The Quenching Phenomena for Fourth-order Nonlinear Parabolic Equationos

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Abstract: In this paper, we investigate the quenching phenomena of initial boundary value problem for the fourth-order nonlinear parabolic equations in bounded domains. First, we not only obtain quenching phenomena in finite time but also estimate the quenching time under some assumptions on the exponents and initial data for a class of equation with the common source term. Then we prove quenching phenomena in finite time and exactly estimate the quenching time for a class of equation with the special source term. Our main tools are maximum principle, the comparision principle and eigenfunction method.

Keywords: Quenching phenomena; fourth-order parabolic equation; Quenching time

0 Introduction

In this paper, we consider the quenching phenomena of the following initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta^2 u = g\left(u\right), & \left(x,t\right) \in \Omega \times \left(0,T\right) \\ u = 0, \Delta u = 0\,, & \left(x,t\right) \in \partial\Omega \times \left(0,T\right) \\ u\left(x,0\right) = \phi\left(x\right), & x \in \Omega \end{cases}$$

where $\Omega \subset R^N$ is an N dimensional domain, $\partial\Omega$ is the boundary of Ω, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x^2}$ is the Laplace operator on Ω, $\Delta^2 u = \Delta\left(\Delta u\right)$, $m \in \left(-\infty, \infty\right)$, $\alpha \in \left(0, \infty\right)$, $\phi\left(x\right)$ is a nonnegative continuous function on Ω with $\sup_{x\in\Omega}\phi(x) < b$, and $\phi(x) \equiv 0$ on $\partial\Omega$.

In [1-3], authors study some fourth-order elliptic equations. 1990, Brown, Russell M and Shen Zhongwei [4] studied the following initial boundary value problem

$$u_t(t,x) + \Delta^2 u(t,x) = 0$$

where $(t,x) \in (0,T) \times D$ and $D \subset R^n$ is a Lipchitz domain, then authors derived the existence and uniqueness of a solution u. Rchke [5] generalized Brown's results and proved the global existence and uniqueness results on an arbitrary domain Ω in R^N by considering thoroughly the following problem

$$u_t + \Delta^2 u = f(\{D^\alpha u, 0 \le |\alpha| \le 4\})\,. \tag{1.2}$$

There are many authors who extend the results of the problem (1.2) in [6, 7]. And in [6], authors determined the existence and uniqueness of a local solution and extended a local solution

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to a global solution by establishing uniform a priori estimate. In [8], Messound investigated the attractors for following equation

$$u_t + \Delta^2 u + q\Delta u + f(u) = g \quad in \quad R^3.$$

Since Kawarada [9] introduced first the concept of quenching for second order nonlinear parabolic equation, many authors have investigated the quenching phenomena for second order nonlinear parabolic equation and derived many interesting results. However, few authors consider

the quenching phenomenon for the fourth-order parabolic equation at present. Here we investigate the quenching phenomena and estimate the quenching time for the fourth-order parabolic equation by using the maximum principle.

For convenience, we first introduce the following definition of the quenching.

Definition. Let $u(x,t)$ be a classical solution of problem (1.1). We say that $u(x,t)$ quenches in finite time if there exists a real number $T \in (0, \ \infty)$ such that

$$\lim_{t\to T}\sup_{x\in\Omega} u(x,t) = b.$$

If we assume that there are constants $c_1 > 0$ and $c_2 \in R$, we can define four conditions for $g(s)$: $[0,b) \mapsto (0,\infty)$.

(G1) $g(s)$ is locally Lipchitz on $[0,b)$ and $g(0) > 0$,

(G2) $\lim_{u\to b^-} g(s) = +\infty$,

(G3) $g(s) \ge c_1 + c_2 s, \quad s \in [0,b)$,

(G4) $c_2 + \left({}^{c_1}\!/_{b} \right) > 0$.

The paper is arranged as follows. In Section 2, we give the main result of my research and prove it. In Section 3, we prove the quenching condition of fourth-order nonlinear parabolic equations with special term.

