On Shear-Free perturbations of FLRW Universes

Anne Marie Nzioki[†], Rituparno Goswami[†], Peter K.S. Dunsby[†‡] and George F. R. Ellis[†]
† *ACGC and Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, 7701, South Africa and*
‡ *South African Astronomical Observatory, Observatory, Cape Town, South Africa**
(Dated: July 28, 2011)

A surprising exact result for the Einstein Field Equations is that if pressure-free matter is moving in a shear-free way, then it must be either expansion-free or rotation-free. It has been suggested this result is also true for any barotropic perfect fluid, but a proof has remained elusive. We consider the case of barotropic perfect fluid solutions linearized about a Robertson-Walker geometry, and prove that the result remains true except for the case of a specific highly non-linear equation of state. We argue that this equation of state is non-physical, and hence the result is true in the linearized case for all physically realistic barotropic perfect fluids. This result, which is not true in Newtonian cosmology, demonstrates that the linearized solutions, believed to result in standard local Newtonian theory, do not always give the usual behaviour of Newtonian solutions.

I. INTRODUCTION

This paper deals with a number of interesting properties of shear-free perfect fluid solutions of General Relativity (GR). The motivation for this work stems from the desire to probe the relationship between relativistic and Newtonian cosmology and their implications for the study of the growth of large-scale structure in the Universe. Of particular importance is understanding the differential properties of time-like geodesics which describe the fluid flow in cosmology. The kinematics of such fluid flows are described by the expansion Θ, shear (or distortion) σ_{ab}, rotation ω^c, and acceleration A_a of the four-velocity field u^a tangent to the fluid flow lines. Their governing equations are obtained by contracting the Ricci identities (applied to u^a) along and orthogonal to u^a, which determine how they couple to gravity via the Einstein Field Equations [1].

Of particular interest is what role the shear plays in the relationship between Newtonian and relativistic cosmologies. For example it has been known for some time that quasi-Newtonian descriptions of cosmology, the so-called *Silent models*, may be constructed for observers which move along geodesics which are both shear-free and irrotational [2]. The intricate relationship between the kinematic quantities in Newtonian and relativistic cosmologies is most strikingly seen in a remarkable result first obtained by one of us in 1967 [3]. In this paper it was found that *if the four velocity vector field of a barotropic perfect fluid with vanishing pressure is shear free, then either the expansion or the rotation of the fluid vanishes.* This is a purely local result to which no corresponding Newtonian equivalent appears to hold, as counter-examples can be explicitly constructed [4]. Given that this theorem and its extensions appear to hold for arbitrarily weak fields and for fluids of arbitrarily low density, one needs to understand why the Newtonian approximation fails.

The result has be extended to general barotropic fluids for number of special cases by Senovilla [5], but has yet to be proved in general. As a first step towards this goal, we examine what ever result holds in situations where the hydrodynamic and gravitational equations have been linearised. Of course there are many ways of doing this, but one way that is cosmologically relevant is to linearise the equations about a Friedmann-Lemaître-Robertson-Walker (FLRW) background [6–10]. These *almost FLRW* models can be thought of as lying somewhere between the full non-linear GR situation and Newtonian theory, at least in the cosmological context, and therefore an analysis of theorem in this context could shed some light on the generality of the result. We show that it remains true for such linearised barotropic perfect fluid solutions, unless the fluid obeys a highly non-linear equation of state (see (55) below) which we argue is non-physical. Hence the result remains true for physically realistic equations of state in an almost-FLRW geometry.

This result will be useful in obtaining and studying new perfect-fluid solutions of Einstein's field equations with a shear-free velocity vector field, and in examining how linearized General Relativity solutions relate to the Newtonian case, which is the foundation of astrophysical studies in cosmology.

II. LINEARISED FIELD EQUATIONS ABOUT FLRW BACKGROUND

To perturb the FLRW spacetime we use the standard 1+3 covariant approach [1], where we must first define a time-like congruence with a unit tangent vector u^a. The natural choice of this vector is tangent to the the matter flow lines. Then the spacetime is split locally in the form $R \otimes V$ where R denotes the worldline along u^a and V is the 3-space perpendicular to u^a. Then any vector X^a can be projected on the 3-space by the projection tensor $h^a{}_b = g^a{}_b + u^a u_b$. At this point, two derivatives are defined: the vector u^a is used to define the *covariant time derivative* along the observers' worldlines(denoted

