非齐次非线性 Schrödinger 方程爆破解的 L^2-集中率

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摘要：本文研究临界幂非线性项的非齐次 Schrödinger 方程的爆破解. 利用对应基态变分特征, 我们得到爆破解的爆破速率以及爆破解的 L^2-集中率.

关键词：非齐次 Schrödinger 方程; 爆破解; 爆破率; L^2-集中率.

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Rate of L^2-Concentration of Blow-up Solutions for the Inhomogeneous Nonlinear Schrödinger Equation

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Abstract: This paper is concerned with the blow-up solutions for inhomogeneous Schrödinger equation with L^2 critical nonlinearity. Using the variational characteristic of the corresponding ground state, the blow-up rate and rate of L^2-concentration of blow-up solutions are obtained.

Key words: Inhomogeneous Schrödinger equation; Blow-up solution; Blow-up rate; Rate of L^2-concentration.

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0 Introduction

In this paper, we study the Cauchy problem of the following inhomogeneous Schrödinger equation with L^2 critical nonlinearity

$$iu_t + \triangle u + |x|^b |u|^{\frac{2b+4}{N}} u = 0,\ t \geq 0,\ x \in \mathbb{R}^N, \tag{0.1}$$

$$u(0, x) = u_0, \tag{0.2}$$

where $i = \sqrt{-1}$; $\triangle = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; $u = u(t,x)$: $[0,T) \times \mathbb{R}^N \to \mathcal{C}$ is the complex valued function and $0 < T \leq +\infty$; N is the space dimension; the parameter $b \geq 0$. A few years ago, it was suggested that stable high power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel(see[1, 2]). In this case, beam propagation can be modeled by the inhomogeneous nonlinear Schrödinger equation in the following form

$$i\phi_t + \triangle \phi + K(x)|\phi|^{p-2}\phi = 0, \quad \phi(0,x) = \varphi \in H^1(\mathbb{R}^N). \tag{0.3}$$

Recently, this type of inhomogeneous nonlinear Schrödinger equations have been widely investigated. When $k_1 \leq K(x) \leq k_2$ with $k_1,\ k_2 > 0$ and $p = 2 + \frac{4}{N}$, Merle[3] proved the existence and nonexistence of blow-up solutions of the Cauchy problem (0.3). When $K(x) = K(\varepsilon|x|) \in C^4(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ with ε small and $p = 2 + \frac{4}{N}$, Fibich, Liu and Wang[2, 4, 5] obtained the stability and instability of standing waves of the Cauchy problem (0.3).

For the Cauchy problem (0.1)-(0.2), Chen and Guo[6] showed the local well-posedness in $H_r^1 = H_r^1(\mathbb{R}^N)$, where $H_r^1(\mathbb{R}^N)$ is the set of radially symmetric functions in $H^1(\mathbb{R}^N)$. Chen[7] showed the sharp conditions of blow-up and global existence of the solutions. On the other hand, in Equation (0.1), the nonlinearity is in the form $|x|^b|u|^{\frac{2b+4}{N}}u$. Due to the unbounded potential $|x|^b$, to our knowledge, there are few results on the blow-up dynamical properties of the blow-up solutions.

Motivated by the studies of the classical homogeneous nonlinear Schrödinger equation(see[8, 9, 10]), we consider the ground state solution of the Cauchy problem (0.1)-(0.2), which is a special periodic solutions of Equation (0.1) in the form $u(t,x) = e^{i\omega t}Q(x)$, where $\omega \in \mathbb{R}$ and $Q(x)$ is called a ground state satisfying

$$-\triangle Q + \frac{b+2}{N} Q - |x|^b |Q|^{\frac{2b+4}{N}} Q = 0, \quad Q \in H_r^1. \tag{0.4}$$