1 Main result

Let $\Omega \subset R^N$ be a bounded domain. In this paper, $\lambda_1(\Omega)$ and $\Psi_1(x)$ denote the first eigenvalue and the first eigenfunction of the following eigenvalue problem

$$\begin{cases} \Delta\Psi + \lambda\Psi = 0, \ x \in \Omega \\ \Psi = 0, \qquad x \in \partial\Omega \end{cases}$$

For convenience, we choose $\Psi_1(x)$ so that

$$\Psi_1(x) > 0, x \in \Omega$$

and

$$\int_\Omega \Psi_1(x)\,dx = 1$$

and sometimes denote $\lambda_1(\Omega)$ simply by λ_1.

Then we have

Theorem 2.1. *Let $\Omega \subset R^N$ be a bounded domain and let $u(x,t)$ be the classical solution of problem (1.1). If $g(s)$ satisfies (G1)-(G4), then $u(x,t)$ must be quenching in a finite time*

$T_{\max}$, *and for* $T_{\max}$ *the estimate is*

$$\int_M^b \frac{ds}{g(s)} \le T_{\max} \le \frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)b}{c_1+(c_2+\lambda_1^2)m}$$

where $M=\sup_{x\in\Omega}\phi(x)<b, m=\int_\Omega \phi(x)\Psi_1(x)dx$.

Proof. Let $(0,T_{\max})$ be the maximum time interval in which the classical solution $u(x,t)$ of problem (1.1) exists. By (G1) and the comparison principle one has

$$C_1 t_1^{-N/2}\ln(1+t) \le I_m(0,t) \le C_2 t^{-N/2}\ln(1+t)$$

Since b is a singular point of $g(s)$ and $g(s)$ is maximal, one can conclude that if $T_{\max}<+\infty$, then

$$\lim_{t\to T^-}\sup_{x\in\Omega} u(x,t)=b$$

Otherwise, $u(x,t)$ can be extended beyond. This is impossible. Now, to prove Theorem 2.1, it is sufficient to prove that $T_{\max}$ is finite and

$$\int_M^b \frac{ds}{g(s)} \le T_{\max} \le \frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)b}{c_1+(c_2+\lambda_1^2)m}$$

Through the research of problem (1.1), we will know

$$\int_\Omega \Psi_1\Delta^2 u dx = \lambda_1^2\int_\Omega \Psi_1 u dx$$

In fact, as $u=0,\ for\ (x,t)\in\partial\Omega\times(0,T)$, we have

$$\begin{aligned}\int_\Omega \Psi_1\Delta u dx &= \Psi_1\frac{\partial u}{\partial x}\Big|_{\partial\Omega} - \int_\Omega \frac{\partial\Psi_1}{\partial x}\frac{\partial u}{\partial x}dx\\ &= -\int_\Omega \frac{\partial\Psi_1}{\partial x}\frac{\partial u}{\partial x}dx\\ &= -\frac{\partial\Psi_1}{\partial x}u\Big|_{\partial\Omega} + \int_\Omega \Delta\Psi_1 u dx\\ &= -\lambda_1\int_\Omega \Psi_1 u dx\end{aligned}$$

If we take $\Delta u = v$, then we have

$$v=0,(x,t)\in\partial\Omega\times(0,T)$$

Hence, we have

$$\begin{aligned}\int_\Omega \Psi_1\Delta^2 u dx &= -\lambda_1\int_\Omega \Psi_1 v dx\\ &= \lambda_1^2\int_\Omega \Psi_1 u dx\end{aligned}$$

To prove Theorem 2.1, multiplying the differential equation in (1.1) by $\Psi_1(x)$ and integration on Ω with respect to x , we have

$$\frac{d}{dt}\int_\Omega u\Psi_1 dx - \lambda_1^2\int_\Omega u\Psi_1 dx = \int_\Omega g(u)\Psi_1 dx \tag{2.1}$$

Since $u(x,t)$ is the classical solution of problem (1.1), one has

$$0<u(x,t)<b,\quad x\in\Omega,\quad t\in\left(0,T_{\max}\right)$$

Hence, by (G3) one has

$$g(u)\geq c_1+c_2u,\quad (x,t)\in\Omega\times\left(0,T_{\max}\right) \tag{2.2}$$

Substituting (2.2) into (2.1), one can obtain that

$$\frac{d}{dt}\int_\Omega u\Psi_1dx-\lambda_1^2\int_\Omega u\Psi_1dx\geq c_1+c_2\int_\Omega u\Psi_1dx \tag{2.3}$$