* anne.nzioki@gmail.com, Rituparno.Goswami@uct.ac.za, George.Ellis@uct.ac.za, Peter.Dunsby@uct.ac.za

by a dot) for any tensor $T^{a..b}{}_{c..d}$, given by

$$\dot{T}^{a..b}{}_{c..d} = u^e \nabla_e T^{a..b}{}_{c..d} \tag{1}$$

and the tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative* D for any tensor $T^{a..b}{}_{c..d}$:

$$D_e T^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r T^{f..g}{}_{p..q} \;, \tag{2}$$

with total projection on all the free indices. Angle brackets denote orthogonal projections of vectors, and the orthogonally *projected symmetric trace-free* PSTF part of tensors:

$$V^{\langle a \rangle} = h^a{}_b V^b \;,\; T^{\langle ab \rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3} h^{ab} h_{cd} \right] T^{cd} \;. \tag{3}$$

This splitting of spacetime also naturally defines the 3-volume element

$$\epsilon_{abc} = -\sqrt{|g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}u^d \;, \tag{4}$$

with the following identities

$$\epsilon_{abc}\epsilon^{def} = 3! h^d_{[a} h^e_b h^f_{c]} \;\;;\;\; \epsilon_{abc}\epsilon^{dec} = 2! h^d_{[a} h^e_{b]}. \tag{5}$$

The covariant derivative of the time-like vector u^a can now be decomposed into the irreducible parts as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c \;, \tag{6}$$

where $A_a = \dot{u_a}$ is the acceleration, $\Theta = D_a u^a$ is the expansion, $\sigma_{ab} = D_{\langle a}u_{b\rangle}$ is the shear tensor and $w^a = \epsilon^{abc}D_b u_c$ is the vorticity vector. Similarly the Weyl curvature tensor can be decomposed irreducibly into the Gravito-Electric and Gravito-Magnetic parts as

$$E_{ab} = C_{abcd}u^c u^d = E_{\langle ab \rangle} \;;\; H_{ab} = \frac{1}{2}\epsilon_{acd}C^{cd}_{be}u^e = H_{\langle ab \rangle} \;, \tag{7}$$

which allows a covariant description of tidal forces and gravitational radiation.

In the 1+3 covariant perturbation theory [6–13], we consider the background to be FLRW where the Hubble scale sets the scale for the perturbations. The quantities that vanish in the background spacetime are considered to be first order and are automatically gauge-invariant by virtue of the Stewart and Walker lemma [14]. In the perturbed spacetime the matter is considered to be a perfect fluid with the Energy Momentum tensor

$$T^{ab} = (\mu + p)u^a u^b + p g^{ab} \;, \tag{8}$$

so the vector field u^a is uniquely defined as the timelike eigenvector of T^{ab} as long as $\mu + p \neq 0$ (the heat flux q^a and the anisotropic stress π_{ab} vanish in the perturbed spacetime). Furthermore, we assume the matter to have a barotropic equation of state $p = p(\mu)$ satisfying the Weak and Dominant energy conditions. We exclude the vacuum case, therefore the energy conditions will be

$$\mu > 0 \;\;;\;\; \mu + p > 0 \;\;;\;\; \mu \geq |p| \tag{9}$$

for both the background spacetime and the perturbed solution (and the Minkowski and De Sitter backgrounds will not occur). The local isentropic sound speed is

$$c_s^2 \equiv \frac{dp}{d\mu} \;\;;\;\; 0 \leq c_s^2 \leq 1 \;. \tag{10}$$

The bound on the local sound speed is required for local stability of matter (lower bound) and causality (upper bound), respectively.

Now we consider shear-free perturbations and hence the shear tensor (σ_{ab}) vanishes identically. With the conditions above, the linearised field equations are then as follows:

Propagation equations

$$\dot{\Theta} = D_a A^a - \frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) \;, \tag{11}$$

$$\dot{\omega}^{\langle a \rangle} = \frac{1}{2}\epsilon^{abc}D_b A_c - \frac{2}{3}\Theta\omega^a \;, \tag{12}$$

$$\dot{H}^{\langle ab \rangle} = -\epsilon^{cd\langle a}D_c E^{\rangle b}_d - \Theta H^{ab} \;, \tag{13}$$

$$\dot{E}^{\langle ab \rangle} = \epsilon^{cd\langle a}D_c H^{\rangle b}_d - \Theta E^{ab} \;, \tag{14}$$