In this paper, we call the solution of Equation (0.4) $Q = Q(x)$ as the ground state solution of Equation (0.4). Sintzoff and Willem[13] proved the existence of the ground state (0.4). Chen[7] showed the following generalized Gagliardo-Nirenberg inequality: $\forall\ f \in H_r^1(\mathbb{R}^N)$ and $b \geq 0$

$$\int |x|^b |f|^{\frac{2N+2b+4}{N}} dx \leq \frac{2N+2b+4}{2N} \left(\frac{\|f\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{2b+4}{N}} \|\nabla f\|_{L^2}^2, \tag{0.5}$$

where Q is the ground state solution of Equation (0.4). In the present paper, we firstly obtain the lower bound of blow-up rate of the solutions to the Cauchy problem (0.1)-(0.2) by the scaling invariance, as follows.

$$\|\nabla u(t,x)\|_{L^2} \geq \frac{K}{\sqrt{T-t}} \quad \text{as} \quad t \to T. \tag{0.6}$$

Moreover, we obtain the rate of L^2-concentration of the radially symmetric blow-up solutions. It reads that if $u(t,x)$ is the radially symmetric blow-up solution of the Cauchy problem (0.1)-(0.2) in finite time $0 < T < +\infty$, then, $\forall\ \varepsilon > 0$, $\exists\ K > 0$ such that

$$\liminf_{t \to T} \int_{|x| \leq K\sqrt{T-t}} |u(t,x)|^2 dx \geq (1-\varepsilon) \int |Q|^2 dx, \tag{0.7}$$

where Q is the ground state solution of Equation (0.4).

The major difficulties in studying the L^2-concentration of the radially symmetric blow-up solutions to the Cauchy problem (0.1)-(0.2) is that the nonlinearity containing a unbounded potential $|x|^b$. Firstly, Although our main arguments are from Merle and Tsutsumi[9, 10], we need some new estimations to deal with the unbounded potential $|x|^b$. Secondly, for the time being, as we have mentioned, the results in the present paper are new for the Cauchy problem (0.1)-(0.2) and the L^2-concentration properties of the radially symmetric blow-up solutions have definite meanings in physics. Finally, since $b > 0$, there is no space transformation invariance for Equation (0.1) and the uniqueness of the ground state solution of Equation (0.4) is still open. The L^2-concentration properties of blow-up solutions to the Cauchy problem (0.1)-(0.2) in the nonradial case is still unknown.

In this paper, we denote $L^q(\mathbb{R}^N)$, $\|\cdot\|_{L^q(\mathbb{R}^N)}$, $H_r^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \cdot dx$ by L^q, $\|\cdot\|_{L^q}$, H_r^1 and $\int \cdot dx$ respectively, and the various positive constants will be simply denoted by C.

1 Preliminary

For the Cauchy problem (0.1)-(0.2), the work space is defined by

$$H_r^1 := \{u \in H^1 |\ u(x) = u(r)\} \quad \text{where} \quad r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2},$$

which is a Hilbert space. Moreover, we define the energy functional $E(u)$ on H_r^1 by

$$E(u) := \frac{1}{2}\int |\nabla u|^2 dx - \frac{N}{2N+2b+4}\int |x|^b |u|^{\frac{2N+2b+4}{N}} dx.$$

The functional $E(u)$ is well-defined according to the Sobolev embedding theorem(see [8]). Chen and Guo[6] showed the local well-posedness for the Cauchy problem (0.1)-(0.2) in H_r^1, as follows.

Proposition 1. *(Chen and Guo[6]) Let $N \geq 3$ and $u_0 \in H_r^1$. There exists an unique solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) on the maximal time interval $[0,T)$ such that $u(t,x) \in$*

$C([0,T);H_r^1)$ and either $T=+\infty$ (global existence), or else $T<+\infty$ and $\lim_{t\to T}\|u(t,x)\|_{H_r^1}=+\infty$ (blow-up). Furthermore, for all $t\in[0,T)$, $u(t,x)$ satisfies the following conservation laws

(i) Conservation of mass

$$\int |u(t,x)|^2dx=\int |u_0|^2dx. \tag{1.1}$$

(ii) Conservation of energy

$$E(u(t,x))=E(u_0). \tag{1.2}$$

At the end of this section, we introduce two lemmas, which are important in studying the radially symmetric functions.

