三维空间中 Davey-Stewartson 系统的最佳爆破准则

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摘要：本文研究如下的 Davey-Stewartson 系统的爆破解

$$iu_t + \triangle u + |u|^{p-1}u + E(|u|^2)u = 0,\ t \geq 0,\ x \in R^3, \qquad (DS)$$

其中 $1 < p < 5$. 首先，利用 $H^1(R^3)$ 中有界序列的图景分解, 我们给出了基态的一些新变分特征以及广义 Gagliardo-Nirenberg 不等式. 进而, 对于 $1+\frac{4}{3} \leq p < 5$, 我们得到 (DS) 爆破解存在的最佳判别准则.

关键词：Davey-Stewartson 系统; 图景分解; 爆破解; 最佳准则.

中图分类号：　175.26

Sharp Blow-up Criteria for the Davey-Stewartson System in R^3

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Abstract: In this paper, we study the blow-up solutions for the Davey-Stewartson system

$$iu_t + \triangle u + |u|^{p-1}u + E(|u|^2)u = 0,\ t \geq 0,\ x \in R^3, \qquad (DS)$$

where $1 < p < 5$. Firstly, using the profile decomposition of the bounded sequences in $H^1(R^3)$, we give some new variational characteristics for the ground states and generalized Gagliardo-Nirenberg inequalities. Moreover, we obtain the precise expressions on the sharp blow-up criteria to (DS) for $1 + \frac{4}{3} \leq p < 5$.

Key words: Davey-Stewartson system; Profile decomposition; Blow-up solution; Sharp criteria.

0　Introduction

This paper is concerned with the Cauchy problem of the following Davey-Stewartson system

$$iu_t + \triangle u + |u|^{p-1}u + E(|u|^2)u = 0, \ t > 0, \ x \in R^N, \tag{0.1}$$

$$u(0, x) = u_0, \tag{0.2}$$

where $i = \sqrt{-1}$; $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$ is the Laplace operator on R^N; $u = u(t, x)$: $[0, T) \times R^N \to \mathcal{C}$ is the complex valued function and $0 < T \le +\infty$; the parameter $1 < p < 2^* - 1$ (where $2^* = +\infty$ for $N = 2$; $2^* = \frac{2N}{N-2}$ for $N \ge 3$); E is the singular integral operator with symbol $\sigma_1(\xi) = \frac{\xi_1^2}{|\xi|^2}, \xi \in R^N$, $E(\psi) = \mathcal{F}^{-1}\frac{\xi_1^2}{|\xi|^2}\mathcal{F}\psi$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and Fourier inverse transform on R^N, respectively.

For the Cauchy problem (0.1)-(0.2), Ghidaglia and Saut[1], Guo and Wang[2] established the local well-posedness in the energy space H^1 for $N = 2$ and $N = 3$ respectively (see[3, 4] for a review). Cipolatti [5] showed the existence of the standing waves. Cipolatti[6], Ohta[7, 8], Gan and Zhang[9] showed the stability and instability of the standing waves. Ghidaglia and Saut[1], Guo and Wang [2] showed the existence of the blow-up solutions. Ozawa[10] gave the exact blow-up solutions. Wang and Guo [11] studied the scattering of solutions. Richards[12], Papanicolaou etal [13], Gan and Zhang[14, 9], Shu and Zhang[15] studied the sharp conditions of blow-up and global existence. Richards [12] obtained the mass-concentration properties of the blow-up solutions in L^2-critical case in R^2.

We note that in R^2, Richards[12] and Papanicolaou etal [13] gave the precise expression on the sharp blow-up criteria in L^2-critical case that is $N = 2$ and $p = 3$. But in R^3 Equation (0.1) has not any L^2-critical case because of influence of the nonlocal term $E(|u|^2)u$. Although in [14, 9, 15] some sharp thresholds of blow-up and global existence are gotten, we also note that the upper bound d of the energy functional $I(u)$ is not determined. This motivates us to investigate the precise expression on the sharp blow-up criteria in R^3.

In the present paper, firstly, we study the sharp blow-up criteria for the Cauchy problem (0.1)-(0.2) in R^3 for $1 + \frac{4}{3} \le p < 5$. Motivated by Holmer and Roudenko's studies[16] for the classical L^2-super nonlinear Schrödinger equation, we consider the following elliptic equations

$$\frac{3}{2}\triangle Q - \frac{1}{2}Q + |Q|^2Q + E(|Q|^2)Q = 0, \ Q \in H^1(R^3) \tag{0.3}$$

and

$$\frac{3}{2}\triangle R - \frac{1}{2}R + E(|R|^2)R = 0, \ \ R \in H^1(R^3). \tag{0.4}$$

Applying the profile decomposition of the bounded sequences in $H^1(R^3)$, we obtain the following generalized Gagliardo-Nirenberg inequalities

$$\int |f|^4 + E(|f|^2)|f|^2 dx \le \frac{2}{\|Q\|_2^2}\|\nabla f\|_2^3\|f\|_2, \ \ \forall \ f \in H^1(R^3) \tag{0.5}$$

and

$$\int E(|f|^2)|f|^2 dx \leq \frac{2}{\|R\|_2^2} \|\nabla f\|_2^3 \|f\|_2, \quad \forall\, f \in H^1(R^3) \tag{0.6}$$

where Q is the solution of Equation (0.3) and R is the solution of Equation(0.4). Using the above Gagliardo-Nirenberg inequalities, we obtain the sharp blow-up criteria to the Cauchy problem (0.1)-(0.2) for $1 + \frac{4}{3} \leq p < 5$ by overcoming the loss of scaling invariance. We remark that we get a clear bound value of energy functional, which corresponds to d in [9]. Furthermore, we prove that there is no L^3 strong limit of the blow-up solutions to the Cauchy problem (0.1)-(0.2) provided $1 < p \leq 3$.

There are two major difficulties in the analysis of blow-up solutions to the Davey-Stewartson system (0.1)-(0.2) in $H^1(R^3)$: one is the nonlinearity containing the singular integral operator E; the other one is the loss of scaling invariance to Equation (0.1) for $p \neq 3$, which destroys the balance between $|u|^{p-1}u$ and $E(|u|^2)u$. Due to the singular integral operator E, we have to establish the corresponding generalized Gagliardo-Nirenberg inequalities and variational structures. Since there is no scaling invariance for $p \neq 3$, improving Holmer and Roudenko's method we use the ground state of the classical nonlinear Schrödinger equation to describe the sharp blow-up criteria to the Davey-Stewartson system (0.1)-(0.2). Finally, for the time being, as we have mentioned, the results in the present paper are new for the Davey-Stewartson system (0.1)-(0.2). In particular, the sharp blow-up criteria are different from [9], and the sharp blow-up criteria obtained in this paper are more precisely, which is very useful from the viewpoint of physics.

We conclude this section with several notations. We abbreviate $L^q(R^3)$, $\|\cdot\|_{L^q(R^3)}$, $H^s(R^3)$ and $\int_{R^3} \cdot dx$ by L^q, $\|\cdot\|_q$, H^s and $\int \cdot dx$. The various positive constants will be simply denoted by C.

1 Preliminary

For the Cauchy problem (0.1)-(0.2), the energy space is defined by

$$H^1 := \{u \in L^2 \,;\, \nabla u \in L^2\},$$

which is a Hilbert space. The norm of H^1 is denoted by $\|\cdot\|_{H^1}$. Moreover, we define a functional $H(u)$ in H^1 by

$$H(u) := \frac{1}{2} \int |\nabla u(t,x)|^2 dx - \frac{1}{p+1} \int |u(t,x)|^{p+1} dx - \frac{1}{4} \int E(|u|^2)|u|^2 dx.$$

The functional H is well-defined according to the Sobolev embedding theorem and the properties of the singular operator E.

