The quenching phenomena for second-order nonlinear parabolic equation with nonlinear source

XU Runzhang, ZHANG Mingyou, LIU Jie
(College of Science, Harbin Engineering University, Harbin 150001)

Abstract: In this paper, we investigate the quenching phenomena of the Cauchy problem for the second-order nonlinear parabolic equation on unbounded domain. It is shown that the solution quenches in finite time under some assumptions on the exponents and the initial data. Our main tools are comparison principle and maximum principle. We extend the result to the case of more generally nonlinear absorption.

Keywords: Parabolic equation; Quenching phenomena; Unbounded domain

0 Introduction

In this context, we investigate the quenching phenomena of the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{\alpha(1+|x|)^m}{(1-u)^n}, & (x,t) \in R^N \times (0,T), \\ u(x,0) = 0, & x \in R^N, \\ u(x,t) \to 0, & |x| \to \infty, \end{cases} \tag{1.1}$$

where $m, n \in (-\infty, +\infty)$ are real number and α is a positive parameter.

For convenience, we first introduce the following definition of the quenching.

Definition. *We say that the solution* $u(x,t)$ *of problem (1.1) quenches in finite time if there exists a real number* $0 < T < \infty$ *such that*

$$\lim_{t \to T_1} \sup_{x \in R^N} u(x,t) = 1.$$

The quenching problem has 36－year history. In 1975, Kawarada first introduced the concept of quenching when considering a famous initial boundary problem for the parabolic equation $u_t = u_{xx} + 1/(1-u)$ and derived many interesting results (see [1-6] and references therein). In [5], Acker and Walter have considerably sharpened and generalized Kawardada's results. Levine[6] thoroughly considered the following problem which greatly generalized the results of [1] and [5]

$$\begin{cases} u_t = u_{xx} + \varepsilon(1-u)^{-\beta}, & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \\ u(x,0) = u_0(x), & u_0 < 1, \quad 0 \le x \le 1, \end{cases} \tag{1.2}$$

then gave the criteria for the quenching, nonquenching and beyond quenching of the solution. There are many authors who extended the results of (1.2) in [7-11]. In [7], Levine determined quenching sets and derived quenching rate estimates. Furthermore, papers [12-14] presented recent progress in the analysis of quenching phenomena on bounded domain. It's clear that all

Foundations: National Natural Science Foundation of China (11101102); Ph.D. Programs Foundation of Ministry of Education of China (20102304120022); Natural Science Foundation of Heilongjiang Province (A201014); Foundational Science Foundation of Harbin Engineering University and Fundamental Research Funds for the Central Universities (HEUCF20111101)

Brief author introduction: Xu Runzhang, (1982-), male, Professor, Nonlinear evolution equation and nonlinear dynamic system. E-mail: xurunzh@yahoo.com.cn

results in the papers above were obtained in bounded domain.

Compared to the case of bounded domain, it seems that very few results concerning the quenching phenomena were obtained on unbounded domain (see [15, 16]). In [17], Dai Quiyi investigated the quenching phenomena of problem (1.1) when $n=1$ and obtained some results that the solution quenches in finite time. In addition, it is shown in [1, 10, 17] that the absorbing term $(1-u)^{-1}$ plays an essential role in quenching phenomena. For the case that the absorbing term is $(1-u)^{-n}$, will the solution quench in finite time? The problem above is still open up to now. We will restrict our attention to more complex problem and the case where $(1-u)^{-n}$. For the problem, we can obtain the following results

Theorem 1. *If $m=0$, $n>0$, then the solution of problem (1.1) always quenches in finite time for any $\alpha>0$.*

Theorem 2. *Assume that $N\ge 3$, $n\in R$. Then we have*

(i)If $m<-2$, then the solution of problem (1.1) exists globally for α small enough and quenches in finite time for α large enough.

(ii)If $m\ge -2$, then the solution of problem (1.1) always quenches in finite time for any $\alpha>0$.

Theorem 3. *If $N\le 2$, $n\in R$, then for any $m\in(-\infty,+\infty)$ and $\alpha\in(0,\infty)$, the solution of problem (1.1) always quenches in finite time.*

The paper is arranged as follows. In Section 2, we introduce three lemmas; In Section 3, we prove the main results.