Lemma 1. *(Strauss[11]) Let $N\ge 2$ and $u(t)\in H_r^1$. Then, for any positive constant R, we have*

(i)

$$|x|^{\frac{N-1}{2}}|u(x)|\le C(N)\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{1.3}$$

(ii)

$$\|u(t)\|^2_{L^\infty(|x|>R)}\le CR^{1-N}\|\nabla u(t)\|_{L^2(|x|>R)}\|\ u(t)\|_{L^2(|x|>R)}. \tag{1.4}$$

Lemma 2. *(Rother[12]) If $N\ge 3$, $1\le p<+\infty$ and $p=2^*+\frac{2c}{N-2}$, then there exists a $C(N,c)>0$ such that for every $u\in D_r^{1,2}(\mathbb{R}^N)=\{u\in L^{2^*};\nabla u\in L^2\}$,*

$$\left(\int |x|^c|u|^pdx\right)^{\frac{2}{p}}\le C(N,c)\int|\nabla u|^2dx. \tag{1.5}$$

Proposition 2. *(Chen[7]) Let $N\ge 3$ and $0\le b<2(N-1)$. If $f\in H_r^1$, then*

$$\int |x|^b|u|^{\frac{2N+2b+4}{N}}dx\le\frac{2N+2b+4}{2N}\left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{2b+4}{N}}\|\nabla u\|^2_{L^2}, \tag{1.6}$$

where Q is the ground state solution of Equation (0.4).

2 Rate of L^2-Concentration

In this section, we first show the lower bound of blow-up rate of the solutions to the Cauchy problem (0.1)-(0.2) by the scaling-invariant of Equation (0.1). Secondly, we obtain the rate of L^2-concentration of the radially symmetric blow-up solutions to the Cauchy problem (0.1)-(0.2) by the generalized Gagliardo-Nirenberg inequality (1.6), as follows.

Theorem 1. *Let $N\ge 3$, $0\le b<2(N-1)$ and $u_0\in H_r^1$ be radially symmetric. Assume that $u(t,x)\in C([0,T);H_r^1)$ is the corresponding blow-up solution of the Cauchy problem (0.1)-(0.2) in finite time $0<T<+\infty$.*

(i)If $a(t)$ is a decreasing function from $[0,T)$ to $\mathbb{R}^+$ such that

$$\lim_{t\to T} a(t) \to 0 \quad \text{and} \quad \lim_{t\to T} \frac{\sqrt{T-t}}{a(t)} \to 0, \tag{2.1}$$

then

$$\liminf_{t\to T} \int_{|x|\le a(t)} |u(t,x)|^2 dx \ge \int |Q|^2 dx. \tag{2.2}$$

(ii)For any $\varepsilon > 0$,there exists a constant $K > 0$ such that

$$\liminf_{t\to T} \int_{|x|<K\sqrt{T-t}} |u(t,x)^2 dx \ge (1-\varepsilon) \int |Q|^2 dx, \tag{2.3}$$

where Q is the ground state solution of Equation (0.4).

In order to prove Theorem 1, we have to establish the lower bound of blow-up rate of the solutions to the Cauchy problem (0.1)-(0.2). Motivated by the study of the classical nonlinear Schrödinger equation(see[14, 15]), we have the following proposition.

Proposition 3. *Let $N \ge 3$, $0 \le b < 2(N-1)$ and $u(t,x)$ be the radially blow-up solution to the Cauchy problem (0.1)-(0.2) in finite time $0 < T < +\infty$. Then, there exists a constant $K = K(\|u_0\|_{L^2}) > 0$ such that*

$$\|\nabla u(t,x)\|_{L^2} \ge \frac{K}{\sqrt{T-t}}, \quad 0 \le t < T. \tag{2.4}$$