Guo and Wang[2] established the local well-posedness of the Cauchy problem (0.1)-(0.2) in energy space H^1.

Proposition 1. *Let $u_0 \in H^1$. There exists an unique solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) on the maximal time $[0,T)$ such that $u(t,x) \in C([0,T);H^1)$ and either $T = +\infty$(global existence), or else $T < +\infty$ and $\lim_{t \to T} \|u(t,x)\|_{H^1} = +\infty$ (blow-up). Furthermore, for all $t \in [0,T)$, $u(t,x)$ satisfies the following conservation laws*

(i) Conservation of mass

$$\|u(t,x)\|_2 = \|u_0\|_2, \tag{1.1}$$

(ii) Conservation of energy

$$H(u(t,x)) = H(u_0). \tag{1.2}$$

For more specific results concerning the Cauchy problem (0.1)-(0.2), we refer the reader to[1, 4]. In addition, by some basic calculations, we have the following proposition(see also Ohta[8]).

Proposition 2. *Assume that $u_0 \in H^1$, $|x|u_0 \in L^2$ and the corresponding solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) on the interval $[0,T)$. Then, for all $t \in [0,T)$ we have $|x|u(t,x) \in L^2$. Moreover, let $J(t) := \int |x|^2 |u(t,x)|^2 dx$, we have*

$$J^{'}(t) = -4\Im \int x u \nabla \overline{u} dx \tag{1.3}$$

and

$$J^{''}(t) = 8 \int |\nabla u|^2 dx - 12\frac{p-1}{p+1} \int |u|^{p+1} dx - 6 \int E(|u|^2)|u|^2 dx. \tag{1.4}$$

We give some known facts of the singular integral operator E (see Cipolatti[5, 6]), as follows.

Lemma 1. *Let E be the singular integral operator defined in Fourier variables by*

$$\mathcal{F}[E(\psi)](\xi) = \sigma_1(\xi)\mathcal{F}[\psi](\xi),$$

where $\sigma_1(\xi) = \frac{\xi_1^2}{|\xi|^2}, \xi \in R^3$ and $\mathcal{F}$ denotes the Fourier transform in R^3. For $1 < p < +\infty$, E satisfies the following properties:

(i) $E \in \mathcal{L}(L^p, L^p)$, where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded linear operators from L^p to L^p.

(ii) If $\psi \in H^s(R^3)$, then $E(\psi) \in H^s(R^3)$, $s \in R$.

(iii) If $\psi \in W^{m,p}$, then $E(\psi) \in W^{m,p}$ and

$$\partial_k E(\psi) = E(\partial_k \psi), \quad k = 1, 2.$$

(iv) E preserves the following operations:

- *translation:* $E(\psi(\cdot + y))(x) = E(\psi)(x + y), \ y \in R^3$;

- *dilatation:* $E(\psi(\lambda\cdot))(x) = E(\psi)(\lambda x), \lambda > 0$;

- *conjugation:* $\overline{E(\psi)} = E(\overline{\psi})$,

where $\overline{\psi}$ *is the complex conjugate of* ψ.

At the end of this section, we shall give the profile decomposition of the bounded sequences in H^1 proposed by Gérard[17], Hmidi and Keraani [18], which is important to study the variational characteristic of the ground state.

Proposition 3. *Let* $\{v_n\}_{n=1}^\infty$ *be a bounded sequence in* H^1. *Then there is a subsequence of* $\{v_n\}_{n=1}^\infty$ *(still denoted by* $\{v_n\}_{n=1}^\infty$ *) and a sequence* $\{V^j\}_{j=1}^\infty$ *in* H^1 *and a family of* $\{x_n^j\}_{j=1}^\infty \subset R^3$ *such that*

(i) for every $j \neq k$, $|x_n^j - x_n^k| \overset{n \to \infty}{\to} +\infty$;

(ii) for every $l \geq 1$ *and every* $x \in R^3$

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x) \tag{1.5}$$

with

$$\lim_{n \to \infty} \sup \|v_n^l\|_r \overset{l \to \infty}{\to} 0, \tag{1.6}$$

for every $r \in (2, 6)$.

Moreover, we have, as $n \to \infty$,

$$\|v_n\|_2^2 = \sum_{j=1}^l \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1) \tag{1.7}$$

and

$$\|\nabla v_n\|_2^2 = \sum_{j=1}^l \|\nabla V^j\|_2^2 + \|\nabla v_n^l\|_2^2 + o(1). \tag{1.8}$$

2 Sharp Gagliardo-Nirenberg Inequalities

In order to study the sharp blow-up criteria to the Cauchy problem (0.1)-(0.2), we have to establish the generalized Gagliardo-Nirenberg inequalities corresponding to the Davey-Stewartson system.

Firstly, we consider the following elliptic equation

$$\frac{3}{2}\triangle Q - \frac{1}{2}Q + |Q|^2 Q + E(|Q|^2)Q = 0, \ Q \in H^1, \tag{2.1}$$

which is the ground state of the Davey-Stewartson system (0.1)-(0.2) for $p = 3$. Cipolatti[5] showed the existence of the elliptic Equation (2.1) in dimension two and three. Papanicolaou etal [13] computed the best constant of the generalized Gagliardo-Nirenberg inequality in dimension two.

In the present paper, applying the profile decomposition of the bounded sequences in $H^1(R^3)$, we obtain a new and simple proof of the existence for Equation (2.1) and the best constant of the generalized Gagliardo-Nirenberg inequality in dimension three. We should point out that the method used in this paper is different from [5, 13]. This method can be used for the dimension $N = 2$.

Theorem 1. *Let $f \in H^1$, then*

$$\int |f|^4 + E(|f|^2)|f|^2 dx \le \frac{2}{\|Q\|_2^2}\|\nabla f\|_2^3\|f\|_2, \tag{2.2}$$

where Q is the solution of Equation (2.1).

In order to prove the best constant of the generalized Gagliardo-Nirenberg inequality (2.2), we consider the variational problem

$$J := \inf\{J(u) : u \in H^1\} \quad \text{where} \quad J(u) := \frac{(\int |u|^2 dx)^{\frac{1}{2}}(\int |\nabla u|^2 dx)^{\frac{3}{2}}}{\int(|u|^4 + E(|u|^2)|u|^2)dx}. \tag{2.3}$$

It is obvious that if W is the minimizer of $J(u)$, then $|W|$ is also a minimizer. Hence we can assume that W is an real positive function. Indeed, $W = |W|e^{i\theta(x)}$ we have

$$|\nabla|W|| \le |\nabla W|$$

in the sense of distribution. On the other hand, if $W \in H^1$, then$|W| \in H^1$ and $J(|W|) \le J(W)$.

By some basic calculations, if W is the minimizer of $J(u)$, we have the following lemma.

