非等温粘性Cahn-Hilliard方程组整体解对平衡态的收敛速率

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摘要： 考察了3维有界区域中一类描述快速相分离过程的非等温粘性Cahn-Hilliard方程组。该方程组由关于序参量的具弛豫项的粘性Cahn-Hilliard方程以及关于相对温度的满足Maxwell - Cattaneo 热传导定律的发展方程组组成。假设变量和化学势满足齐次的Neumann边界条件。证明了当时间趋于无穷大时方程组整体解对平衡态的收敛速率，改进了以前文献的结果。

关键词： 偏微分方程，Cahn-Hilliard 方程，Maxwell-Cattaneo 定律，收敛速率

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On the rate of convergence to equilibrium for a non-isothermal viscous Cahn-Hilliard equation

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Abstract: In this short note, a model describing non-isothermal fast phase separation processes taking place in a three-dimensional bounded domain is considered. The model consists of a viscous Cahn-Hilliard equation for the order parameter characterized by the presence of an inertial term and evolution equations for the (relative) temperature due to the Cattaneo–Maxwell heat conduction law. The state variables and the chemical potential are subject to the homogeneous Neumann boundary conditions. An improved estimate on the convergence rate of global solutions to steady states is proved.

Key words: Partial differential equation, Cahn-Hilliard equation, Maxwell-Cattaneo's law, convergence rate.

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0 Introduction

Let $\Omega \in \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. We consider the following nonisothermal Cahn-Hilliard system with inertial term (cf. [4])

$$(\theta + \chi)_t + \nabla \cdot \mathbf{q} = 0, \tag{0.1}$$

$$\sigma \mathbf{q}_t + \mathbf{q} = -\nabla \theta, \tag{0.2}$$

$$\varepsilon \chi_{tt} + \chi_t - \Delta \mu = 0, \tag{0.3}$$

$$\mu = -\Delta \chi + \alpha \chi_t + \phi(\chi) - \theta, \tag{0.4}$$

subject to the boundary conditions

$$\mathbf{q} \cdot \nu = \partial_\nu \mu = \partial_\nu \chi = 0, \qquad (x,t) \in \Omega \times (0, \infty), \tag{0.5}$$

and the initial data

$$\theta|_{t=0} = \theta_0(x),\ \mathbf{q}|_{t=0} = \mathbf{q}_0(x),\ \chi|_{t=0} = \chi_0(x),\ \chi_t|_{t=0} = \chi_1(x), \quad x \in \Omega. \tag{0.6}$$

Here θ denotes the (relative) temperature around a given critical one, χ represents the order parameter (or phase-field). $\varepsilon > 0$ is a small inertial parameter and $\alpha \geq 0$ is a viscosity coefficient, $\sigma \in [0, 1]$. ν stands for the outward normal derivative on the boundary Γ.

System (0.1)-(0.3) can be viewed as an evolution system which describes a two-phase system subject to nonisothermal phase separation. The inertial term $\varepsilon \chi_{tt}$ accounts for fast phase separation processes (cf. [2]). (0.2) represents the so-called Maxwell-Cattaneo's law of heat conduction modeling thermal disturbance as wave-like pulses propagating at finite speed (cf. [1]). The standard Fourier law is obtained when $\sigma = 0$.

We recall that in the isothermal case, the singular perturbation/hyperbolic relaxation of viscous/nonviscous Cahn-Hilliard equation has been investigated in several papers (see e.g., [3, 7, 9, 10] and the references therein). As far as the nonisothermal case (0.1)-(0.3) is concerned, the well-posedness of the initial and boundary value problem was obtained in the recent paper [4]. Moreover, the authors proved that the corresponding dynamical system is dissipative and possesses a global attractor. Besides, the convergence of each trajectory to a single steady state was established, assuming that the nonlinear potential is real analytic. (We also refer to [6] for a fully hyperbolic phase-field model that includes a damped hyperbolic equation of second order with respect to the phase function, instead of the Cahn-Hilliard one (0.3)).

The aim of this short note is to improve the convergence rate obtained in [4] for system (0.1)-(0.6). In the next section we first recall some known results in the literature. In Section 3, we provide the proof of our main result (see Theorem 2.1).

1 Functional Settings and Known Results

Let $H = L^2(\Omega)$ and $\mathbf{H} = (L^2(\Omega))^3$. These spaces are endowed with the natural inner product denoted by $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For the sake of simplicity, we will assume $\varepsilon = 1$. Then, we set $V = H^1(\Omega), \mathbf{V} = (H^1(\Omega))^3$ and $W = H^2(\Omega)$, both endowed with their standard inner products, and we define the subspace of H of the null mean functions $H_0 = \{v \in H : \langle v, 1\rangle = 0\}$. We also introduce the linear nonnegative operator $A = -\Delta : D(A) \subset H \to H_0$ with domain $D(A) = \{v \in W : \partial_\nu v = 0|_\Gamma\}$, and denote by A_0 its restriction to H_0. Note that A_0 is a positive linear operator; hence, for any $r \in \mathbb{R}$, we can define its powers A^r and, consequently, set $V_0^r = D(A_0^{r/2})$ endowed with the inner product $\langle v_1, v_2\rangle_{V_0^r} = \langle A_0^{\frac{r}{2}} v_1, A_0^{\frac{r}{2}} v_2\rangle$. We also denote the dual space of V by V^* whose norm can be defined as $\|v\|_{V^*}^2 = \|A_0^{-1/2}(v - \langle v, 1\rangle)\|^2 + \langle v, 1\rangle^2$. Moreover, we denote