$$\dot{\mu} = -\Theta(\mu + p) \;, \tag{15}$$

Constraint equations

$$(C_0)^{ab} := E^{ab} - D^{\langle a}A^{b\rangle} = 0 \;, \tag{16}$$

$$(C_1)^a := D^a\Theta - \frac{3}{2}\epsilon^{abc}D_b\omega_c = 0 \;, \tag{17}$$

$$(C_2) := D^a\omega_a = 0 \;, \tag{18}$$

$$(C_3)^{ab} := H^{ab} + D^{\langle a}\omega^{b\rangle} = 0 \;. \tag{19}$$

$$(C_4)^a := D^a p + (\mu + p)A^a = 0 \;, \tag{20}$$

$$(C_5)^a := D_b E^{ab} - \frac{1}{3}D^a\mu = 0 \;, \tag{21}$$

$$(C_6)^a := D_b H^{ab} + (\mu + p)\omega^a = 0 \;. \tag{22}$$

We note that the constraints $(C_1)^a$, (C_2), $(C_3)^{ab}$, $(C_5)^a$ and $(C_6)^a$ are the constraints of the Einstein field equations for general matter motion specialized to the shear-free case and are known to be consistently *time propagated* along u^a locally. However the conditions $\sigma_{ab} = 0$ and $q^a = 0$ give the two new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively.

We also use the following linearised commutation relations for shear-free congruences: For any scalar 'f'

$$[D_a D_b - D_b D_a] f = 2\epsilon_{abc}\omega^c \dot{f} \;,$$
$$\epsilon^{abc} D_b D_c f = 2\omega^a \dot{f} \;. \tag{23}$$

If the gradient of the scalar is of the first order, we then have

$$[D^a D_b D_a - D_b D^2] f = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) D_b f \;, \tag{24}$$

$$[D^2 D_b - D_b D^2] f = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) D_b f + 2\epsilon_{dbc} D^d(\omega^c \dot{f}) \;. \tag{25}$$

Also for any first order 3-vector $V^a = V^{\langle a \rangle}$, we have

$$[D^a D_b - D_b D^a] V_a = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) h^a{}_{[a} V_{b]} \;, \tag{26}$$

$$h^a{}_c h^d{}_b (D_d V^c)^{\cdot} = D_b \dot{V}^{\langle a \rangle} - \frac{1}{3}\Theta D_b V^a \tag{27}$$

$$h^a{}_c (D^2 V^c)^{\cdot} = D_b (D^{\langle b} V^{a \rangle})^{\cdot} - \frac{1}{3}\Theta D^2 V^a \;. \tag{28}$$

Using the field equations and identities of this section we will now investigate the compatibility of the new constraints with the existing ones in terms of the consistency up to the linear order of their spatial and temporal propagation.

III. CONSISTENCY OF THE NEW CONSTRAINTS

The conditions of shear-free perturbations and the matter being a perfect fluid in the perturbed spacetime give the new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively. To check their compatibility with the linearised existing constraints of Einstein field equations (henceforth all the equations are up to the linear order), we plug $(C_0)_{bd}$ in $(C_5)_b$ to get

$$D^d D_{\langle b} A_{d \rangle} - \frac{1}{3} D_b \mu = 0 \;. \tag{29}$$

Now from the constraint $(C_4)_b$ we have

$$A_b = -\frac{c_s^2}{\mu + p} D_b \mu \tag{30}$$

Using equation (30) in (29) we get the constraint

$$(C_7)_b := \frac{c_s^2}{\mu + p} D^d D_{\langle b} D_{d \rangle} \mu + \frac{1}{3} D_b \mu = 0 \;. \tag{31}$$

For the new constraints $(C_0)^{ab}$ and $(C_4)^a$ to be compatible with the existing ones, the constraint $(C_7)_b$ must be satisfied.

To check the spatial consistency of $(C_7)_b$ on any initial hypersurface we take the curl of (31) to get

$$\frac{c_s^2}{\mu + p}\epsilon^{acb} D_c D^d D_{\langle b} D_{d \rangle} \mu + \frac{1}{3}\epsilon^{acb} D_c D_b \mu = 0 \;, \tag{32}$$

which using (23) gives

$$\frac{c_s^2}{\mu + p}\epsilon^{acb} D_c D^d D_{\langle b} D_{d \rangle} \mu + \frac{2}{3}\omega^a \dot{\mu} = 0 \;. \tag{33}$$