Lemma 2. *If W is the minimizer of $J(u)$, then W satisfies*

$$\frac{3}{2}\|W\|_2\|\nabla W\|_2\triangle W - \frac{1}{2}\frac{\|\nabla W\|_2^3}{\|W\|_2}W + 2J(|W|^2 + E(|W|^2))W = 0. \tag{2.4}$$

Proof. Since W is a minimizing function of $J(u)$ in H^1, we have

$$\frac{d}{d\varepsilon}J(W + \varepsilon v) \mid_{\varepsilon=0} = 0. \tag{2.5}$$

By some basic calculations, we have

$$\frac{d}{d\varepsilon}\{\|(W + \varepsilon v)\|_2\|\nabla(W + \varepsilon v)\|_2^3\} \mid_{\varepsilon=0} = \frac{1}{2}\frac{\|\nabla W\|_2^3}{\|W\|_2}\int 2\Re W\overline{v}dx - \frac{3}{2}\|W\|_2\|\nabla W\|_2\int 2\Re\triangle W\overline{v}dx \tag{2.6}$$

and

$$\frac{d}{d\varepsilon}\{\int |(W + \varepsilon v)|^4 + E(|W + \varepsilon v|^2)|W + \varepsilon v|^2 dx\} \mid_{\varepsilon=0} = 4\int(|W|^2 + E(|W|^2))\Re W\overline{v}dx. \tag{2.7}$$

By (2.5)-(2.7), we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\|\nabla W\|_2^3}{\|W\|_2}\int 2\Re W\overline{v}dx - \frac{3}{2}\|W\|_2\|\nabla W\|_2\int 2\Re\triangle W\overline{v}dx \\
= \ & 2\frac{\|\nabla W\|_2^3\|W\|_2}{\int |W|^4 + E(|W|^2)|W|^2 dx}\int(|W|^2 + E(|W|^2))2\Re W\overline{v}dx,
\end{aligned}
\tag{2.8}
$$

which implies that (2.4) is true.

Now, we use the profile decomposition of the bounded sequences in H^1 to prove the following proposition, and Theorem 1 is a direct conclusion of the following proposition.

Proposition 4. *J is attained at a function $U(x)$ with the following properties:*

$$
U(x) = aQ(\lambda x + b) \quad \text{for} \quad \text{some} \quad a \in C^*, \ \lambda > 0 \ \text{and} \ b \in R^3 \tag{2.9}
$$

where Q is the solution of Equation(2.1). Moreover,

$$
J = \frac{\|Q\|_2^2}{2}. \tag{2.10}
$$

Proof. If we set $u^{\lambda,\mu} = \mu u(\lambda x)$, where $\lambda = \frac{\|u\|_2}{\|\nabla u\|_2}$, $\mu = \frac{\|u\|_2^{\frac{1}{2}}}{\|\nabla u\|_2^{\frac{3}{2}}}$, we have

$$
\|u^{\lambda,\mu}\|_2 = 1, \quad \|\nabla u^{\lambda,\mu}\|_2 = 1 \ \text{and} \ J(u^{\lambda,\mu}) = J(u).
$$

Now, choosing a minimizing sequence $\{u_n\}_{n=1}^\infty \subset H^1$ such that $J(u_n) \to J$ as $n \to \infty$, after scaling, we may assume

$$
\|u_n\|_2 = 1 \ \text{and} \ \|\nabla u_n\|_2 = 1, \tag{2.11}
$$

and we have

$$
J(u_n) = \frac{1}{\int |u_n|^4 + E(|u_n|^2)|u_n|^2 dx} \to J, \ \text{as} \ n \to \infty. \tag{2.12}
$$

Note that $\{u_n\}_{n=1}^\infty$ is bounded in H^1. It follows form the profile decomposition (Proposition 3) that

$$
u_n(x) = \sum_{j=1}^l U^j(x - x_n^j) + r_n^l(x),
$$

$$
\sum_{j=1}^l \|U_n^j\|_2^2 \leq 1 \ \text{and} \ \sum_{j=1}^l \|\nabla U_n^j\|_2^2 \leq 1, \tag{2.13}
$$

where $U_n^j = U^j(x - x_n^j)$. Moreover, using the Hölder's inequality for r_n^l and the properties of E, we have

$$
\begin{aligned}
\int(|r_n^l|^4 + E(|r_n^l|^2)|r_n^l|^2)dx \ & \leq C(\|r_n^l\|_4^4 + \|E(|r_n^l|^2)\|_2\|r_n^l\|_4^2) \\
& \leq C(\|r_n^l\|_4^4 + \|r_n^l\|_4^4) \to 0, \ \text{as} \ l \to \infty.
\end{aligned}
\tag{2.14}
$$

Applying the orthogonal conditions and the properties of E, we have the following claims:

(i)

$$\int |\sum_{j=1}^{l} U^j(x - x_n^j)|^4 dx \to \sum_{j=1}^{l} \int |U^j|^4 dx, \quad \text{as} \quad n \to \infty. \tag{2.15}$$

(ii)

$$\int E(|\sum_{j=1}^{l} U^j(x-x_n^j)|^2)|\sum_{j=1}^{l} U^j(x-x_n^j)|^2 dx \to \sum_{j=1}^{l} \int E(|U^j|^2)|U^j|^2 dx, \quad \text{as} \quad n \to \infty. \tag{2.16}$$

Indeed, for (2.15), it suffices to show that

$$I_n = \int U_n^{j_1} U_n^{j_2} U_n^{j_3} U_n^{j_4} dx \to 0, \quad \text{as} \quad n \to \infty \tag{2.17}$$

for $1 \le j_k \le l$ and at least two j_k are different. Assuming for example $j_1 \ne j_2$, by the Hölder's inequality, we can estimate

$$|I_n|^2 \le C^2 \|U_n^{j_1} U_n^{j_2}\|_2^2,$$

where $C = \Pi_{k=3}^{4} \|U_n^{j_k}\|_4$. Without loss of generality, we can assume that both U^{j_1} and U^{j_2} are continuous and compactly supported. Now, we use the pairwise orthogonal conditions, and we have the following estimation

$$|I_n|^2 \le C^2 \int |U^{j_1}(y)U^{j_2}(y - (x_n^{j_2} - x_n^{j_1}))|^2 dy \to 0, \quad \text{as } n \to \infty. \tag{2.18}$$

This completes the proof of Claim (i).

For (2.16), we have

$$\int E(|\sum_{j=1}^{l} U^j(x - x_n^j)|^2)|\sum_{j=1}^{l} U^j(x - x_n^j)|^2 dx$$

$$\le C \quad (\int [\sum_{j=1}^{l} E(|U_n^j|^2) + \sum_{1 \le j,k \le l, j \ne k} E(|U_n^j U_n^k|)][\sum_{j=1}^{l} |U_n^j|^2 + \sum_{1 \le j,k \le l, j \ne k} |U_n^j U_n^k|] dx),$$

which implies that

$$\int E(|\sum_{j=1}^{l} U^j(x - x_n^j)|^2)|\sum_{j=1}^{l} U^j(x - x_n^j)|^2 dx - \sum_{j=1}^{l} \int E(|U^j|^2)|U^j|^2 dx$$

$$\le \quad C \sum_{j=1}^{l} \int E(|U_n^j|^2)|U_n^j|^2 dx + \sum_{1 \le j,k \le l, j \ne k} \int E(|U_n^j|^2)|U_n^k|^2 dx$$

$$+ C \sum_{1 \le i,j,k \le l, i \ne j} \int E(|U_n^i U_n^j|)|U_n^k|^2 dx + \sum_{1 \le i,j,k \le l, j \ne k} \int E(|U_n^i|^2)|U_n^j U_n^k| dx \tag{2.19}$$

$$+ C \sum_{1 \le i,j,k,m \le l, i \ne j, k \ne m} \int E(|U_n^i U_n^j|)|U_n^k U_n^m| dx$$

$$= \quad I + II + III.$$

Without loss of generality, we can assume that U^i, U^j, U^k and U^m are continuous and compactly supported. Using the orthogonal conditions and the properties of the singular operator $E(u)$, we have

$$I = C \sum_{1 \le j,k \le l, j \ne k} \int E(|U_n^j|^2)|U_n^k|^2 dx = \sum_{1 \le j,k \le l, j \ne k} \int E(|U^j|^2)(x - x_n^j)|U^k(x - x_n^k)|^2 dx$$

$$= C \sum_{1 \le j,k \le l, j \ne k} \int E(|U^j|^2)(x)|U^k(x - (x_n^k - x_n^j))|^2 dx$$

$$\to 0, \text{ as } n \to \infty,$$

$$II \le C \sum_{1 \le i,j,k \le l, i \ne j} [\|E(|U_n^i U_n^j|)\|_2 \|U_n^k\|_4^2 + \|E(|U_n^k|^2)\|_{L^2} \|U_n^i U_n^j\|_2]$$

$$\le C \sum_{1 \le i,j,k \le l, i \ne j} \|U_n^i U_n^j\|_2 \|U_n^k\|_4^2$$

$$\to 0, \text{ as } n \to \infty$$

and

$$III \le C \sum_{1 \le i,j,k,m \le l, i \ne j, k \ne m} \|E(|U_n^i U_n^j|)\|_2 \|U_n^k U_n^m\|_2 \to 0, \text{ as } n \to \infty.$$

The last step estimations of I, II and III follows from the proof of Claim (i) and this completes the proof of Claim (ii).