1 Three main lemmas

This section gives three lemmas needed for proving the main results. To this end, let

$$p(t,x,y)=(4\pi t)^{-N/2}\exp\left(-\frac{|x-y|^2}{4t}\right)$$

denote the fundamental solution of the heat operator and set

$$I_m(x,t)=\int_{R^N} p(t,x,y)(1+|y|)^m\,dy.$$

Then, we have

Lemma 2.1[9, Lemma 1]. *If $m\le 0$, then $I_m(x,t)\le 1$ for $t\ge 0$.*

Lemma 2.2[10, Lemma 6]. *If $m\le 0$ and $t>0$, then the function $I_m(x,t)$ attains its maximum at $x=0$.*

Lemma 2.3[9, Lemma 3]. *Assume that $m\in R$ and $t\ge 1$. Then we have*

(i)If $m>-N$, then there exist two positive constants C_1,C_2 such that

$$C_1t^{m/2}\le I_m(0,\ t)\le C_2t^{m/2};$$

(ii)If $m=-N$, then there exist two positive constants C_1,C_2 such that

$$C_1t_1^{-N/2}\ln(1+t)\le I_m(0,t)\le C_2t^{-N/2}\ln(1+t);$$

(iii)If $m\le -N$, then there exist two positive constants C_1,C_2 such that

$$C_1t^{-N/2}\le I_m(0,t)\le C_2t^{-N/2}.$$

2 The proof of the main theorems

Proof of Theorem 1. If we let $m=0$, $n>0$, then the absorption of problem (1.1) is $\frac{\alpha}{(1-u)^n}$, which can be defined by

$$g(u)=\frac{\alpha}{(1-u)^n}.$$

By taking $b=1, c_1=\alpha \ and \ c_2=n\alpha$, we have $g(u):[0,b)\mapsto(0,\infty)$ satisfies

(G1) $g(u)$ is a locally Lipschitzian function over $[0,b)$ and $g(0)>0$,

(G2) $\lim_{u\to b^-} g(u)=+\infty$,

(G3) $g(u)\geq c_1+c_2u, \quad u\in[0,b)$,

(G4) $c_2+\left(c_1/b\right)>0$.

In view of [11], we can know the solution of problem (1.1) always quenches in finite time for any $\alpha>0$. In fact, we take $u(x,t)\to 0$ as $|x|\to\infty$, then we can get a real number $M>0$ so that $u(x,t)<\varepsilon$ as $x>M$. Hence,

$$u_x\to 0 \quad as \quad |x|\mapsto\infty.$$

We assume by contradiction that there exist two points $x_1,x_2>M$ such that for all $x\in[x_1,x_2]$

$$u_x>\delta>0 \quad or \quad u_x<\delta<0.$$

Without loss of generality, we may assume that $u_x>\delta>0.$ Then, we infer

$$u>u_x(x_2-x_1)>\delta(x_2-x_1)>0.$$

By taking eventually $\varepsilon=\delta(x_2-x_1)$ we conclude $u(x,t)>\varepsilon$, which is impossible. Moreover, similar calculations lead to

$$u_{xx}\to 0 \quad as \quad |x|\mapsto\infty. \tag{2.1}$$

To prove that $\frac{\partial u}{\partial t}\to 0$ as $|x|\to\infty$, we assume by contradiction that there exists $\varepsilon>0$ such that

$$\frac{\partial u}{\partial t}>\varepsilon.$$

We integrate by parts over $[0,t]$ to obtain

$$u>\varepsilon t+C,$$

which contradicts the fact that $u(x,t)<\varepsilon$. Thus, we prove that $\frac{\partial u}{\partial t}\to 0$ as $|x|\to\infty$. Therefore, we combine this with (2.1), we get

$$\frac{\partial u}{\partial t}-\Delta u\to 0 \quad as \quad |x|\mapsto\infty,$$

which is contradictory to problem (1.1). So we imply that the solution of problem (1.1) does not exist globally if $m=0, n=0$.

If we let $m=0, n<0$, then problem (1.1) implies the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t}-\Delta u=\alpha(1-u)^{-n}, & (x,t)\in R^N\times(0,T), \\ u(x,0)=0, & x\in R^n, \\ u(x,t)\to 0, & |x|\to\infty. \end{cases}$$

We can get $(1-u)^{-n}\to 1$ as $|x|\to\infty$, thus $\dfrac{\partial u}{\partial t}-\Delta u\to\alpha\neq 0$ as $|x|\to\infty$. This contradicts (1.1) and Theorem 1 is proved.