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} \cdot \nu|_\Gamma = 0\}$$

and the product spaces

$$\mathcal{H}_\sigma = H \times \mathbf{H} \times V \times V^*, \quad \mathcal{V}_\sigma = V \times \mathbf{V}_0 \times D(A) \times H,$$

with the following norms

$$\|(z_1, \mathbf{z}_2, z_3 z_4)\|_{\mathcal{H}_\sigma}^2 = \|z_1\|^2 + \sigma\|\mathbf{z}_2\|^2 + \|z_3\|_V^2 + \|z_4\|_{V^*}^2,$$

$$\|(z_1, \mathbf{z}_2, z_3 z_4)\|_{\mathcal{V}_\sigma}^2 = \|z_1\|_V^2 + \sigma\|\mathbf{z}_2\|_{\mathbf{V}}^2 + \|z_3\|_W^2 + \|z_4\|^2,$$

if $\sigma > 0$. For $\sigma = 0$ we can simply set

$$\mathcal{H}_0 = H \times V \times V^*, \qquad \mathcal{V}_0 = V \times D(A) \times H.$$

The assumptions on the nonlinearity ϕ and its potential $\Phi(y) = \int_0^y f(z)dz$ are
(H1) $\Phi \in C^3(\mathbb{R})$ such that $\Phi(y) \geq -c_0, \quad \forall y \in \mathbb{R}$,
(H2) $|\phi''(y)| \leq c_1(1 + |y|), \quad \forall y \in \mathbb{R}$,
(H3) $\forall \epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$|\phi(y)| \leq \epsilon\Phi(y) + c_\epsilon, \quad \forall y \in \mathbb{R},$$

(H4) $\forall \zeta \in \mathbb{R}$ there exist $c_2 > 0$ and $c_3 \geq 0$ such that

$$(y - \zeta)\phi(y) \geq c_2\Phi(y) - c_3, \quad \forall y \in \mathbb{R},$$

(H5) For all $y \in \mathbb{R}$, $\phi'(y) \geq -c_4$. Here c_0, c_1, c_4 are positive constants.

Remark 1.1. *Although assumptions (H1)–(H5) look complicated, one can verify that the double well potential $\Phi(y) = \frac{1}{4}(y^2 - 1)^2$ and the corresponding function $\phi(y) = y^3 - y$ satisfy all the assumptions stated above.*

The wellposedness of problem (0.1)–(0.3) has been proved in [4], in particular, we have

Proposition 1.1. *Assume $\alpha > 0$ and (H1)–(H5) hold. For any initial data $(\theta_0, \sigma \boldsymbol{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, system (0.1)-(0.6) admits a unique global solution*

$$\theta \in C([0,\infty), H), \quad \sigma \boldsymbol{q} \in C([0,\infty), \boldsymbol{H}), \quad \boldsymbol{q} \in L^2(0,\infty; \boldsymbol{H}),$$

$$\chi \in C([0,\infty), V), \quad \chi_t \in C([0,\infty), V^*) \cap L^2(0,\infty, H).$$

Then they proceeded to prove the convergence to equilibrium of the global solution with an estimate on the convergence rate provided that the function ϕ is real analytic.

Proposition 1.2. *Let $\alpha > 0$, $\sigma > 0$ be fixed, ϕ be real analytic and (H1)–(H5) hold. For any $(\theta_0, \sigma \boldsymbol{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, the ω-limit set of the trajectory $(\theta(t), \boldsymbol{q}(t), \chi(t), \chi_t(t))$ consists of a single point such that*

$$\omega(\theta_0, \boldsymbol{q}_0, \chi_0, \chi_1) = (\theta_\infty, \mathbf{0}, \chi_\infty, 0)$$

where $(\theta_\infty, \chi_\infty)$ satisfies

$$\begin{cases} \theta_\infty = \int_\Omega (\theta_0 - \chi_1) dx, \\ \int_\Omega \chi_\infty = \int_\Omega (\chi_0 + \chi_1) dx, \\ A(A\chi_\infty + \phi(\chi)) = 0. \end{cases} \tag{1.1}$$

Moreover,

$$\lim_{t\to\infty} \|\chi(t) - \chi_\infty\|_V = 0,$$

and there exist $t^* > 0$ and a positive constant C such that

$$\|\theta(t) - \theta_\infty\|_{V^*} + \|\chi(t) - \chi_\infty\|_{V^*} \le C t^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge t^*. \tag{1.2}$$

If $\sigma = 0$ a similar result holds.

In the subsequent proof, we shall make use of the following uniform estimates of global solutions to (0.1)-(0.6) (cf. [4]):

Lemma 1.1. *Assume $\alpha > 0$ and (H1)–(H5) hold. For any $(\theta_0, \sigma \boldsymbol{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, there exists a constant C depending only on the $\mathcal{H}_\sigma$ norm of initial data and ϕ such that for $t \ge 0$,*

$$\|(\theta(t), \boldsymbol{q}(t), \chi(t), \chi(t))\|^2_{\mathcal{H}_\sigma} \le C, \tag{1.3}$$

and

$$\int_0^\infty (\|\theta(\tau) - (\theta(\tau), 1)\|^2 + \|\boldsymbol{q}(t)\|^2 + \|\chi_t\|^2) d\tau \le C. \tag{1.4}$$

Moreover, for the case $\sigma = 0$,

$$\|(\theta(t), \chi(t), \chi_t(t))\|_{\mathcal{V}_0} \le C, \qquad t \ge t_1 = t_1(R) > 0, \tag{1.5}$$

provided that $\|(\theta_0, \chi_0, \chi_1)\|_{\mathcal{H}_0} \le R$.

2 Improved Convergence Rate

The main result of this paper is as follows.

Theorem 2.1. *Under the same assumptions as in Proposition 1.2,*
(1) If $\sigma > 0$, for any $(\theta_0, \sigma \boldsymbol{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, there holds

$$\|\theta(t) - \theta_\infty\| + \|\boldsymbol{q}(t)\| + \|\chi(t) - \chi_\infty\|_V + \|\chi_t(t)\|_{V^*} \le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 0.$$

(2) If $\sigma = 0$, for any $(\theta_0, \chi_0, \chi_1) \in \mathcal{H}_0$, there holds

$$\|\theta(t) - \theta_\infty\|_V + \|\chi(t) - \chi_\infty\|_{H^2} + \|\chi_t(t)\| \le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 1.$$

Proof of Theorem 2.1.