Therefore, by (2.12) and (2.14)- (2.16), we have

$$\sum_{j=1}^{l} \int (|U^j|^4 + E(|U^j|^2)|U^j|^2) dx \to \frac{1}{J}, \text{ as } n \to \infty. \tag{2.20}$$

On the other hand, by the definition of J, we have

$$J \int (|U^j|^4 + E(|U^j|^2)|U^j|^2) dx \le \|U^j\|_2 \|\nabla U^j\|_2^3. \tag{2.21}$$

Since the series $\sum_j \|U^j\|_2^2$ is convergent, there exists a $j_0 \ge 1$ such that

$$\|U^{j_0}\|_2 = \sup\{\|U^j\|_2 \ ; \ j \ge 1\}. \tag{2.22}$$

It follows from (2.20)-(2.22) that

$$1 \le J \left(\sum_{j=1}^{l} \int (|U^j|^4 + E(|U^j|^2)|U^j|^2) dx \right) \le \sup\{\|U^j\|_2 \ ; \ j \ge 1\} \left(\sum_{j=1}^{l} \|\nabla U^j\|_2^3 \right)$$

$$\le \|U^{j_0}\|_2 \left(\sum_{j=1}^{l} \|\nabla U^j\|_2^2 \right) \le \|U^{j_0}\|_2. \tag{2.23}$$

It follows from (2.13) that $\|U^{j_0}\|_2 = 1$, which implies that there exists only one term $U^{j_0} \ne 0$ such that

$$\|U^{j_0}\|_2 = 1, \ \|\nabla U^{j_0}\|_2 = 1 \text{ and } \int (|U^{j_0}|^4 + E(|U^{j_0}|^2)|U^{j_0}|^2) dx = \frac{1}{J}. \tag{2.24}$$

Therefore, we show that U^{j_0} is the minimizer of $J(u)$. It follows from Lemma 2 that

$$\frac{3}{2}\triangle U^{j_0} - \frac{1}{2}U^{j_0} + 2J(|U^{j_0}|^2 + E(|U^{j_0}|^2))U^{j_0} = 0. \tag{2.25}$$

We may assume that U^{j_0} is an real positive function by the definition of $J(u)$.

On the other hand, if Q is the solution of Equation (2.1), we claim

$$-\frac{1}{2}\int Q^2 dx - \frac{3}{2}\int |\nabla Q|^2 dx + \int Q^4 + E(|Q|^2)Q^2 dx = 0 \tag{2.26}$$

and

$$\int Q^2 dx + \int |\nabla Q|^2 dx - \int Q^4 + E(|Q|^2)Q^2 dx = 0. \tag{2.27}$$

Indeed, multiplying (2.1) by Q and integrating by parts, we have that (2.26) is true.

For another thing, multiplying (2.1) by $x \cdot \nabla Q$ and integrating by parts, we have

$$\frac{3}{2}\int \triangle Q x \cdot \nabla Q dx - \frac{1}{2}\int Q x \cdot \nabla Q dx + \int Q^2 Q x \cdot \nabla Q dx + \int E(|Q|^2)Q x \cdot \nabla Q dx = 0.$$

It follows from some basic calculations that

$$-\int \triangle Q x \cdot \nabla Q dx = \int |\nabla Q|^2 dx + \int x \cdot \nabla(\frac{|\nabla Q|^2}{2})dx = -\frac{1}{2}\int |\nabla Q|^2 dx,$$

$$\int Q x \cdot \nabla Q dx = -\frac{3}{2}\int Q^2 dx,$$

$$\int Q^2 Q x \cdot \nabla Q dx = -\frac{3}{4}\int Q^4 dx$$

and

$$\int E(|Q|^2)Q x \cdot \nabla Q dx = -\frac{3}{4}\int E(|Q|^2)Q^2 dx.$$

Collecting the above identities, we have that (2.27) is true.

Now, we return to the proof of Proposition 4, and we take $U^{j_0} = aQ(\lambda x + b)$ with Q is the positive solution of (2.1). By some computations, we have that $\|U^{j_0}\|_2^2 = \frac{a^2}{\lambda^3}\|Q\|_2^2 = 1$, $\|\nabla U^{j_0}\|_2^2 = \frac{a^2}{\lambda}\|\nabla Q\|_2^2 = 1$ and $\int(|U^{j_0}|^4 + E(|U^{j_0}|^2)|U^{j_0}|^2)dx = \frac{a^4}{\lambda^3}\int(|Q|^4 + E(|Q|^2)|Q|^2)dx = \frac{1}{J}$. Applying Claim (2.26) and (2.27), we have

$$\int(|Q|^4 + E(|Q|^2)|Q|^2)dx = 2\int Q^2 dx = 2\int |\nabla Q|^2 dx,$$

which implies that

$$J = \frac{\lambda^3}{a^4}\frac{1}{\int(|Q|^4 + E(|Q|^2)|Q|^2)dx} = \frac{1}{2a^2} = \frac{\|Q\|_2^2}{2}. \tag{2.28}$$

This completes the proof Proposition 4.

In the end of this section, we consider the following elliptic equation

$$\frac{3}{2}\triangle R - \frac{1}{2}R + E(|R|^2)R = 0, \quad R \in H^1. \tag{2.29}$$

By the same argument in Theorem 1, we have the following theorem.

Theorem 2. *Let $f \in H^1$, then*

$$\int E(|f|^2)|f|^2 dx \leq \frac{2}{\|R\|_2^2} \|\nabla f\|_2^3 \|f\|_2, \tag{2.30}$$

where R is the solution of Equation(2.29).

Remark 1. *The best constant of the generalized Gagliardo-Nirenberg inequalities (2.2) and (2.30) are dependent on space dimension N, but it is independent of the choice of the ground state solution $Q(x)$ and $R(x)$. Since the effect of singular integral operator E, the uniqueness of elliptic Equations (2.1) and (2.29) is still open. In this paper, for the elliptic Equations (2.1) and (2.29), we suppose all minimizers of the corresponding variational problem have the same L^2 norm.*

In the end, we collect Weinstein's[22] results, and we consider the following elliptic equation

$$\frac{3(p-1)}{4}\triangle P - \frac{5-p}{4}P + |P|^{p-1}P = 0, \ P \in H^1. \tag{2.31}$$

Strauss[21] showed the existence of equation (2.31). Weinstein[22] showed the best constant of the Gagliardo-Nirenberg inequality, as follows

Proposition 5. *Let $f \in H^1$ and $1 < p < 5$, then*

$$\|f\|_{p+1}^{p+1} \leq \frac{p+1}{2\|P\|_2^{p-1}} \|\nabla f\|_2^{\frac{3(p-1)}{2}} \|f\|_2^{\frac{5-p}{2}}, \tag{2.32}$$

where P is the solution of Equation (2.31).

