t_0,x_0) can be chosen anywhere in $(0,t^+)\times\Gamma_*$, this implies

$$\rho\in C^\omega((0,t^+)\times\Gamma_*), \tag{47}$$

that is, the time-space manifold $\bigcup_{t\in(0,t^+)}(\{t\}\times\Gamma(t))$ is real analytic.

To verify that the components $\sigma(t,\cdot)$ and $p(t,\cdot)$ belong to $C^\omega(\bar{\Omega}(t))$ for each $t\in(0,t^+)$, we first consider the boundary value problem (7). It follows from Section 1 that the problem (7) has a unique solution $w=\mathcal{Q}_1(\rho)$ satisfying (11), which combined with (47) and the inclusion $h^{3+\alpha}(\bar{\Omega}_*)\subset C(\bar{\Omega}_*)$ yields $w(t,\cdot)\in C^\omega(\bar{\Omega}_*)$ for each $t\in(0,t^+)$. Since $\beta(t,\cdot)=\Theta_*^\rho w(t,\cdot)$, we conclude from (3), (47) and the above relation that $\beta(t,\cdot)\in C^\omega(\bar{\Omega}(t))$. Similarly we can prove that $\sigma(t,\cdot),p(t,\cdot)\in C^\omega(\bar{\Omega}(t))$. This completes the proof of Theorem 1. □

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