Proof. Motivated by the study of classical nonlinear Schrödinger equation (see [15]), for a fixed $0 \le t < T$, one defines

$$\psi^t(s,x) = \lambda^{\frac{N}{2}}(t) u(t + \lambda^2(t)s, \lambda(t)x) \tag{2.5}$$

with $\lambda(t) = \frac{1}{\|\nabla u\|_{L^2}}$. Note that $\psi^t(s,x)$ is defined for

$$t + \lambda^2(t)s < T \quad \Leftrightarrow \quad s < S_c = \frac{T-t}{\lambda^2(t)},$$

and for all $s \in [0, S_c)$, $\psi^t(s,x)$ satisfies

$$i\psi^t_s + \triangle\psi^t + |x|^b |\psi^t|^{\frac{2b+4}{N}} \psi^t = 0. \tag{2.6}$$

Moreover, since $\|\nabla u\|_{L^2} \to 0$ as $t \to T$, one has

$$\|\nabla \psi^t\|_{L^2} \to +\infty, \quad \text{as} \quad s \to S_c, \tag{2.7}$$

and

$$\|\psi^t(s=0,x)\|_{L^2} = \|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{2.8}$$

By the definition of $\lambda(t)$, one has

$$\|\nabla \psi^t(s=0,x)\|^2_{L^2} = \lambda^2(t) \|\nabla u(t)\|^2_{L^2} = 1, \tag{2.9}$$

which implies that

$$\|\psi^t(s=0,x)\|_{H_r^1}^2 = \|u_0\|_{L^2}^2 + 1. \tag{2.10}$$

On the other hand, by resolution of the Cauchy problem locally in time by fixed point arguments (see[6]), for all $c_1 > 0$, there exists a $t_1(c_1) > 0$ such that if $\|\psi^t(s=0,x)\|_{H_r^1}^2 \le c_1$, then there exists a $c_2 > 0$ such that $\|\psi^t(s,x)\|_{H_r^1}^2 \le c_2$ in the interval $t \in [0, t_1]$. Therefore, applying this statement with $c_1 = \|u_0\|_{L^2}^2 + 1$ (independent of t), one obtains that $\forall\ t \in [0,T)$

$$\frac{T-t}{\lambda^2(t)} = S_c \ge t_1, \tag{2.11}$$

which implies that

$$\|\nabla u(t)\|_{L^2}^2 \ge \frac{t_1}{T-t}.$$

This completes the proof of Proposition 3.2.

Now, using the generalized Gagliardo-Nirenberg inequality (1.6), we have the following proposition and Theorem 1 is a direct application.

Proposition 4. *Let $N \ge 3$, $0 \le b < 2(N-1)$ and $u_0 \in H_r^1$ be radially symmetric. Assume that $u(t,x) \in C([0,T); H_r^1)$ is the corresponding blow-up solution of the Cauchy problem (0.1)-(0.2) in finite time $0 < T < +\infty$. Set $\lambda(t) = \|\nabla u(t)\|_{L^2}$. Then,*

(i)If $a(t)$ is a decreasing function from $[0,T)$ to $\mathbb{R}^+$ such that

$$\lim_{t\to T} a(t) \to 0 \quad \text{and} \quad \lim_{t\to T} \frac{1}{\lambda(t)a(t)} \to 0, \tag{2.12}$$

then

$$\liminf_{t\to T} \int_{|x|\le a(t)} |u(t,x)|^2 dx \ge \int |Q|^2 dx, \tag{2.13}$$

(ii)For any $\varepsilon > 0$,there exists a constant $K > 0$ such that

$$\liminf_{t\to T} \int_{|x|<\frac{K}{\lambda(t)}} |u(t,x)^2 dx \ge (1-\varepsilon) \int |Q|^2 dx, \tag{2.14}$$

where Q is the ground state solution of Equation (0.4).

Proof. (i) Let $\rho(x) = \rho(|x|) \in C_0^\infty(\mathbb{R}^N)$ be a radially symmetric function such that

$$\rho(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| \ge 2, \end{cases}$$

and $\forall\ r > 0$, one denotes $\rho_r(x) = \rho(\frac{x}{r})$. Using these notations, one takes $\rho_a(x) = \rho(\frac{x}{a(t)})$, $\lambda(t) = \|\nabla u(t)\|_{L^2}$ and $\lambda_a(t) = \|\nabla \rho_a u(t)\|_{L^2}$. Since the initial data $u_0 \in H_r^1$, one has that the corresponding solution $u(t,x)$ is also radially symmetric according to the local well-posedness.