3   Sharp Blow-up Criteria

In this section, using the sharp Gagliardo-Nirenberg inequalities obtained in Section 2, we obtain the sharp blow-up criteria to the Cauchy problem (0.1)-(0.2). More precisely, establishing four classes of invariant evolution flows according to the value of p, we obtain the sharp blow-up criteria to the Cauchy problem (0.1)-(0.2) for all $1 + \frac{3}{4} \leq p < 5$.

• **Sharp Criteria for** $p = 3$

Theorem 3. *Let $p = 3$, $u_0 \in H^1$ and satisfy*

$$H(u_0)\|u_0\|_2^2 < \frac{2}{27}\|Q\|_2^4. \tag{3.1}$$

Then, we have that

(i) *If*

$$\|\nabla u_0\|_2 \|u_0\|_2 < \frac{2}{3}\|Q\|_2^2, \tag{3.2}$$

then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) exists globally. Moreover, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2 \|u(t,x)\|_2 < \frac{2}{3}\|Q\|_2^2. \tag{3.3}$$

(ii) *If*

$$\|\nabla u_0\|_2 \|u_0\|_2 > \frac{2}{3}\|Q\|_2^2, \tag{3.4}$$

and $|x|u_0 \in L^2$, then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$,

where Q is the solution of Equation (2.1).

Proof. Applying the generalized Gagliardo-Nirenberg inequality (Theorem 1), we have

$$
\begin{aligned}
H(u) &= \tfrac{1}{2}\int |\nabla u|^2 dx - \tfrac{1}{4}\int |u|^4 + E(|u|^2)|u|^2 dx \\
&\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \frac{\|u\|_2}{2\|Q\|_2^2}\|\nabla u\|_2^3.
\end{aligned} \tag{3.5}
$$

Now, we define a function $f(y)$ on $[0,+\infty)$ by

$$f(y) = \frac{1}{2}y^2 - \frac{\|u_0\|_2}{2\|Q\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0,+\infty)$ and

$$f'(y) = y - \frac{3\|u_0\|_2}{2\|Q\|_2^2}y^2 = y(1 - \frac{3\|u_0\|_2}{2\|Q\|_2^2}y). \tag{3.6}$$

It is obvious that there are two roots for equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \frac{2\|Q\|_2^2}{3\|u_0\|_2}$. Hence, we have that y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0,y_2)$ and decreasing on the interval $[y_2,+\infty)$.

Note that $f(0) = 0$ and $f_{max} = f(y_2) = \frac{2\|Q\|_2^4}{27\|u_0\|_2^2}$. By the conservation of energy and assumption (3.1), we have

$$f(\|\nabla u\|_2) \leq H(u) = H(u_0) < \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2} = f(y_2). \tag{3.7}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (0.1)-(0.2), as follows.

$$K_1 := \{u \in H^1 \mid 0 < \|\nabla u\|_2\|u\|_2 < \frac{2}{3}\|Q\|_2^2, \ 0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}\}$$

and

$$K_2 := \{u \in H^1 \mid \|\nabla u\|_2\|u\|_2 > \frac{2}{3}\|Q\|_2^2, \ 0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}\}.$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_1$, we have $0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}$ and $\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\|Q\|_2^2$, which implies that $\|\nabla u_0\|_2 < y_2$. Since $f(y)$ is continuous and increasing on $[0, y_2)$ and $f(y) < f_{max} = \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t, x)\|_2 < y_2,$$

which implies that K_1 is invariant.

If $u_0 \in K_2$, we have $\|\nabla u_0\|_2\|u_0\|_2 > \frac{2}{3}\|Q\|_2^2$, which implies that $\|\nabla u_0\|_2 > y_2$. Since $f(y)$ is continuous and decreasing on $[y_2, +\infty)$ and $f(y) < f_{max} = \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t, x)\|_2 > y_2 \quad \text{and} \quad \|\nabla u(t, x)\|_2\|u(t, x)\|_2 > \frac{2}{3}\|Q\|_2^2, \tag{3.8}$$

which implies that K_2 is invariant.

Now, we return to the proof the Theorem 3. By (3.1) and (3.2), we have $u_0 \in K_1$. Applying the invariant of K_1, we have that (3.3) is true and the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) exists globally. This completes the part (i) of the proof.

By (3.1) and (3.4), we have $u_0 \in K_2$. Applying the invariant of K_2, we have (3.8) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t, x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$\begin{aligned} J''(t) \quad &= \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx \\ &= 8 \int |\nabla u|^2 dx - 6 \int |u|^4 + E(|u|^2)|u|^2 dx \\ &= 24H(u_0) - 4\|\nabla u\|_2^2. \end{aligned} \tag{3.9}$$

Multiplying both side of (3.9) by $\|u_0\|_2^2$, applying the conservation laws, (3.1) and (3.8), we have

$$\begin{aligned} \|u_0\|_2^2 \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx \quad &= 24H(u_0)\|u_0\|_2^2 - 4\|\nabla u\|_2^2\|u_0\|_2^2 \\ &< \frac{16}{9}\|Q\|_2^4 - \frac{16}{9}\|Q\|_2^4 = 0. \end{aligned} \tag{3.10}$$

By the classical analysis identity

$$J(t) = J(0) + J'(0)t + \int_0^t J''(s)(t - s)ds, \tag{3.11}$$

we have that the maximal existence interval I of $u(t, x)$ must be finite, which implies that the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$. This completes the proof.

- **Sharp Criteria for** $p = 1 + \frac{4}{3}$

Theorem 4. *Let $p = 1 + \frac{4}{3}$, $u_0 \in H^1$ and satisfy*

$$\|u_0\|_2 < \|P\|_2 \quad \text{and} \quad H(u_0) < \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}. \tag{3.12}$$

Then, we have

  (i) If

$$\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{3.13}$$

then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) exists globally. Moreover, for all time t, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2\|u(t,x)\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}. \tag{3.14}$$

  (ii) If

$$\|\nabla u_0\|_2\|u_0\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{3.15}$$

and $|x|u_0 \in L^2$, then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$,

where P is the solution of Equation (2.31) and R is the solution of Equation (2.29).

    Proof. Applying Theorem 2 and Proposition 5, we have

$$\begin{aligned} H(u) \quad &= \tfrac{1}{2}\int|\nabla u|^2 dx - \tfrac{1}{2+\frac{4}{3}}\int|u|^{2+\frac{4}{3}}dx - \tfrac{1}{4}\int E(|u|^2)|u|^2 dx \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \frac{\|u\|_2^{\frac{4}{3}}}{2\|P\|_2^{\frac{4}{3}}}\|\nabla u\|_2^2 - \frac{\|u\|_2}{2\|R\|_2^2}\|\nabla u\|_2^3. \end{aligned} \tag{3.16}$$

Now, we define a function $f(y)$ on $[0, +\infty)$ by

$$f(y) = \left(\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{3}}}{2\|P\|_2^{\frac{4}{3}}}\right)y^2 - \frac{\|u_0\|_2}{2\|R\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0, +\infty)$ and

$$f'(y) = \left(1 - \frac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}\right)y - \frac{3\|u_0\|_2}{2\|R\|_2^2}y^2. \tag{3.17}$$

It is obvious that there are two roots for equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2\|P\|_2^{\frac{4}{3}}} > 0$. Hence, we have that y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_2)$ and decreasing on the interval $[y_2, +\infty)$.