Set $y(t)=\int_\Omega u\Psi_1dx$, and (2.3) can be read as

$$\frac{dy}{dt}\geq c_1+\left(c_2+\lambda_1^2\right)y,\ t\in\left(0,T_{\max}\right) \tag{2.4}$$

Since

$$0<y(t)=\int_\Omega u\Psi_1dx<b,t\in(0,T_{\max})$$

from the condition $0<c_2+(c_1/b)$, i.e., $c_1+c_2b>0$, one has

$$c_1+(c_2+\lambda_1^2)y>0,\quad t\in(0,T_{\max}) \tag{2.5}$$

Taking into account (2.4) and (2.5), one has

$$\frac{dy}{c_1+(c_2+\lambda_1^2)y}\geq dt$$

and this implies that

$$t\leq\frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)y(t)}{c_1+(c_2+\lambda_1^2)y(0)} \tag{2.6}$$

Let $t\to T_{\max}$. From (2.6) it follows that

$$T_{\max}\leq\frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)y(T_{\max})}{c_1+(c_2+\lambda_1^2)y(0)} \tag{2.7}$$

Due to

$$0<y(T_{\max})=\int_\Omega u(x,T_{\max})\Psi_1dx\leq b$$

from (2.7), one can obtain that

$$T_{\max}\leq\frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)b}{c_1+(c_2+\lambda_1^2)m} \tag{2.8}$$

where $m=y(0)=\int_\Omega\phi(x)\Psi_1(x)dx$.

It is obvious that

$$0<\frac{1}{c_2+\lambda_1^2}\ln\frac{c_1+(c_2+\lambda_1^2)b}{c_1+(c_2+\lambda_1^2)m}<+\infty$$

thus $u(x,t)$ quenches in finite time.

To obtain a lower bound of $T_{\max}$, let us consider the initial value problem

$$\begin{cases}\dfrac{d\eta(t)}{dt}=g(\eta)\\ \eta(0)=M\end{cases} \tag{2.9}$$

where $M = \sup_{x\in\Omega} \varphi(x) < b$.

Scince $g(s) > 0$, by (2.9) one has

$$\int_M^{\eta(T)} \frac{ds}{g(s)} = t \tag{2.10}$$

Let t^* be the time for which $\lim_{t\to t^*} \eta(t) = b$, and from (2.10) we have

$$t^* = \int_M^b \frac{ds}{g(s)}$$

Obviously, $\eta(t)$ is a superfunction concerning $u(x,t)$, and thus

$$T_{\max} \ge t^* = \int_M^b \frac{ds}{g(s)} \tag{2.12}$$

so Theorem 2.1 is proved.

2 The quenching condition of fourth-order nonlinear parabolic equations with special team First-order headline

Consider the initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta^2 u = \dfrac{\alpha\left(1+|x|\right)^m}{1-u}, & (x,t) \in R^N \times (0,T) \\ u(x,0) = 0, & x \in R^N \\ u(x,t) \to 0, & |x| \to \infty \end{cases} \tag{3.1}$$

where $m \ge 0$, $\alpha > 0$, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x^2}$, $\Delta^2 u = \Delta(\Delta u)$.

If $u(x,t) \to 0$ as $|x| \to \infty$, then $\Delta u \to 0$ as $|x| \to \infty$ and $\frac{\partial u}{\partial t} \to 0$ as $|x| \to \infty$.

In fact, we have

$$\frac{\partial u}{\partial t} = \lim_{\Delta t\to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

If $\frac{\partial u}{\partial t} \ne 0$, then there must be a real number $\varepsilon > 0$ so that $\left|\frac{\partial u}{\partial t}\right| > \varepsilon$. Then we have

$$|u(x,t+\Delta t) - u(x,t)| > \varepsilon \Delta t > 0$$

Hence, we have $u(x,t) \ne 0$. This is impossible.

So we have $\Delta u \to 0$ and $\frac{\partial u}{\partial t} \to 0$ as $|x| \to \infty$.

Now I state my main result.