First we consider the case $\sigma > 0$ and for the sake of simplicity we take $\sigma = 1$. Notice that in the assumption of Proposition 1.2, $\alpha > 0$ is fixed. This fact will provide us a better lower-order estimate on the convergence rate. Set

$$\tilde{\theta} = \theta - \langle \theta, 1 \rangle, \quad \tilde{\chi} = \chi - \langle \chi, 1 \rangle, \tag{2.1}$$
$$h(\tilde{\chi}) = \tilde{\phi}(\tilde{\chi}) - \tilde{\phi}(\tilde{\chi} - \langle \chi_1, e^{-t} \rangle).$$

Let

$$\mathcal{L}(t) = \frac{1}{2}\left(\|\tilde{\theta}(t)\|^2 + \|\mathbf{q}(t)\|^2 + \|\nabla \tilde{\chi}(t)\|^2 + 2\langle \tilde{\Phi}(\tilde{\chi}(t)), 1 \rangle + \|\tilde{\chi}_t(t)\|_{V^*}^2 \right).$$

A direct computation shows that

$$\begin{aligned} & \frac{d}{dt}\left(\mathcal{L}(t) + \mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\theta} \rangle \right) + \|\mathbf{q}\|^2 + \|\tilde{\chi}_t\|_{V^*}^2 + \alpha \|\tilde{\chi}_t\|^2 + \mu \|\tilde{\theta}\|^2 \\ = \;& -\mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\theta} \rangle - \mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\chi}_t \rangle + \mu \|A_0^{-1/2} \nabla \cdot \mathbf{q}\|^2 + \langle h(\tilde{\chi}), \tilde{\chi}_t \rangle. \end{aligned} \tag{2.2}$$

where $\mu > 0$ is some constant to be chosen later. The R.H.S. of (2.2) can be estimated as follows

$$-\mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\theta} \rangle - \mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\chi}_t \rangle \le C\mu (\|\mathbf{q}\|^2 + \|\theta\|^2 + \|A_0^{-1/2} \tilde{\chi}_t\|^2),$$
$$\langle h(\tilde{\chi}), \tilde{\chi}_t \rangle \le C_\alpha e^{-2t} + \frac{\alpha}{2} \|\tilde{\chi}_t\|^2.$$

As a result,

$$\begin{aligned} & \frac{d}{dt}\left(\mathcal{L}(t) + \mu \langle \mathbf{q}, \nabla A_0^{-1} \tilde{\theta} \rangle \right) + \|\mathbf{q}\|^2 + \|\tilde{\chi}_t\|_{V^*}^2 + \frac{\alpha}{2} \|\tilde{\chi}_t\|^2 + \mu \|\tilde{\theta}\|^2 \\ \le \;& C\mu (\|\mathbf{q}\|^2 + \|\theta\|^2 + \|A_0^{-1/2} \tilde{\chi}_t\|^2) + \mu \|A_0^{-1/2} \nabla \cdot \mathbf{q}\|^2 + C_\alpha e^{-2t}. \end{aligned} \tag{2.3}$$

Denote

$$\mathcal{G}(t) = \left\langle A_0^{-1} \tilde{\chi}_t, A_0^{-1} \left(A_0 \tilde{\chi} + \tilde{\phi}(\tilde{\chi}) - \overline{\tilde{\phi}(\tilde{\chi})} \right) \right\rangle,$$

where

$$\overline{\tilde{\phi}(\tilde{\chi})} = \langle \tilde{\phi}(\tilde{\chi}), 1\rangle.$$

We introduce

$$\mathcal{M}(t) = \mathcal{L}(t) - E(\tilde{\chi}_\infty) + \mu\langle \mathbf{q}, \nabla A_0^{-1}\tilde{\theta}\rangle + \mu_1 \mathcal{G}(t) + Ce^{-2t}$$

and

$$\mathcal{N}^2(t) = \|\mathbf{q}(t)\|^2 + \|\tilde{\chi}_t(t)\|^2_{V^*} + \|\tilde{\chi}_t(t)\|^2 + \|\theta(t)\|^2 + \|A_0^{-1/2}(A_0\tilde{\chi} + \tilde{\phi}(\tilde{\chi}) - \overline{\tilde{\phi}(\tilde{\chi})})\|^2.$$

We can take $\mu, \mu_1 > 0$ sufficiently small that there exists $\gamma > 0$ depending on μ, μ_1, α such that (cf. [4, (4.16)])

$$\frac{d}{dt}\mathcal{M}(t) + \gamma \mathcal{N}^2(t) \le 0, \qquad t \ge 0. \tag{2.4}$$

For our system (0.1)-(0.6), a special type of Łojasiewicz-Simon inequality has been obtained in [4] (cf. [4, Lemma 4.1]),

Lemma 2.1 (Łojasiewicz-Simon type inequality). *Let*

$$\tilde{\phi}(y) = \phi(y + \langle \chi_0 + \chi_1, 1\rangle), \quad \textit{and} \quad \tilde{\Phi}(y) = \int_0^y \tilde{\phi}(\xi)d\xi, \quad \forall\, y \in \mathbb{R},$$

$$E(v) = \frac{1}{2}\|\nabla v\|^2 + \langle \tilde{\Phi}(v), 1\rangle.$$

Suppose that ϕ is real analytic and (H2), (H5) hold. Let $v_\infty \in V_0^2$ be a function satisfying

$$A(A_0 v_\infty + \tilde{\phi}(v_\infty)) = 0.$$

Then there exist $\rho \in (0, \frac{1}{2})$, $\eta > 0$ and a positive constant L such that

$$|E(v) - E(v_\infty)|^{1-\rho} \le L\|A_0 v + \tilde{\phi}(v) - \langle \tilde{\phi}(v), 1\rangle\|_{V_0^{-1}}, \tag{2.5}$$

for all $v \in V_0^1$ such that $\|v - v_\infty\|_{V_0^1} \le \eta$.