Note that $f(y_1) = 0$ and

$$f_{max} = f(y_2) = \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}.$$

By the conservation of energy and the assumption (3.12), we have

$$f(\|\nabla u\|_2) \leq H(u) = H(u_0) < f(y_2). \tag{3.18}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (0.1)-(0.2), as follows.

$$K_3 := \{u \in H^1 \mid 0 < \|\nabla u\|_2\|u\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \ \|u\|_2 < \|P\|_2, \ 0 < H(u) < D\}$$

and

$$K_4 := \{u \in H^1 \mid \|\nabla u\|_2\|u\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \ \|u\|_2 < \|P\|_2, \ 0 < H(u) < D\},$$

where $D = \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}$. Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_3$, we have $\|u\|_2 < \|P\|_2$, $0 < H(u) < D$ and $\|\nabla u_0\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2\|P\|_2^{\frac{4}{3}}}$, which implies that $\|\nabla u_0\|_2 < y_2$. Since $f(y)$ is increasing on $[0, y_2)$ and $0 < f(y) < D$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2\|u(t,x)\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}},$$

which implies that K_3 is invariant.

If $u_0 \in K_4$, we have $\|u\|_2 < \|P\|_2$, $0 < H(u) < D$ and $\|\nabla u_0\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2\|P\|_2^{\frac{4}{3}}}$, which implies $\|\nabla u_0\|_2 > y_2$. Since $f(y)$ is decreasing on $[y_2, \infty)$ and $0 < f(y) < D$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 > y_2 \ \text{ and } \ \|\nabla u(t,x)\|_2\|u(t,x)\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{3.19}$$

which implies that K_4 is invariant.

Now, we return to the proof the Theorem 4. By (3.12) and (3.13), we have $u_0 \in K_3$. Applying the invariant of K_3, we have that (3.14) is true and the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) exists globally. This completes the part (i) of the proof.

By (3.12) and (3.15), we have $u_0 \in K_4$. Applying the invariant of K_4, we have (3.19) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus,

we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$
\begin{aligned}
J''(t) \quad &= \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \\[2mm]
&= 8 \int |\nabla u|^2 dx - \tfrac{16}{2+\frac{4}{3}} \int |u|^{2+\frac{4}{3}} - 6 \int E(|u|^2)|u|^2 dx \\[2mm]
&= 24H(u_0) - 4 \int |\nabla u|^2 dx + \tfrac{8}{2+\frac{4}{3}} \int |u|^{2+\frac{4}{3}} dx \\[2mm]
&\le 24H(u_0) - 4[1 - \tfrac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}]\|\nabla u\|_2^2.
\end{aligned}
\tag{3.20}
$$

By the assumption (3.12), it follows from (3.19) and (3.20) that

$$
\begin{aligned}
\|u_0\|_2^2 \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \quad &\le 24H(u_0)\|u_0\|_2^2 - 4[1 - \tfrac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}]\|\nabla u\|_2^2\|u_0\|_2^2 \\[2mm]
&< \tfrac{16\|R\|_2^4(\|P\|_2^{\frac{4}{3}}-\|u_0\|_2^{\frac{4}{3}})^3}{9\|P\|_2^4} - \tfrac{16\|R\|_2^4(\|P\|_2^{\frac{4}{3}}-\|u_0\|_2^{\frac{4}{3}})^3}{9\|P\|_2^4} = 0,
\end{aligned}
\tag{3.21}
$$

which implies that the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$. This completes the proof.

In order to study the sharp thresholds of blow-up and global existence for the Cauchy problem (0.1)-(0.2) for $1 + \frac{4}{3} < p < 3$ and $3 < p < 5$, we need the following preparations.

Let us define a function $g(y)$ on $[0, +\infty)$

$$
g(y) = 1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}} y^{\frac{3(p-1)}{2}-2} - \frac{3\|u_0\|_2}{2\|P\|_2^2} y,
\tag{3.22}
$$

where P is the solution of Equation (2.31). We claim that there exists an unique positive solution y_0 for the equation $g(y) = 0$. Indeed, by some computations, we have for $y > 0$

$$
g'(y) = -\frac{3(p-1)(3p-7)\|u_0\|_2^{\frac{5-p}{2}}}{8\|P\|_2^{p-1}} y^{\frac{3(p-1)}{2}-3} - \frac{3\|u_0\|_2}{2\|P\|_2^2} < 0,
\tag{3.23}
$$

which implies that $g(y)$ is decreasing on $[0, +\infty)$. Notice that

$$
g(0) = 1 > 0,
$$

and

$$
g(\frac{2\|P\|_2^2}{3\|u_0\|_2}) = -\frac{3(p-1)}{4}(\frac{2}{3})^{\frac{3p-7}{2}}\frac{\|u_0\|_2^{6-2p}}{\|P\|_2^{6-2p}} < 0.
$$

Since $g(y)$ is continuous on $[0, +\infty)$, there exists an unique positive $y_0 \in [0, \frac{2\|P\|_2^2}{3\|u_0\|_2}]$ such that $g(y_0) = 0$.

- **Sharp Criteria for** $1 + \frac{4}{3} < p < 3$

Theorem 5. *Let* $1 + \frac{4}{3} < p < 3$, $u_0 \in H^1$ *and satisfy*

$$0 < H(u_0) < \frac{3p - 7}{6(p - 1)} y_0^2. \tag{3.24}$$

Then, we have

(i) If

$$\|\nabla u_0\|_2 < y_0, \tag{3.25}$$

then the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) exists globally. Moreover, for all time t, $u(t, x)$ satisfies

$$\|\nabla u(t, x)\|_2 < y_0. \tag{3.26}$$

(ii) If

$$\|\nabla u_0\|_2 > y_0, \tag{3.27}$$

and $|x|u_0 \in L^2$, then the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$,

where y_0 is the unique positive solution of the equation $g(y) = 0$ and $g(y)$ is defined in (3.22).

Proof. Applying the Gagliardo-Nirenberg inequality (Proposition 5), we have

$$
\begin{aligned}
H(u) \quad &= \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{1+p} \int |u|^{p+1} dx - \frac{1}{4} \int E(|u|^2)|u|^2 dx \\
&\geq \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{1+p} \int |u|^{p+1} dx - \frac{1}{4} \int |u|^4 dx \\
&\geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{\|u\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}} \|\nabla u\|_2^{\frac{3(p-1)}{2}} - \frac{\|u\|_2}{2\|P\|_2^2} \|\nabla u\|_2^3.
\end{aligned}
\tag{3.28}
$$

Now, we define a function $f(y)$ on $[0, +\infty)$ by

$$f(y) = \frac{1}{2} y^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}} y^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2} y^3,$$

then we have $f(y)$ is continuous on $[0, +\infty)$ and

$$f'(y) = y[1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}} y^{\frac{3(p-1)}{2} - 2} - \frac{3\|u_0\|_2}{2\|P\|_2^2} y] = yg(y). \tag{3.29}$$

By the properties of $g(y)$, we have

$$f'(0) = f'(y_0) = y_0 g(y_0) = 0 \quad \text{and} \quad f''(y_0) = g(y_0) + y_0 g'(y_0) < 0, \tag{3.30}$$

which implies that 0 and y_0 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_0)$ and decreasing on the interval $[y_0, +\infty)$.