Breaking the PSTF part according to equation (3) and using the commutators (24), (25) we have

$$\frac{c_s^2}{\mu + p}\epsilon^{acb}\left[\frac{2}{3} D_c D_b D^2 \mu + \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) D_c D_b \mu + \dot{\mu}\epsilon_{dbk} D_c D^d \omega^k\right] + \frac{2}{3}\omega^a \dot{\mu} = 0 \;. \tag{34}$$

Again using (23) and (5) in the above equation we get

$$\frac{c_s^2}{\mu + p}\left[\frac{4}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\omega^a \dot{\mu} - \dot{\mu} D_k D^a \omega^k + \dot{\mu} D^2 \omega^a\right] + \frac{2}{3}\omega^a \dot{\mu} = 0 \;. \tag{35}$$

Now from the relation (25) and using (18) we know

$$D_k D^a \omega^k = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\omega^a \;, \tag{36}$$

Plugging (36) and (15) in (35) and simplifying we finally get

$$(C_8)^a := \Theta\left[\frac{2}{3}\omega^a Y + c_s^2 D^2 \omega^a\right] = 0 \;, \tag{37}$$

where

$$Y = \mu + p + c_s^2\left(\mu - \frac{1}{3}\Theta^2\right) \;. \tag{38}$$

From $(C_8)^a$ we can immediately see that for matter with constant pressure ($p = \text{constant} \Rightarrow c_s^2 = 0$), shear-free perturbations are consistent iff $\Theta\omega^a = 0$ (as according to the second condition of (9), $\mu + p > 0$). That is, if the geodesics of the matter congruence in the perturbed spacetime are shear-free then they should be either expansion-free or vorticity-free (or both). This shows that *the results of [3] and [5] for pressure-free matter are true for the linearized theory.*

However for a general equation of state, all we can say from the equation (37) is, either the matter congruence is expansion free ($\Theta = 0$), or the vorticity vector must satisfy

$$(C_9)^a := \frac{2}{3}\omega^a Y + c_s^2 D^2\omega^a = 0 \,, \tag{39}$$

for the new constraints to be spatially consistent on any initial hypersurface.

Now let us check the temporal consistency of the constraint (39). Propagating it along u^a we get

$$(c_s^2 D^2\omega^a)\dot{} + \frac{2}{3}(\omega^a Y)\dot{} = 0 \,. \tag{40}$$

We can easily see that

$$\dot{c_s^2} = -\Theta(\mu+p)\frac{d^2p}{d\mu^2} \,. \tag{41}$$

Now from (28)we have

$$c_s^2(D^2\omega^a)\dot{} = c_s^2[D_b(D^{\langle b}\omega^{a\rangle})\dot{} - \frac{1}{3}\Theta D^2\omega^a] \,. \tag{42}$$

We know from the constraint (18) that

$$D_b(D^{\langle b}\omega^{a\rangle})\dot{} = \frac{1}{2}D_b[(D^b\omega^a)\dot{} + (D^a\omega^b)\dot{}] \,. \tag{43}$$

Using (27) the equation (43) becomes

$$D_b(D^{\langle b}\omega^{a\rangle})\dot{} = \frac{1}{2}D_b\left[D^b\dot{\omega}^{\langle a\rangle} - \frac{1}{3}\Theta D^b\omega^a + D^a\dot{\omega}^{\langle b\rangle} - \frac{1}{3}\Theta D^a\omega^b\right] \,. \tag{44}$$

Simplifying the above equation using (12), (20) and (23), we get

$$D_b(D^{\langle b}\omega^{a\rangle})\dot{} = -\frac{1}{2}\Theta(1-c_s^2)(D^2\omega^a + D_bD^a\omega^b) \,. \tag{45}$$

Putting equation (45) in (42), we have

$$c_s^2(D^2\omega^a)\dot{} = -\Theta\alpha c_s^2 D^2\omega^a - \Theta\beta D_bD^a\omega^b \,, \tag{46}$$

where

$$\alpha = -\frac{c_s^2}{2} + \frac{5}{6} \;\; ; \;\; \beta = \frac{c_s^2}{2}(1-c_s^2) \,. \tag{47}$$

Using (39) and (36), (46) becomes

$$c_s^2(D^2\omega^a)\dot{} = \frac{2}{3}\omega^a\Theta\left[\alpha Y - \beta\left(\mu - \frac{1}{3}\Theta^2\right)\right] \,. \tag{48}$$