From Proposition 1.2, we can see that there exists $t_* > 0$ such that for all $t \ge t_*$ the condition in Lemma 2.1 is satisfied. Namely, for the constant $\eta > 0$ in Lemma 2.1, there holds $\|\tilde{\chi} - \tilde{\chi}_\infty\|_{V_0^1} \le \eta$ for $t \ge t_*$.

It is well-known that the Łojasiewicz-Simon type inequality can be used to obtain (lower-order) convergence rate of global solutions to equilibrium (cf. e.g., [5]). Actually, after a refinement of the argument in [4], we can show that

$$\int_0^\infty \mathcal{N}(\tau)d\tau \le Ct^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge t_*.$$

By Proposition 1.1, after adjusting C properly, we have

$$\int_0^\infty \mathcal{N}(\tau)d\tau \le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 0, \tag{2.6}$$

which entails that (since now $\alpha > 0$)

$$\int_t^\infty \|\tilde{\chi}_t(\tau)\| d\tau \le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 0. \tag{2.7}$$

On the other hand, from the boundary condition we have

$$\langle (\theta+\chi)(t), 1\rangle = \langle \theta_0 + \chi_0, 1\rangle,$$

$$\langle \chi(t), 1\rangle = \langle \chi_0 + \chi_1, 1\rangle - \langle \chi_1, e^{-t}\rangle, \quad \langle \chi_t(t), 1\rangle = \langle \chi_1, 1\rangle e^{-t}.$$

As a result,

$$\begin{aligned} \|\chi - \chi_\infty\| &\le \|\tilde{\chi}(t) - \tilde{\chi}_\infty\| + \|\langle \chi(t), 1\rangle - \langle \chi_\infty, 1\rangle\| \\ &\le \int_t^\infty \|\tilde{\chi}_t(\tau)\| d\tau + C\left|\int_\Omega (\chi(t) - \chi_\infty) dx\right| \\ &\le C(1+t)^{-\frac{\rho}{1-2\rho}} + Ce^{-t}\left|\int_\Omega \chi_1 dx\right| \\ &\le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 0. \end{aligned} \tag{2.8}$$

Here $\tilde{\chi}_\infty = \chi_\infty - \langle \chi_\infty, 1\rangle$.

Remark 2.1. *We notice that estimate* (2.8) *is better than Proposition 1.2 (cf.* (1.2)*).*

Next, we proceed to get further estimates on convergence rate in higher-order norms. The calculations are performed in a formal way, but they can be justified by a standard density argument.

Under changes of variable (2.1), system (0.1)–(0.3) can be written as

$$\langle (\tilde{\theta} + \tilde{\chi})_t, v\rangle - \langle \mathbf{q}, \nabla v\rangle = 0, \tag{2.9}$$

$$\langle \sigma \mathbf{q}_t + \mathbf{q}, \mathbf{v}\rangle = \langle \tilde{\theta}, \nabla \cdot \mathbf{v}\rangle, \tag{2.10}$$

$$\langle \tilde{\chi}_{tt} + \tilde{\chi}_t, w\rangle + \langle A\tilde{\chi} + \phi(\chi) + \alpha\tilde{\chi}_t - \tilde{\theta}, Aw\rangle = 0, \tag{2.11}$$

for all $v \in V$, $\mathbf{v} \in \mathbf{V}_0$ and $w \in D(A)$. On the other hand, we can write the stationary problem (1.1) in the following form

$$\begin{cases} \theta_\infty = \int_\Omega (\theta_0 - \chi_1) dx, \\ \int_\Omega \chi_\infty = \int_\Omega (\chi_0 + \chi_1) dx, \\ \langle A\tilde{\chi}_\infty + \phi(\chi), Aw\rangle = 0, \end{cases} \tag{2.12}$$

for all $w \in D(A)$.

Subtracting the third equation in (2.12) from (2.11), we get

$$\langle \tilde{\chi}_{tt} + \tilde{\chi}_t, w\rangle + \langle A(\tilde{\chi} - \tilde{\chi}_\infty) + \phi(\chi) - \phi(\chi_\infty) + \alpha\tilde{\chi}_t - \tilde{\theta}, Aw\rangle = 0, \tag{2.13}$$

for all $w \in D(A)$.

Take $v=\tilde{\theta}$ in (2.9), $\mathbf{v}=\mathbf{q}$ in (2.10) and $w=A_0^{-1}(\tilde{\chi}_t+\beta(\tilde{\chi}-\tilde{\chi}_\infty))$. Adding together the resultants, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2+\sigma\|\mathbf{q}\|^2+\|A_0^{-1/2}\tilde{\chi}_t\|^2+\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2\\
&\quad+2\langle\Phi(\chi)-\phi(\chi_\infty)\chi-\Phi(\chi_\infty)+\phi(\chi_\infty)\chi_\infty,1\rangle+2\beta\langle A_0^{-1/2}\tilde{\chi}_t,A_0^{-1/2}(\tilde{\chi}-\tilde{\chi}_\infty)\rangle\\
&\quad+\beta\|A_0^{-1/2}(\tilde{\chi}-\tilde{\chi}_\infty)\|^2+\alpha\beta\|\tilde{\chi}-\tilde{\chi}_\infty\|^2\Big)\\
&+\|\mathbf{q}\|^2+(1-\beta)\|A_0^{-1/2}\tilde{\chi}_t\|^2+\alpha\|\tilde{\chi}_t\|^2+\beta\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2\\
=\;&\langle\phi(\chi),\langle\chi_t,1\rangle\rangle-\langle\phi(\chi_\infty),\langle\chi_t,1\rangle\rangle-\beta\langle\phi(\chi)-\phi(\chi_\infty),\tilde{\chi}-\tilde{\chi}_\infty\rangle\\
&+\beta\langle\tilde{\theta},\tilde{\chi}-\tilde{\chi}_\infty\rangle. \qquad (2.14)
\end{aligned}$$