Note that $f_{max} = f(y_0)$ and $f(0) = 0$. Since $g(y_0) = 0$, we have

$$
\begin{aligned}
f_{max} = f(y_0) \quad &= \tfrac{1}{2}y_0^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}} y_0^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2} y_0^3 \\
&= [\tfrac{1}{2} - \tfrac{2}{3(p-1)}]y_0^2 + [\tfrac{1}{p-1} - \tfrac{1}{2}]\frac{\|u_0\|_2}{\|P\|_2^2} y_0^3 \\
&\geq \tfrac{3p-7}{6(p-1)} y_0^2.
\end{aligned}
\tag{3.31}
$$

By the conservation of energy and the assumption (3.24), we have

$$
H(u) = H(u_0) < \frac{3p-7}{6(p-1)} y_0^2 < f_{max}.
\tag{3.32}
$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (0.1)-(0.2), as follows. We set

$$
K_5 := \{u \in H^1 \mid 0 < \|\nabla u\|_2 < y_0,\ 0 < H(u) < \frac{3p-7}{6(p-1)} y_0^2\}
$$

and

$$
K_6 := \{u \in H^1 \mid \|\nabla u\|_2 > y_0,\ 0 < H(u) < \frac{3p-7}{6(p-1)} y_0^2\}.
$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_5$, we have $0 < H(u) < \frac{3p-7}{6(p-1)} y_0^2$ and $\|\nabla u_0\|_2 < y_0$. Since $f(y)$ is continuous and increasing on $[0, y_0)$ and $f(y) < \frac{3p-7}{6(p-1)} y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$
\|\nabla u(t,x)\|_2 < y_0,
$$

which implies that K_5 is invariant.

If $u_0 \in K_6$, we have $0 < H(u) < \frac{3p-7}{6(p-1)} y_0^2$ and $\|\nabla u_0\|_2 > y_0$. Since $f(y)$ is continuous and decreasing on $[y_0, +\infty)$ and $f(y) < \frac{3p-7}{6(p-1)} y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$
\|\nabla u(t,x)\|_2 > y_0,
\tag{3.33}
$$

which implies that K_6 is invariant.

Now, we return to the proof the Theorem 5. By (3.24) and (3.25), we have $u_0 \in K_5$. Applying the invariant of K_5, we have that (3.26) is true and the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) exists globally . This completes the part (i) of the proof.

By (3.24) and (3.27), we have $u_0 \in K_6$. Applying the invariant of K_6, we have (3.33) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$
\begin{aligned}
J''(t) \quad &= \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \\
&= 8\int |\nabla u|^2 dx - \tfrac{12(p-1)}{p+1} \int |u|^{p+1} dx - \tfrac{1}{4}\int E(|u|^2)|u|^2 dx \\
&= 12(p-1)H(u_0) - [6(p-1)-8]\|\nabla u\|_2^2 + [3(p-1)-6]\int E(|u|^2)|u|^2 dx \\
&\leq 2[3(p-1)-4]y_0^2 - [6(p-1)-8]y_0^2 = 0,
\end{aligned}
\tag{3.34}
$$

for $1 + \frac{4}{3} < p < 3$, which implies that the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$. This completes the proof.

- **Sharp Criteria for** $3 < p < 5$

Theorem 6. *Let* $3 < p < 5$, $u_0 \in H^1$ *and satisfy*

$$0 < H(u_0) < \frac{1}{6}y_0^2. \tag{3.35}$$

Then, we have that

(i) If

$$\|\nabla u_0\|_2 < y_0, \tag{3.36}$$

then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) exists globally . Moreover, for all time t, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2 < y_0. \tag{3.37}$$

(ii) If

$$\|\nabla u_0\|_2 > y_0, \tag{3.38}$$

and $|x|u_0 \in L^2$, then the solution $u(t,x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$,

where y_0 is the unique positive solution of the equation $g(y) = 0$ and $g(y)$ is defined in (3.22).

Proof. Applying the Gagliardo-Nirenberg inequality (Proposition 5), we have

$$
\begin{aligned}
H(u) \quad &= \tfrac{1}{2}\int |\nabla u|^2 dx - \tfrac{1}{1+p}\int |u|^{p+1} - \tfrac{1}{4}\int E(|u|^2)|u|^2 dx \\
&\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \frac{\|u\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}\|\nabla u\|_2^{\frac{3(p-1)}{2}} - \frac{\|u\|_2}{2\|P\|_2^2}\|\nabla u\|_2^3.
\end{aligned}
\tag{3.39}
$$

Now, we define a function $f(y)$ on $[0, +\infty)$

$$f(y) = \frac{1}{2}y^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0, +\infty)$ and

$$f'(y) = y[1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}-2} - \frac{3\|u_0\|_2}{2\|P\|_2^2}y] = yg(y), \tag{3.40}$$

where $g(y)$ is defined in (3.22). By the properties of $g(y)$, we have

$$f'(0) = f'(y_0) = y_0 g(y_0) = 0 \ \text{ and } \ f''(y_0) = g(y_0) + y_0 g'(y_0) < 0, \tag{3.41}$$

which implies that 0 and y_0 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_0)$ and decreasing on the interval $[y_0, +\infty)$.

Note that $f(0) = 0$ and $f_{max} = f(y_0)$. Since $g(y_0) = 0$, we have

$$
\begin{aligned}
f_{max} = f(y_0) \quad &= \tfrac{1}{2}y_0^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2}y_0^3 \\
&= [\tfrac{1}{2} - \tfrac{1}{3}]y_0^2 + [\tfrac{p-1}{4} - \tfrac{1}{2}]\frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{2}} \\
&\geq \tfrac{1}{6}y_0^2.
\end{aligned}
\tag{3.42}
$$

By the conservation of energy and the assumption (3.35), we have

$$
0 < H(u) = H(u_0) < \frac{1}{6}y_0^2 < f_{max}.
\tag{3.43}
$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (0.1)-(0.2), as follows. We set

$$
K_7 := \{u \in H^1 \mid 0 < \|\nabla u\|_2 < y_0,\ 0 < H(u) < \frac{1}{6}y_0^2\}
$$

and

$$
K_8 := \{u \in H^1 \mid \|\nabla u\|_2 > y_0,\ 0 < H(u) < \frac{1}{6}y_0^2\}.
$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_7$, we have $\|\nabla u_0\|_2 < y_0$. Since $f(y)$ is continuous and increasing on $[0, y_0)$ and $\forall\, y \in [0, +\infty)$, $f(y) < \frac{1}{6}y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$
\|\nabla u(t, x)\|_2 < y_0,
$$

which implies that K_7 is invariant.

If $u_0 \in K_8$, we have $\|\nabla u_0\|_2 > y_0$. Since $f(y)$ is continuous and decreasing on $[y_0, +\infty)$ and $\forall\, y \in [0, +\infty)$, $f(y) < \frac{1}{6}y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$
\|\nabla u(t, x)\|_2 > y_0,
\tag{3.44}
$$

which implies that K_8 is invariant.

Now, we return to the proof the Theorem 6. By (3.35) and (3.36), we have $u_0 \in K_7$. Applying the invariant of K_7, we have that (3.37) is true and the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) exists globally . This completes the part (i) of the proof.

By (3.35) and (3.38), we have $u_0 \in K_8$. Applying the invariant of K_8, we have (3.44) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t, x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$
\begin{aligned}
J''(t) \quad &= \tfrac{d^2}{dt^2}\int |x|^2|u(t, x)|^2 dx \\
&= 8\int |\nabla u|^2 dx - \frac{12(p-1)}{p+1}\int |u|^{p+1}dx - \tfrac{1}{4}\int E(|u|^2)|u|^2 dx \\
&= 24H(u_0) - 4\|\nabla u\|_2^2 + \frac{24-12(p-1)}{p+1}\int |u|^{p+1}dx \\
&< 4y_0^2 - 4y_0^2 = 0,
\end{aligned}
\tag{3.45}
$$

for $3 < p < 5$, which implies that the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) blows up in finite time $T < +\infty$. This completes the proof.

4　Properties of Blow-up Solutions

In this section, we shall investigate the blow-up properties of the solutions to the Cauchy problem (0.1)-(0.2). We prove the nonexistence of the L^3 strong limit to the blow-up solutions of the Cauchy problem (0.1)-(0.2) for $1 < p \leq 3$, as follows.