Proof of Theorem 2. The mild solution of problem (1.1) is

$$u(x,t)=\alpha\int_0^t\int_{R^N}p(t-s,x,y)\frac{(1+|y|)^m}{(1-u)^n}dyds.$$

(i)In order to obtain global solution for α small enough, which suffices to prove by the priori assumption that

$$u(x,t)\le\frac{1}{2},\qquad for\ all\ \ (x,t)\in R^N\times(0,T). \tag{2.2}$$

Indeed, it is valid for α small enough, since the solution may be found by successive substitution in right hand side of the last equation.

By comparison principle, we obtain

$$u(x,t)\ge 0,\qquad for\ (x,t)\in R^N\times(0,\infty). \tag{2.3}$$

We combine (2.2) and (2.3), which gives

$$\left(\frac{1}{2}\right)^{-n}\le(1-u)^{-n}\le 1.$$

Then we have

$$\begin{aligned} u(x,t)&=\alpha\int_0^t\int_{R^N}p(t-s,x,y)\frac{(1+|y|)^m}{1-u}dyds \\ &\le\alpha\int_0^t\int_{R^N}p(t-s,x,y)(1+|y|)^m dyds \\ &=\alpha\int_0^t I_m(x,t-s)ds, \end{aligned}$$

where $I_m(x,t-s)=\int_{R^N}p(t-s,x,y)(1+|y|)^m dy$. We distinguish the following two cases.

Case 1: If $t\le 1$ occurs, by Lemma 2.1we conclude

$$\begin{aligned} u(x,t)&\le\alpha\int_0^t I_m(x,t-s)ds \\ &\le\alpha\int_0^t ds\le\alpha. \end{aligned}$$

Case2: We now suppose that $t>1$ occurs, then

$$u(x,t)\le \alpha\int_0^t I_m(x,t-s)ds$$
$$=\alpha\int_0^{t-1} I_m(x,t-s)ds+\alpha\int_{t-1}^t I_m(x,t-s)ds.$$

By Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$u\left(x,t\right)\le a+a\int_0^{t-1} I_m(0,t-s)ds$$
$$\le a\left(1+\int_1^{\infty} I_m(0,\theta)d\theta\right).$$

Hence, there exists sufficiently small a such that

$$u(x,t)<\frac{1}{2},\left(x,t\right)\in R^N\times\left(0,\infty\right),$$

so we obtain that the problem (1.1) has a global solution for $\alpha>0$ small enough.

Next, we are going to prove that the solution of problem (1.1) quenches in finite time for α large enough. By taking $t>1$ we have

$$u\left(x,t\right)=\alpha\int_0^t\int_{R^N} p\left(t-s,x,y\right)\frac{\left(1+|y|\right)^m}{\left(1-u\right)^n}dyds$$
$$\ge\alpha\int_0^{t-1} I_m\left(x,t-s\right)\left(\frac{1}{2}\right)^{-n}ds. \tag{2.4}$$

Moreover, by (2.4) we obtain

$$u(0,t)\ge\alpha\left(\frac{1}{2}\right)^{-n}\int_0^{t-1} I_m(0,t-s)ds$$
$$=\alpha\left(\frac{1}{2}\right)^{-n}\int_1^t I_m(0,\theta)d\theta. \tag{2.5}$$

By Lemma 2.3, this implies

$$0<\int_1^{\infty} I_m(0,\theta)d\theta=J_m<+\infty,\qquad for\ N\ge 3, m<-2.$$

Combining (2.5) and by taking $\alpha>\dfrac{\left(1/2\right)^n}{J_m}$, we get $u(0,t)>1$, which is impossible. This implies that $u\left(x,t\right)$ quenches in finite time for α large enough. The proof is finished.

(ii) We now prove Theorem 2(ii). By contradiction, we assume that

$$u(x,t)<1,\qquad for\ (x,t)\in R^N\times(0,\infty).$$

Since $t>1$, we deduce

$$u(0,t)\ge\alpha\left(\frac{1}{2}\right)^{-n}\int_0^{t-1} I_m(0,t-s)ds$$
$$=\alpha\left(\frac{1}{2}\right)^{-n}\int_1^t I_m(0,\theta)d\theta. \tag{2.6}$$

By applying Lemma 2.3, we abtain

$$\lim_{t\to\infty}\int_1^t I_m(0,\theta)d\theta = +\infty, \quad for\ m \ge -2, N \ge 3. \qquad (2.7)$$

Hence, combining (2.6) and (2.7), we infer

$$\lim_{t\to\infty} u(0,t) = +\infty, \quad for\ \ any\ \ \alpha > 0, m \ge -2, N \ge 3.$$

This is a contradiction. This implies that $u(x,t)$ quenches in finite time for all $\alpha > 0$.