Combining (41) and (48) and using (39) we get

$$(c_s^2 D^2\omega^a)\dot{} = \frac{2}{3}\omega^a\Theta\left[\frac{Y}{c_s^2}(\mu+p)\frac{d^2p}{d\mu^2} + \alpha Y - \beta\left(\mu - \frac{1}{3}\Theta^2\right)\right] \,. \tag{49}$$

Also from (11), (15) and (41) we have

$$\dot{Y} = -\Theta\left[(\mu+p)\left(\mu - \frac{1}{3}\Theta^2\right)\frac{d^2p}{d\mu^2} + Z\right] \,, \tag{50}$$

where

$$Z = (\mu+p)(1+c_s^2) + \frac{2}{3}c_s^2\left(\mu - \frac{1}{3}\Theta^2\right) \,. \tag{51}$$

Now using (12), (20), (23) and (50) we get

$$\frac{2}{3}(\omega^a Y)\dot{} = -\frac{2}{3}\omega^a\Theta\left[\left(-c_s^2 + \frac{2}{3}\right)Y + (\mu+p)\left(\mu - \frac{1}{3}\Theta^2\right)\frac{d^2p}{d\mu^2} + Z\right] \tag{52}$$

Finally using (49) and (52) in (40) and simplifying, we get

$$\frac{2}{3}\omega^a\Theta(\mu+p)\left[(\mu+p)\frac{d^2p}{d\mu^2} - c_s^2\left(\frac{5}{6} + \frac{c_s^2}{2}\right) - \frac{{}^3R}{2(\mu+p)}c_s^4\left(1-c_s^2\right)\right] = 0 \,. \tag{53}$$

where ${}^3R = 2[\mu - (1/3)\Theta^2]$ is the spatial curvature. In FLRW spacetimes it can be written in term of the scale factor '$a(t)$' as,

$${}^3R = \frac{k}{a(t)^2} = k\exp\left\{\frac{2}{3}\int\frac{d\mu}{\mu+p}\right\} \,, \tag{54}$$

where $k = -1, 0, +1$ denotes open, flat and closed universes respectively. Thus we can easily see that for the new constraints to be spatially and temporaly consistent we must have either $\omega^a\Theta = 0$ or the barotropic equation of state must satisfy the following non-linear higher order DE:

$$(\mu+p)\frac{d^2p}{d\mu^2} - \frac{dp}{d\mu}\left(\frac{5}{6} + \frac{1}{2}\frac{dp}{d\mu}\right) - k\frac{\exp\left\{\frac{2}{3}\int\frac{d\mu}{\mu+p}\right\}}{2(\mu+p)}\left(\frac{dp}{d\mu}\right)^2\left(1 - \frac{dp}{d\mu}\right) = 0 \,. \tag{55}$$

We see that *the shear-free results of [3] and [5] are avoided, at least at the linearised level, if the equation of state of the matter solves (55).* However, *a priori* it seems highly unlikely that any realistic barotropic equation of state will obey this extremely non-linear equation. We now try to find solutions of this equation, under various simplified assumptions or realistic initial conditions, to confirm it is nonphysical.

1. Flat universe ($k = 0$) with $c_s^2 = \text{constant} \neq 0$:

   This is the simplest case in which the equation (55) reduces to a simple algebraic equation

   $$\left(\frac{5}{6} + \frac{1}{2}c_s^2\right) = 0 \,, \tag{56}$$

   which gives $c_s^2 = -5/3$. This is physically not possible as the lower bound on the local sound speed (10) is violated.

2. Closed/open universe with $c_s^2 = \text{constant} \neq 0$: In this case also, the equation (55) reduces to an algebraic equation, and we get the relation

$$^3R = -2\frac{\left(\frac{5}{6}+\frac{1}{2}c_s^2\right)}{c_s^2\,(1-c_s^2)}(\mu+p) \tag{57}$$

Differentiating (57) with respect to μ and using (54) we get

$$\frac{2}{3}\frac{^3R}{(\mu+p)} = -2\frac{\left(\frac{5}{6}+\frac{1}{2}c_s^2\right)}{c_s^2\,(1-c_s^2)}(1+c_s^2)\;. \tag{58}$$

Eliminating $^3R/(\mu+p)$ from (57) and (58) we get the solution $c_s^2 = -1/3$, which again violates the lower bound of the local sound speed.