The R.H.S. can be estimated as follows

$$|\langle\phi(\chi)-\phi(\chi_\infty),\langle\chi_t,1\rangle\rangle|\le\left|\int_\Omega(\phi(\chi)-\phi(\chi_\infty))dx\right|\left|\int_\Omega\chi_1dx\right|e^{-t}\le Ce^{-t}. \qquad (2.15)$$

$$\begin{aligned}
&|-\beta\langle\phi(\chi)-\phi(\chi_\infty),\tilde{\chi}-\tilde{\chi}_\infty\rangle|\\
\le\;&\beta\|\phi'\|_{L^3}\|\chi-\chi_\infty\|_{L^6}\|\tilde{\chi}-\tilde{\chi}_\infty\|\\
\le\;&C\beta\|\chi-\chi_\infty\|_{H^1}\|\tilde{\chi}-\tilde{\chi}_\infty\|\\
\le\;&C\beta\|\nabla(\chi-\chi_\infty)\|\|\tilde{\chi}-\tilde{\chi}_\infty\|+C\beta\|\chi-\chi_\infty\|\|\tilde{\chi}-\tilde{\chi}_\infty\|\\
\le\;&\frac{1}{2}\beta\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2+C\beta\|\tilde{\chi}-\tilde{\chi}_\infty\|^2+C\beta\|\chi-\chi_\infty\|^2\\
\le\;&\frac{1}{2}\beta\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2+C\beta(1+t)^{-\frac{2\rho}{1-2\rho}}.
\end{aligned}$$

$$\Big|\beta\langle\tilde{\theta},\tilde{\chi}-\tilde{\chi}_\infty\rangle\Big|\le\beta\|\tilde{\chi}-\tilde{\chi}_\infty\|\|\tilde{\theta}\|\le\frac{1}{2}\beta\|\tilde{\chi}-\tilde{\chi}_\infty\|^2+\frac{1}{2}\beta\|\tilde{\theta}\|^2\le C\beta(1+t)^{-\frac{2\rho}{1-2\rho}}+\frac{1}{2}\beta\|\tilde{\theta}\|^2. \qquad (2.16)$$

On the other hand, we have (cf. [4, (2.29)])

$$\begin{aligned}
\frac{d}{dt}\langle\mathbf{q},\nabla A_0^{-1}\tilde{\theta}\rangle\;&=\;-\frac{1}{\sigma}\langle\mathbf{q},\nabla A_0^{-1}\tilde{\theta}\rangle-\frac{1}{\sigma}\|\tilde{\theta}\|^2-\langle\mathbf{q},\nabla A_0^{-1}\tilde{\chi}_t\rangle+\|A_0^{-1/2}\nabla\cdot\mathbf{q}\|^2\\
&\le\;-\frac{1}{2\sigma}\|\tilde{\theta}\|^2+\|A_0^{-1/2}\tilde{\chi}_t\|^2+\kappa_1\left(1+\frac{1}{\sigma}\right)\|\mathbf{q}\|^2. \qquad (2.17)
\end{aligned}$$

Multiplying (2.17) by γ_1 and adding it to (2.14), we infer from (2.15)–(2.16) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2+\sigma\|\mathbf{q}\|^2+\|A_0^{-1/2}\tilde{\chi}_t\|^2+\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2\\
&\quad+2\langle\Phi(\chi)-\phi(\chi_\infty)\chi-\Phi(\chi_\infty)+\phi(\chi_\infty)\chi_\infty,1\rangle+2\beta\langle A_0^{-1/2}\tilde{\chi}_t,A_0^{-1/2}(\chi-\chi_\infty)\rangle\\
&\quad+\beta\|A_0^{-1/2}(\tilde{\chi}-\tilde{\chi}_\infty)\|^2+\alpha\beta\|\tilde{\chi}-\tilde{\chi}_\infty\|^2+\gamma_1\langle\mathbf{q},\nabla A_0^{-1}\tilde{\theta}\rangle\Big)+(1-\beta-\gamma_1)\|A_0^{-1/2}\tilde{\chi}_t\|^2\\
&+\left[1-\gamma_1\kappa_1\left(1+\frac{1}{\sigma}\right)\right]\|\mathbf{q}\|^2+\alpha\|\tilde{\chi}_t\|^2+\frac{1}{2}\beta\|\nabla(\tilde{\chi}-\tilde{\chi}_\infty)\|^2+\left(\frac{\gamma_1}{2\sigma}-\frac{\beta}{2}\right)\|\tilde{\theta}\|^2\\
\le\;&Ce^{-t}+C\beta(1+t)^{-\frac{2\rho}{1-2\rho}}. \qquad (2.18)
\end{aligned}$$

Denote

$$\begin{aligned}\Upsilon(t) &= \frac{1}{2}\Big(\|\tilde\theta\|^2+\sigma\|\mathbf{q}\|^2+\|A_0^{-1/2}\tilde\chi_t\|^2+\|\nabla(\tilde\chi-\tilde\chi_\infty)\|^2+\beta\|A_0^{-1/2}(\tilde\chi-\tilde\chi_\infty)\|^2\\ &\quad +2\langle\Phi(\chi)-\phi(\chi_\infty)\chi-\Phi(\chi_\infty)+\phi(\chi_\infty)\chi_\infty,1\rangle+2\beta\langle A_0^{-1/2}\tilde\chi_t,A_0^{-1/2}(\chi-\chi_\infty)\rangle\\ &\quad +\alpha\beta\|\tilde\chi-\tilde\chi_\infty\|^2+\gamma_1\langle\mathbf{q},\nabla A_0^{-1}\tilde\theta\rangle\Big).\end{aligned}$$