By the conservation of energy, one has

$$\frac{1}{2}\int |\nabla u|^2 dx - \frac{N}{2N+2b+4}\int_{|x|\le a(t)} |x|^b |u|^{\frac{2N+2b+4}{N}} dx = \frac{N}{2N+2b+4}\int_{|x|>a(t)} |x|^b |u|^{\frac{2N+2b+4}{N}} dx + E(u_0). \tag{2.15}$$

Since

$$-\frac{N}{2N+2b+4}\int|x|^b|\rho_a u|^{\frac{2N+2b+4}{N}}dx \le -\frac{N}{2N+2b+4}\int_{|x|\le a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx \tag{2.16}$$

and

$$\begin{aligned}\lambda_a^2(t) &= \|\nabla(\rho_a u(t))\|_{L^2}^2\\ &= \|\nabla\rho_a u(t)+\rho_a\nabla u(t)\|_{L^2}^2\\ &\le (\tfrac{C}{a(t)}\|u(t)\|_{L^2}+\|\nabla u(t)\|_{L^2})^2\\ &\le \lambda^2(t)+\tfrac{C}{a(t)}\lambda(t)+\tfrac{C}{a^2(t)},\end{aligned} \tag{2.17}$$

one has

$$\begin{aligned}&\tfrac12\int|\nabla(\rho_a u)|^2dx-\tfrac{N}{2N+2b+4}\int|x|^b|\rho_a u|^{\frac{2N+2b+4}{N}}dx\\ \le\ &\tfrac{N}{2N+2b+4}\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx+\tfrac{C}{a(t)}\lambda(t)+\tfrac{C}{a^2(t)}+C.\end{aligned} \tag{2.18}$$

On the other hand, by Lemma 2 and the conservation of mass, one has

$$\begin{aligned}\tfrac{N}{2N+2b+4}\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx &= \tfrac{N}{2N+2b+4}\int_{|x|>a(t)}(|x|^{\frac{N-1}{2}}|u(x)|)^{\frac{2b}{N-1}}|u|^{\frac{2N+2b+4}{N}-\frac{2b}{N-1}}dx\\ &\le C\|u\|_{L^2}^{\frac{b}{N-1}+2}\|\nabla u\|_{L^2}^{\frac{b}{N-1}}\|u\|_{L^\infty(|x|>a(t))}^{\frac{2N+2b+4}{N}-\frac{2b}{N-1}-2}\\ &\le \frac{C}{a^{\frac{2N-b-2}{N}}(t)}\|u\|_{L^2}^{\frac{2N+b+2}{N}}\|\nabla u\|_{L^2}^{\frac{b+2}{N}}\\ &\le \frac{C}{a^{\frac{2N-b-2}{N}}(t)}\|\nabla u\|_{L^2}^{\frac{b+2}{N}}.\end{aligned} \tag{2.19}$$

One claims that, for $0\le b<2(N-1)$

$$\lim_{t\to T}\sup\frac{\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx}{\|\nabla(\rho_a u)\|_{L^2}^2}=0. \tag{2.20}$$

Indeed, since $0\le b<2(N-1)$, one has $\frac{b+2}{N}<2$. It follows from the conservation laws and (1.6) that

$$\begin{aligned}\tfrac12\int|\nabla u|^2dx &= \tfrac{N}{2N+2b+4}\int_{|x|<a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx+\tfrac{N}{2N+2b+4}\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx+E(u_0)\\ &\le C\int|x|^b|\rho_a u|^{\frac{2N+2b+4}{N}}dx+\frac{C}{(a(t)\|\nabla u\|_{L^2})^{\frac{2N-b-2}{N}}}\|\nabla u\|_{L^2}^2+C\\ &\le C\|\rho_a u\|_{L^2}^{\frac{2b+4}{N}}\|\nabla(\rho_a u)\|_{L^2}^2+\frac{C}{(a(t)\|\nabla u\|_{L^2})^{\frac{2N-b-2}{N}}}\|\nabla u\|_{L^2}^2+C\\ &\le C\|\nabla(\rho_a u)\|_{L^2}^2+\frac{C}{(a(t)\lambda(t))^{\frac{2N-b-2}{N}}}\|\nabla u\|_{L^2}^2+C\\ &\le C\|\nabla(\rho_a u)\|_{L^2}^2+\varepsilon\|\nabla u\|_{L^2}^2+C,\end{aligned} \tag{2.21}$$