Theorem 3.1. *Let $u(x,t)$ be the classical solution of problem (3.1), then $u(x,t)$ must be quenching in a finite time $T_{\max}$ and for $T_{\max}$ the estimate is*

$$0 < T_{\max} \le \frac{1}{\alpha + \lambda_1^2} \ln \frac{2\alpha + \lambda_1^2}{\alpha}$$

Proof. If $m \ge 0$, we have

$$g(u)=\frac{\alpha\left(1+|x|\right)^m}{1-u}\ge\frac{\alpha}{1-u}$$

then we have

$$\lim_{u\to1^-} g(u)\to\infty$$

So we know $g(u)$ satisfies (G2).

If we known $h(u)=\dfrac{\alpha}{1-u}$, $c_1=\alpha$, $c_2=\alpha$, $b=1$, then we have

$$h(0)=\frac{\alpha}{1-0},h'(0)=\frac{\alpha}{(1-0)^2},h''(\eta)=\frac{2\alpha}{(1-\eta)^3}>0,\eta\in\left(0,u\right)$$

By Taylor's expansion theorem, there is a number $\varepsilon\in\left(0, u\right)$ such that

$$h(u)=h(0)+h'(0)u+\frac{1}{2}h''(\varepsilon)u^2>\alpha+\alpha u,u\in\left[0,1\right)$$

Since $g(u)=\dfrac{\alpha\left(1+|x|\right)^m}{1-u}\ge\dfrac{\alpha}{1-u}$, we have

$$g(u)\ge\alpha+\alpha=2\alpha>0$$

Then we know $g(u)$ satisfies

(G3) $g(u)\ge c_1+c_2u$, $u\in\left[0,b\right)$

(G4) $c_2+\left({c_1}/{b}\right)>0$.

So $g(u)$ satisfies (G1)-(G4).

By Theorem 2.1, we know if $u(x,t)$ is the classical solution of problem (3.1), then $u(x,t)$ quenches in finite time.

Through the definition of quenching, we know if $u(x,t)$ quenches, then $u(x,t)\to1\ne0$, then we have $|x|<+\infty$. So we can get a large enough number $L>0$ so that $u(x,t)$ quenches as $|x|<L<+\infty$.

Then we have

$$\frac{\alpha}{1-u}\le g\left(u\right)=\frac{\alpha\left(1+|x|\right)^m}{1-u}\le\frac{\alpha\left(1+L\right)^m}{1-u}$$

then

$$\frac{1-u}{\alpha(1+L)^m}\le\frac{1}{g(u)}\le\frac{1-u}{\alpha}$$

then we known

$$\int_0^1\frac{1-s}{\alpha(1+L)^m}ds\le\int_0^1\frac{ds}{g(s)}\le\int_0^1\frac{1-s}{\alpha}ds \tag{3.2}$$

Through calculation, we have

$$\int_0^1 \frac{ds}{g(s)} \geq \frac{\int_0^1 (1-s)ds}{\alpha(1+L)^m} = \frac{1}{2\alpha(1+L)^m} > 0$$

and

$$\int_0^1 \frac{ds}{g(s)} \leq \frac{\int_0^1 (1-s)ds}{\alpha} \leq \frac{1}{2\alpha}$$

Hence, by Theorem 2.1, we have

$$\begin{aligned} T_{\max} &\leq \frac{1}{c_2+\lambda_1^2} \ln \frac{c_1+c_2+\lambda_1^2}{c_1} \\ &= \frac{1}{\alpha+\lambda_1^2} \ln \frac{2\alpha+\lambda_1^2}{\alpha} \\ &< +\infty \end{aligned}$$

and

$$\begin{aligned} T_{\max} &\geq \int_0^1 \frac{ds}{g(s)} \\ &\geq \frac{1}{2\alpha(1+L)^m} \\ &> 0 \end{aligned}$$

So Theorem 2.1 is proved.

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四阶非线性抛物方程的淬火现象

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摘要：在这篇文章里，我们研究了四阶非线性抛物方程在有界域上的初边值问题的淬火现象。首先，通过对具有一般源项的四阶非线性抛物方程中的某些指数和初值添加约束条件，我们不仅得到了方程的解在有限时间内淬火而且对方程的解的淬火时间进行了估计。之后我们又证明了具有特殊源项的四阶非线性抛物方程的解在有限时间内淬火，并且更精确的估计了解
210 的淬火时间。我们的主要研究方法是极值原理，比较原理和特征函数法。

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