By the Taylor's expansion, we have

$$\Phi(\chi)=\Phi(\chi_\infty)+\phi(\chi_\infty)(\chi-\chi_\infty)+f'(\xi)(\chi-\chi_\infty)^2, \tag{2.19}$$

where $\xi=a\chi+(1-a)\chi_\infty$ with $a\in[0,1]$.
Then we deduce that

$$\begin{aligned} &|\langle\Phi(\chi)-\phi(\chi_\infty)\chi-\Phi(\chi_\infty)+\phi(\chi_\infty)\chi_\infty,1\rangle|\\ =\ & \left|\int_\Omega\phi'(\xi)(\chi-\chi_\infty)^2dx\right|\\ \le\ & \|\phi'(\xi)\|_{L^3}\|\chi-\chi_\infty\|_{L^3}^2\\ \le\ & C\|\nabla(\chi-\chi_\infty)\|\|\chi-\chi_\infty\|+C\|\chi-\chi_\infty\|^2\\ \le\ & \frac{1}{4}\|\nabla(\tilde\chi-\tilde\chi_\infty)\|^2+C\|\chi-\chi_\infty\|^2.\end{aligned}$$

$$\left|2\beta\langle A_0^{-1/2}\tilde\chi_t,A_0^{-1/2}(\chi-\chi_\infty)\rangle\right|\le 2\beta\|A_0^{-1/2}\tilde\chi_t\|^2+\frac{1}{2}\beta\|A_0^{-1/2}(\tilde\chi-\tilde\chi_\infty)\|^2,$$

$$\left|\gamma_1\langle\mathbf{q},\nabla A_0^{-1}\tilde\theta\rangle\right|\le\gamma_1\|\mathbf{q}\|^2+\kappa_2\gamma_1\|\tilde\theta\|^2.$$

As a result, if β,γ_1 are chosen sufficiently small that

$$0<\beta\le\min\left\{\frac{1}{4},\frac{\gamma_1}{2\sigma}\right\},\quad 0<\gamma_1\le\min\left\{\frac{1}{4},\frac{\sigma}{2\kappa_1(\sigma+1)},\frac{\sigma}{2},\frac{1}{\kappa_2}\right\}, \tag{2.20}$$

then there exist constants $C_1,C_2>0$ such that

$$\Upsilon(t)+C_1\|\chi-\chi_\infty\|^2\ge C_2\left(\|\tilde\theta\|^2+\|\mathbf{q}\|^2+\|A_0^{-1/2}\tilde\chi_t\|^2+\|\nabla(\tilde\chi-\tilde\chi_\infty)\|^2\right). \tag{2.21}$$

Moreover, we can deduce from (2.18) that

$$\frac{d}{dt}\Upsilon(t)+C_3\Upsilon(t)\le Ce^{-t}+C\beta(1+t)^{-\frac{2\rho}{1-2\rho}}, \tag{2.22}$$

where $C_3>0$ is a constant depending on $\beta,\gamma_1,\alpha,|\Omega|$ as well as the initial data.
Solving the differential inequality (2.22) (cf. e.g., [8]), we get

$$\Upsilon(t)\le C(1+t)^{-\frac{2\rho}{1-2\rho}},\quad\forall t\ge 0. \tag{2.23}$$

Here we use the fact that e^{-t} decays faster than $(1+t)^{-\frac{2\rho}{1-2\rho}}$ for large t and adjust the constant C properly. It follows from (2.21) that

$$\|\theta(t)-\theta_\infty\|\le\|\tilde\theta(t)\|+|\langle\theta(t),1\rangle|\le C(1+t)^{-\frac{\rho}{1-2\rho}}+Ce^{-t}\le C(1+t)^{-\frac{\rho}{1-2\rho}},$$

$$\|\mathbf{q}\| \le C(1+t)^{-\frac{\rho}{1-2\rho}},$$
$$\|\chi_t(t)\|_{V^*} \le \|A_0^{-1/2}\tilde{\chi}_t(t)\| + |\langle \chi_t(t), 1\rangle| \le C(1+t)^{-\frac{\rho}{1-2\rho}} + Ce^{-t} \le C(1+t)^{-\frac{\rho}{1-2\rho}},$$
$$\|\chi(t)-\chi_\infty\|_{H^1} \le \|\nabla(\tilde{\chi}(t)-\tilde{\chi}_\infty)\| + \|\chi(t)-\chi_\infty\| \le C(1+t)^{-\frac{\rho}{1-2\rho}}.$$

We have finished the proof for the case $\sigma > 0$. The case $\sigma = 0$ is easier and can be obtained in a similar way. In summary, the proof of Theorem 2.1 is complete.

Actually, we can say more about the convergence rate. When the authors tried to prove precompactness of trajectories in [4], a decomposition technique was used. Thanks to the assumptions of ϕ, there is a constant $\iota > 2c_4$ which also depends on the norms of initial data such that

$$\frac{1}{2}\|\nabla z\|^2 + (\iota - 2c_4)\|z\|^2 - \langle \phi'(\chi(t))z, z\rangle \ge 0,$$

for all $z \in V$ and $t \ge 0$. Consequently, set

$$\psi(r) = \phi(r) + \iota r, \quad \forall\, r \in \mathbb{R}. \tag{2.24}$$

Then one can split the solution in the following way

$$(\theta, \mathbf{q}, \chi) = (\theta^d, \mathbf{q}^d, \chi^d) + (\theta^c, \mathbf{q}^c, \chi^c),$$

where

$$\begin{cases} \langle (\theta^d + \chi^d)_t, v\rangle - \langle \mathbf{q}^d, \nabla v\rangle = 0, \\ \langle \sigma \mathbf{q}^d_t + \mathbf{q}^d, \mathbf{v}\rangle = \langle \theta^d, \nabla\cdot \mathbf{v}\rangle, \\ \langle \chi^d_{tt} + \chi^d_t, w\rangle + \langle A\chi^d + \psi(\chi) - \psi(\chi^c) + \alpha\chi^d_t - \theta^d, Aw\rangle = 0, \\ \theta^d(0) = \tilde{\chi}_0, \quad \sigma\mathbf{q}^d(0) = \sigma\mathbf{q}_0, \quad \chi^d(0) = \tilde{\chi}_0, \quad \chi^d_t(0) = \tilde{\chi}_1, \end{cases} \tag{2.25}$$