3. Flat universe with varying sound speed: In this case the equation (55) becomes

$$(\mu+p)\frac{d^2p}{d\mu^2} - \frac{dp}{d\mu}\left(\frac{5}{6}+\frac{1}{2}\frac{dp}{d\mu}\right) = 0\;. \tag{59}$$

To solve (59), if we choose the initial epoch ($\mu = \mu_0$) to be a radiation dominated one (which is quite realistic in view of our current understanding of the universe) with $c_s^2 \approx 1/3$, then from (59) we can easily see that c_s^2 monotonically increases with μ. And in the interval ($\mu_0 \leq \mu < \infty$) the function $p(\mu)$ is concave upwards. Therefore there must exist an earlier epoch at which $p(\mu) > \mu$, which violates (9).

4. Closed/open universe with varying sound speed: This is the most general case and let us try to find a solution with similar initial conditions as in the previous case. Since we know that very early universe was radiation dominated, let us suppose that there exists an epoch ($a_0 << 1$) with density μ_0 and pressure p_0 where $(\mu_0, p_0) \approx 1/a_0^4$. As we have already seen, $^3R \approx 1/a_0^2$, hence the last term on the LHS of (55) becomes suppressed and in this case also one can easily show that c_s^2 monotonically increases with μ. Therefore there must exist an earlier epoch $a_1 < a_0$ with $\mu_1 > \mu_0$, where $p(\mu) > \mu$ and (9) is violated. In other words, no solution satisfying (9) exists for (55) that gives a radiation dominated era in the early universe.

Hence for any physically realistic barotropic equation of state, if the new constraints are to be consistently propagated, we must have $\omega^a\Theta = 0$. We thus proved an important theorem for shear-free perturbations of FLRW spacetimes:

For an "almost" homogeneous and isotropic Universe filled with a barotropic perfect fluid subject to a physically realistic equation of state, if the fluid congruence is shear-free in a domain U, then it must be either vorticity-free or expansion-free in U.

IV. DISCUSSIONS

This result gives an interesting scenario. The linearised shearfree solutions - almost universally used to study the formation of structure by gravitational instability in the expanding universe, and believed to result in standard local Newtonian theory - do not have the same behaviour as shearfree Newtonian solutions. This may affect simple structure formation scenarios for rotating matter.

Another interesting point that emerged from our analysis is that there exists a class of barotropic equation of state (however unphysical that may be) for which the usual shear-free result can be avoided in the linearised case. It would be an interesting problem to see whether this same class of equations of state (or some similar class) allows shear free rotating and expanding solutions for the full non-linear Einstein equations for a barotropic perfect fluid.

[1] G. F. R. Ellis & H van Elst, "Cosmological Models", Cargèse Lectures 1998, in Theoretical and Observational Cosmology, Ed. M Lachze-Rey, (Dordrecht: Kluwer 1999), 1. [arXiv:gr-qc/9812046].

[2] S. Matarrese, O. Pantano and D. Saez, Phys. Rev. Lett. **72**, 320 (1994).

[3] G. F. R. Ellis, "The dynamics of pressure-free matter in general relativity", J. Math. Phys. **8**, 1171-1194, (1967).

[4] O. Heckmann and E. Schücking, Handbuch der Physik LIII Edited by S. Flügge. (Springer Verlag, Berlin-Götingin-Heidelberg) 489 (1959).

[5] J. M. M. Senovilla, C. F. Sopuerta and P. Szekeres, Gen. Rel. Grav. **30**, 389-411, (1998), [arXiv:gr-qc/9702035v1].

[6] G. F. R. Ellis & M. Bruni Phys Rev D **40** 1804 (1989).

[7] G. F. R. Ellis, M. Bruni and J. Hwang, Phys. Rev. D **42** 1035 (1990).

[8] M. Bruni, P. K. S. Dunsby & G. F. R. Ellis, Ap. J. **395** 34 (1992).

[9] P. K. S. Dunsby, M. Bruni and G. F. R. Ellis, Astrophys. J. **395**, 54 (1992)

[10] M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, Class. Quant. Grav. **9**, 921 (1992).

[11] R. Maartens and J. Triginer, Phys. Rev. D **56** 4640 (1997).

[12] P. K. S. Dunsby, B. A. C. Bassett and G. F. R. Ellis, Class. Quant. Grav. **14**, 1215 (1997) [arXiv:gr-qc/9811092].

[13] R. Maartens, T. Gebbie and G. F. R. Ellis, Phys. Rev. D **59** 083506 (1999).

[14] J. M. Stewart and M. Walker, *Proc. R. Soc.* London A341, 49 (1974).