Proof of Theorem 3. We may now conclude the proof of Theorem. By supposing $1 \le N \le 2$ and $n < 0$, we obtain

$$\int_1^t I_m(0,\theta)d\theta \to +\infty \ \ as\ t \to \infty, \quad for\ any\ m \in (-\infty, \infty).$$

A similar argument as the proof of Theorem 2(ii) immediately deduce Theorem 3. If we let $n = 0$, then $u(x,t) = \alpha\int_0^t I_m(x,t-s)ds$. We have Theorem 3 by the same proof. If $n > 0$, then by a priori assume

$$u(x,t) \le \frac{1}{2}, \quad for\ (x,t) \in R^N \times (0,\infty),$$

we have $1 \le \dfrac{1}{(1-u)^n} \le 2^n$. Similarly, for $n > 0$, we may prove Theorem 3.

References

[1] Kawarada H. On solutions of initial-boundary problem for u_t=u_xx+1/(1-u)[J]. Publications of the Research Institute for Mathematical Science, 1975, 10(3): 729-736.

[2] Levine H, John M. The quenching of solutions of some nonlinear parabolic equations[J]. SIAM Journal on Mathematical Analysis, 1980, 11(5): 842-847.

[3] Chiy Y C, Man K K. Quenching phenomena for singular nonlinear parabolic equations[J]. Nonlinear Analysis, 1988, 12(12): 1377-1383.

[4] Chiu Y C, Hans K. Quenching for semilinear singular parabolic problems[J]. SIAM Journal on Mathematical Analysis, 1989, 20(3): 558-566.

[5] Pinsky R. Existence and nonexistence of global solution[J]. Journal of Differential Equations. 1997, 33(6): 152-177.

[6] Acker A, Water W. The quenching problem for nonlinear parabolic differential equations.1-12: Lecture Notes in Mathematics, 1976.

[7] Levin H. Quenching, nonquenching and beyond quenching for solution of some parabolic equation[J]. Annali di Matematica Puraed Applicata, 1989, 155(4): 243-260.

[8] Yuan H Z, Mu C L. The quenching behavior of a nonlinear parabolic equation with nonliear boundary outfux[J]. Applied Mathematics and Computation, 2007, 184(2): 624-630.

[9] Levin H, Marek H. Quenching for quasilinear equations[J]. Communications in Partial Differential Equations, 1992, 17(3): 593-614.

[10] Boni T. On quenching of solutions for some semilinear parabolic equations of second order[J]. Bulletin of the Belgian Mathematical Society, 2000, 7(1): 73-95.

[11] Levin H. On the blow up of u_t at quenching[J]. Journal of Differential Equations, 1989, 106(6): 1049-1056.

[12] Robert C. Quenching for the heat equation with a nonlocal nonlinearity[J]. Nonlinear Problems in Applied Mathematics, 1995, 133(1): 199-205.

[13] Chan C. New results in quenching[J]. 1st World Congress Nonlinear Anal, 1996, 7(3): 427-434.

[14] Levin H. Advances in quenching[J]. Nonlinear Differential Equation, 1992, 155(4): 319-346.

[15] Levin H. The phenomenon of quenching: A survey[J]. Journal of Differential Equations, 1985, 17(3): 275-286.

[16] Dai Q, Gu Y. A short note on quenching phenomena for semilinear parabolic equations[J]. Journal of Differential Equations, 1997, 137(5): 240-250.

[17] Dai Q, Zhong Z. The quenching phenomena for the Cauchy problem of semilinear parabolic equations[J]. Journal of Differential Equations, 2001, 175(3): 163-174.

含非线性源的二阶非线性抛物方程的淬火现象

徐润章，张明有，刘杰

（哈尔滨工程大学理学院，哈尔滨 150001）

摘要：在这篇文章里，作者研究了一类二阶非线性抛物方程在无界域上的柯西问题的淬火现象。通过对具有非线性源项的二阶非线性抛物方程中的某些指数和初值添加约束条件，我们得到了方程的解在有限时间内淬火。本文的主要研究方法是比较原理和极值原理，特别值得注意的是在定理的证明过程中比较原理有很重要的作用。我们将结果拓展到更广泛的非线性项。

关键词：抛物方程；淬火现象；无界域

中图分类号：O175.29