Theorem 7. *Let $1 < p \leq 3$ and the initial data $u_0 \in H^1$. If the solution of the Cauchy problem (0.1)-(0.2) $u(t, x)$ blows up in finite time $T < +\infty$, Then for any sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to T$ as $n \to \infty$, $\{u(t_n, x)\}_{n=1}^\infty$ does not have any strong limit in L^3 as $n \to \infty$.*

　　Proof. We prove this result by contradiction. Suppose that $\{u(t_n, x)\}_{n=1}^\infty$ has a strong limit in L^3 along a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to T$ as $n \to \infty$. Since the solution $u(t, x)$ of the Cauchy problem (0.1)-(0.2) blows up at finite time T in H^1, we have $\|\nabla u(t_n)\|_2 \to +\infty$ as $n \to +\infty$. By the conservation of energy

$$H(u) := \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int |u(t, x)|^{p+1} dx - \frac{1}{4} \int E(|u|^2)|u|^2 dx = H(u_0),$$

for $1 < p \leq 3$, we claim $\forall n \neq m$,

$$\|\nabla u(t_n)\|_2^2 \leq C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C. \tag{4.1}$$

Indeed, if $p = 3$, by the conservation of energy, we have

$$\begin{aligned}
\|\nabla u(t_n)\|_{L^2}^2 & \leq 2H(u_0) + \frac{1}{2}\|u(t_n)\|_4^4 + \frac{1}{2} \int E(|u(t_n)|^2)|u(t_n)|^2 dx \\
& \leq 2H(u_0) + C\|u(t_n)\|_4^4 + C \\
& \leq C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C.
\end{aligned}$$

If $1 < p < 3$, using the Gagliardo-Nirenberg inequality and Hölder inequality, we have $\forall\, \varepsilon > 0$

$$\|u(t_n)\|_{p+1}^{p+1} \leq C\|\nabla u(t_n)\|_2^{\frac{3(p-1)}{2}} \|u_0\|_2^{\frac{5-p}{2}} \leq \varepsilon \|\nabla u(t_n)\|_2^2 + C(\varepsilon).$$

By the conservation of energy, we have

$$\begin{aligned}
\|\nabla u(t_n)\|_2^2 & \leq 2H(u_0) + \varepsilon\|\nabla u(t_n)\|_2^2 + C\|u(t_n)\|_4^4 + C(\varepsilon) \\
& \leq \varepsilon\|\nabla u(t_n)\|_2^2 + C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C,
\end{aligned}$$

for $\varepsilon < 1$, which implies that Claim (4.1) is true.

　　Since $3 < 4 < 6$, applying the Hölder's inequality for $\frac{1}{4} = \frac{\theta}{3} + \frac{1-\theta}{6}$, $\theta \in (0, 1)$, we have

$$\begin{aligned}
\|u(t_n) - u(t_m)\|_4^4 & \leq C\|u(t_n) - u(t_m)\|_3^{4\theta} \|u(t_n) - u(t_m)\|_6^{4(1-\theta)} \\
& \leq C\|u(t_n) - u(t_m)\|_3^2 \|\nabla(u(t_n) - u(t_m))\|_2^2.
\end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that for $m \neq n$ large enough

$$\|\nabla u(t_n)\|_2^2 \leq C\|u(t_n) - u(t_m)\|_3^2\|\nabla(u(t_n)\|_2^2 + C_m, \tag{4.3}$$

where C_m depends on m.

On the other hand, since the sequence $\{u(t_n)\}_{n=1}^{\infty}$ converges strongly in L^3, there is a positive integer k such that for all $n \geq k, m \geq k$

$$C\|u(t_n) - u(t_m)\|_3^2 \leq \frac{1}{2}.$$

Therefore, choosing $m = k$ in the inequality (4.3), we obtain that for all $n \geq n_k$

$$\|\nabla u(t_n)\|_2^2 \leq \frac{1}{2}\|\nabla u(t_n)\|_2^2 + C_k, \tag{4.4}$$

which implies that the sequence $\{\nabla u(t_n)\}_{n=1}^{\infty}$ is bounded in L^2. This is contradictory to that $u(t, x)$ blows up in finite time $T < +\infty$. This completes the proof.

参考文献（**References**）

[1] Ghidaglia, J.-M. and Saut, J. C. On the initial value problem for the Davey-Stewartson systems[J], Nonlinearity, 3(1990), 475-506.

[2] Guo, B. L. and Wang, B. X. The Cauchy problem for Davey-Stewartson systems[J], Commun. Pure Appl. Math., 52(1999), 1477-1490.

[3] Cazenave, T. Semilinear Schrödinger equations[M], Courant Lecture Notes in Mathematics, 10, NYU, CIMS, AMS 2003.

[4] Sulem, C. and Sulem, P. L. The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse[M], Appl. Math. Sci., vol. 139, Springer-Verlag, New York, 1999.

[5] Cipolatti, R. On the existence of standing waves for a Davey-Stewartson system[J], Commun. in PDE, 17(1992), 967-988.

[6] Cipolatti, R. On the instability of ground states for a Davey-Stewartson system, Ann. Inst. Henri Poincar$é$[J]. Phys. Theor., 58(1993), 85-104.

[7] Ohta, M. Stability of standing waves for the generalized Davey-Stewartson system[J], J. Dynam. Differ. Eqs., 6(1994), 325-334.

[8] Ohta, M. Instability of standing waves the generalized Davey-Stewartson systems[J], Ann. Inst. Henri Poincare, Phys. Theor., 63(1995), 69-80.

[9] Gan, Z. H. and Zhang, J. Sharp threshold of global existence and instability of standing wave for a Davey-Stewartson system[J], Commun. Math. Phys., 283(2008), 93-125.

[10] Ozawa, T. Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems[J], Proc. Roy. Soc. London Ser. A, 436(1992), 345-349.

[11] Wang, B. X. and Guo, B. L. On the initial value problem and scattering of solutions for the generalized Davey-Stewartson systems[J], Science in China, 44(2001), 994-1002.

[12] Richards, G. Mass concentration for the Davey-Stewartson system[J], http://arxiv.org/abs /0909.0492v1

[13] Papanicolaou, G.C., Sulem, C., Sulem, P-L. and Wang, X.P. The focusing singularity of the Davey-Stewartson equations for gravity-capillary surface waves[J], Physica D, 72(1994), 61-86.

[14] Gan, Z. H. and Zhang, J. Sharp conditions of global existence for the generalized Davey-Stewartson system in three dimensional space[J], Acta Math. Scientia, Ser. A, 26(2006), 87-92.

[15] Shu, J. and Zhang, J. Sharp conditions of global existence for the generalized Davey-Stewartson system[J], IMA J. Appl. Math., 72(2007), 36-42.

[16] Holmer, J. and Roudenko, S. On blow-up solutions to the 3D cubic nonlinear Schrödinger equation[J], Appl. Math. Research eXpress, Vol. 2007, Article ID 004, 29 pages.

[17] Gerard, P. Description du defaut de compacite de l'injection de Sobolev[J], ESAIM Control Optim. Calc. Var., 3(1998), 213-233.

[18] Hmidi, T. and Keraani, S. Blowup theory for the critical nonlinear Schrödinger equations revisited[J], Internat. Math. Res. Notices, 46(2005), 2815-2828.

[19] Weinstein, M. I. Nonlinear Schrödinger equations and sharp interpolation estimates[J], Commun. Math. Phys., 87(1983), 567-576.

[20] Davey, A. and Stewartson, K. On three-dimensional packets of surfaces waves[J], Proc. R. Soc. A., 338(1974), 101-110.

[21] Merle, F. and Raphaël, P. Blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation[J], Annals of Math., 16(2005), 157-222.

[22] Strauss, W. A. Existence of solitary waves in higher dimensions[J], Commun. Math. Phys., 55(1977), 149-162.