where $0<\varepsilon<\frac{1}{2}$ and in the last step, one uses (2.12) and $0\le b<2(N-1)$. Therefore, one has that $\exists K<+\infty$ such that

$$\limsup_{t\to T}\frac{\|\nabla u\|_{L^2}^2}{\|\nabla(\rho_a u)\|_{L^2}^2}=K. \tag{2.22}$$

It follows from (2.19) and (2.22) that

$$\begin{aligned}\frac{\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx}{\|\nabla(\rho_a u)\|_{L^2}^2} &\le C\left(\frac{\|\nabla u\|_{L^2}}{\|\nabla(\rho_a u)\|_{L^2}}\right)^{\frac{b+2}{N}}\frac{1}{a^{\frac{2N-b-2}{N}}(t)\|\nabla(\rho_a u)\|_{L^2}^{2-\frac{b+2}{N}}}\\ &\le C\frac{1}{(a(t)\lambda(t))^{\frac{2N-b-2}{N}}}.\end{aligned} \tag{2.23}$$

Since $0\le b<2(N-1)$, one has $2-\frac{b+2}{N}>0$, which implies that Claim (2.20) is true for $\frac{1}{a(t)\lambda(t)}\to 0$ as $t\to T$.

Applying the generalized Gagliardo-Nirenberg inequality (1.6), it follows from (2.18) and (2.20) that

$$[1-\left(\frac{\|\rho_a u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{2b+4}{N}}]\|\nabla(\rho_a u)\|_{L^2}^2\le\frac{2N}{2N+2b+4}\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx+\frac{C}{a(t)}\lambda(t)+\frac{C}{a^2(t)}+C,$$

where Q is the ground state solution of Equation (0.4).

Therefore, by (2.22), one has

$$[1-\left(\frac{\|\rho_a u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{2b+4}{N}}]\le C\frac{\int_{|x|>a(t)}|x|^b|u|^{\frac{2N+2b+4}{N}}dx}{\lambda_a^2(t)}+\frac{C}{a(t)\lambda_a(t)}+\frac{C}{a^2(t)\lambda_a^2(t)}+\frac{C}{\lambda_a^2(t)}, \tag{2.24}$$

It follows from the Claim (2.12) and (2.20) that

$$\limsup_{t\to T}\left(1-\left(\frac{\|\rho_a u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{2b+4}{N}}\right)\le 0.$$

Therefore, one has

$$\liminf_{t\to T}\int_{|x|\le 2a(t)}|u(t,x)|^2dx\ge\liminf_{t\to T}\int|\rho_a u(t,x)|^2dx\ge\int|Q|^2dx, \tag{2.25}$$

which implies that (2.13) is true.

(ii) The proof is similar with (i). Taking $a(t)=\frac{K}{\lambda(t)}$, where K is an arbitrary positive constant. By (2.24), one has

$$[1-\left(\frac{\|\rho(\frac{\lambda(t)}{K}x)u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{2b+4}{N}}]\le CK^{-\frac{2N-b-2}{N}}+CK^{-1}+CK^{-2}. \tag{2.26}$$

Taking K sufficient large and letting $t\to T$ in (2.26), one has that (2.14) is true.

At the end of this section, we shall give the proof of Theorem 1.

Proof of Theorem 1. By Proposition 3, one has that there exists a constant $M>0$ such that

$$\lambda(t) \leq M\sqrt{T-t}, \quad 0 \leq t < T. \tag{2.27}$$

Applying Proposition 4, one has that the conclusions in Theorem 1 hold. This completes the proof.

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