and

$$\begin{cases} \langle (\theta^c + \chi^c)_t, v\rangle - \langle \mathbf{q}^c, \nabla v\rangle = 0, \\ \langle \sigma \mathbf{q}^c_t + \mathbf{q}^c, \mathbf{v}\rangle = \langle \theta^c, \nabla\cdot \mathbf{v}\rangle, \\ \langle \chi^c_{tt} + \chi^c_t, w\rangle + \langle A\chi^c + \psi(\chi^c) + \alpha\chi^c_t - \theta^c, Aw\rangle = \langle \iota\chi, Aw\rangle, \\ \theta^c(0) = \langle \theta_0, 1\rangle, \quad \sigma\mathbf{q}^c(0) = \mathbf{0}, \quad \chi^c(0) = \langle \chi_0, 1\rangle, \quad \chi^c_t(0) = \langle \chi_1, 1\rangle \end{cases} \tag{2.26}$$

for all $v \in V$, $\mathbf{v} \in \mathbf{V}$ and $w \in D(A)$.

It has been proven in [4, Section 3] that $(\theta^d(t), \mathbf{q}^d(t), \chi^d(t), \chi^d_t(t))$ decays to 0 exponentially fast in $\mathcal{H}_\sigma$ as time goes to infinity while $(\theta^c(t), \mathbf{q}^c(t), \chi^c(t), \chi^c_t(t))$ is uniformly bounded in $\mathcal{V}_\sigma$. Namely,

Lemma 2.2. *If $\|(\theta_0, \boldsymbol{q}_0, \chi_0, \chi_1)\|_{\mathcal{H}_\sigma} \le R$, then for any $t \ge 0$ we have*

$$\|(\theta^d(t), \boldsymbol{q}^d(t), \chi^d(t), \chi^d_t(t))\|_{\mathcal{H}_\sigma} \le C(R)e^{-Ct}, \tag{2.27}$$

and

$$\|(\theta^c(t), \boldsymbol{q}^c(t), \chi^c(t), \chi^c_t(t))\|_{\mathcal{V}_\sigma} \le C(R). \tag{2.28}$$

We notice that (2.27) provides a faster convergence rate for the decay part than the whole trajectory. In what follows, we shall show that the uniform bound (2.28) will enable us to obtain the same convergence rate for the compact part as for the whole trajectory but in the higher order norm in $\mathcal{V}_\sigma$. More precisely, we have

Theorem 2.2. *Under the same assumptions as in Proposition 1.2, if* $\sigma > 0$, *for any* $(\theta_0, \sigma \boldsymbol{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, *there holds*

$$\|\theta^c(t) - \theta_\infty\|_V + \|\boldsymbol{q}^c(t)\|_{\boldsymbol{V}_0} + \|\chi^c(t) - \chi_\infty\|_{H^2} + \|\chi_t^c(t)\| \le C(1+t)^{-\frac{\rho}{1-2\rho}}, \qquad \forall\, t \ge 0. \quad (2.29)$$

Proof. We take $v = A\theta^c$ in the first equation of (2.26) and $\mathbf{v} = -\nabla\nabla\cdot\mathbf{q}^c$ in the second equation. Adding together the resulting identities, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla\theta^c\|^2 + \frac{\sigma}{2}\|\nabla\cdot\mathbf{q}^c\|^2\right) + \|\nabla\cdot\mathbf{q}^c\|^2 + \langle\chi_t^c, A\theta^c\rangle = 0. \quad (2.30)$$

Subtracting the third equation in (2.12) from (2.26), we have

$$\langle\chi_{tt}^c + \chi_t^c, w\rangle + \langle A(\chi^c - \chi_\infty) + \psi(\chi^c) - \psi(\chi_\infty) + \alpha\chi_t^c - \theta^c, Aw\rangle = \langle\iota(\chi - \chi_\infty), Aw\rangle.$$

Taking $w = \chi_t^c + \beta(\chi^c - \chi_\infty)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\chi_t^c\|^2 + \frac{1}{2}\|A(\chi^c - \chi_\infty)\|^2 + \beta\langle\chi_t^c, \chi^c - \chi_\infty\rangle + \frac{\beta}{2}\|\chi^c - \chi_\infty\|^2\right.\\
&\quad\left.+\frac{\alpha\beta}{2}\|\nabla(\chi^c - \chi_\infty)\|^2\right) + \alpha\|\nabla\chi_t^c\|^2 + (1-\beta)\|\chi_t^c\|^2 + \beta\|A(\chi^c - \chi_\infty)\|^2\\
&\quad-\langle\theta^c, A\chi_t^c\rangle\\
=&\ -\langle\psi(\chi^c) - \psi(\chi_\infty), A\chi_t^c\rangle + \iota\langle\chi - \chi_\infty, A\chi_t^c\rangle + \beta\langle\theta^c, A(\chi^c - \chi_\infty)\rangle\\
&\ -\beta\langle\psi(\chi^c) - \psi(\chi_\infty), A(\chi^c - \chi_\infty)\rangle + \iota\langle\chi - \chi_\infty, A(\chi - \chi_\infty)\rangle. \qquad (2.31)
\end{aligned}$$

By the uniform bound (2.28) for $(\theta^c(t), \mathbf{q}^c(t), \chi^c(t), \chi_t^c(t))$ and the uniform bound (1.3) for $(\theta(t), \mathbf{q}(t), \chi(t), \chi_t(t))$, we can estimate the R.H.S. of (2.31) as follows.

$$\begin{aligned}
&-\langle\psi(\chi^c) - \psi(\chi_\infty), A\chi_t^c\rangle\\
=&\ \langle\psi'(\chi^c)\nabla\chi^c - \psi'(\chi_\infty)\nabla\chi_\infty, \nabla\chi_t^c\rangle\\
\le&\ \|\psi'(\chi^c)\nabla(\chi^c - \chi_\infty)\|\|\nabla\chi_t^c\| + \|(\psi'(\chi^c) - \psi'(\chi_\infty))\nabla\chi_\infty\|\|\nabla\chi_t^c\|\\
\le&\ \|\psi'(\chi_c)\|_{L^\infty}\|\nabla(\chi^c - \chi_\infty)\|\|\nabla\chi_t^c\| + \|\psi''(\xi)\|_{L^6}\|\chi^c - \chi_\infty\|_{L^6}\|\nabla\chi_\infty\|_{L^6}\|\nabla\chi_t^c\|\\
\le&\ C(1 + \|\chi^c\|_{H^2}^2)\|\nabla(\chi^c - \chi_\infty)\|\|\nabla\chi_t^c\| + C(1 + \|\xi\|_{L^6})\|\chi_\infty\|_{H^2}\|\chi^c - \chi_\infty\|_{H^1}\|\nabla\chi_t^c\|\\
\le&\ \frac{\alpha}{8}\|\nabla\chi_t^c\|^2 + C\|\chi^c - \chi_\infty\|_{H^1}^2,
\end{aligned}$$

where $\xi = a\chi^c + (1-a)\chi_\infty$ with $a \in [0,1]$ and χ_∞ is bounded in H^2. Similarly, we have

$$\begin{aligned}
& -\beta\langle \psi(\chi^c) - \psi(\chi_\infty), A(\chi^c - \chi_\infty)\rangle \\
= \; & \beta\langle \psi'(\chi^c)\nabla\chi^c - \psi'(\chi_\infty)\nabla\chi_\infty, \nabla(\chi^c - \chi_\infty)\rangle \\
\le \; & \|\psi'(\chi^c)\nabla(\chi^c - \chi_\infty)\|\|\nabla(\chi^c - \chi_\infty)\| + \|(\psi'(\chi^c) - \psi'(\chi_\infty))\nabla\chi_\infty\|\|\nabla(\chi^c - \chi_\infty)\| \\
\le \; & C\|\chi^c - \chi_\infty\|_{H^1}^2.
\end{aligned}$$

$$\begin{aligned}
\beta\langle \theta^c, A(\chi^c - \chi_\infty)\rangle &= \beta\langle \theta^c - \theta_\infty, A(\chi^c - \chi_\infty)\rangle \\
&\le \frac{\beta}{4}\|A(\chi^c - \chi_\infty)\|^2 + \beta\|\theta^c - \theta_\infty\|^2,
\end{aligned}$$

where we use the fact that θ_∞ is a constant and as a result $\langle \theta_\infty, A(\chi^c - \chi_\infty)\rangle = 0$.

$$\begin{aligned}
\iota\langle \chi - \chi_\infty, A\chi_t^c\rangle + \iota\langle \chi - \chi_\infty, A(\chi - \chi_\infty)\rangle &\le -\iota\langle \nabla(\chi - \chi_\infty), \nabla\chi_t^c\rangle - \iota\|\nabla(\chi - \chi_\infty)\|^2 \\
&\le \frac{\alpha}{8}\|\nabla\chi_t^c\|^2 + C\|\chi - \chi_\infty\|_{H^1}^2. \qquad (2.32)
\end{aligned}$$

It follows from (2.31)–(2.32) that

$$\begin{aligned}
& \frac{d}{dt}\left(\frac{1}{2}\|\chi_t^c\|^2 + \frac{1}{2}\|A(\chi^c - \chi_\infty)\|^2 + \beta\langle \chi_t^c, \chi^c - \chi_\infty\rangle + \frac{\beta}{2}\|\chi^c - \chi_\infty\|^2 \right. \\
& \left. + \frac{\alpha\beta}{2}\|\nabla(\chi^c - \chi_\infty)\|^2\right) + \frac{3\alpha}{4}\|\nabla\chi_t^c\|^2 + (1-\beta)\|\chi_t^c\|^2 + \frac{3\beta}{4}\|A(\chi^c - \chi_\infty)\|^2 \\
& -\langle \theta^c, A\chi_t^c\rangle \\
\le \; & C\|\chi^c - \chi_\infty\|_{H^1}^2 + C\|\chi - \chi_\infty\|_{H^1}^2 + \beta\|\theta^c - \theta_\infty\|^2. \qquad (2.33)
\end{aligned}$$

We also have (cf. (2.17))

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{q}^c, \nabla\theta^c\rangle + \frac{1}{\sigma}\|\nabla\theta^c\|^2 &= -\frac{1}{\sigma}\langle \mathbf{q}^c, \nabla\theta^c\rangle - \langle \mathbf{q}^c, \nabla\chi_t^c\rangle + \|\nabla\cdot\mathbf{q}^c\|^2 \\
&\le \frac{1}{2\sigma}\|\nabla\theta^c\|^2 + \frac{1}{2\sigma}\|\mathbf{q}^c\|^2 + \alpha\|\nabla\chi_t^c\|^2 + \frac{\alpha}{4}\|\mathbf{q}^c\|^2. \qquad (2.34)
\end{aligned}$$

Multiplying (2.34) by $\gamma_2 > 0$ and adding it to the sum of (2.30) (2.31), it follows from (2.32)–(2.33) that

$$\begin{aligned}
& \frac{d}{dt}\left(\frac{1}{2}\|\nabla\theta^c\|^2 + \frac{\sigma}{2}\|\nabla\cdot\mathbf{q}^c\|^2 + \frac{1}{2}\|\chi_t^c\|^2 + \frac{1}{2}\|A(\chi^c - \chi_\infty)\|^2 + \beta\langle \chi_t^c, \chi^c - \chi_\infty\rangle \right. \\
& \left. + \frac{\beta}{2}\|\chi^c - \chi_\infty\|^2 + \frac{\alpha\beta}{2}\|\nabla(\chi^c - \chi_\infty)\|^2 + \gamma_2\langle \mathbf{q}^c, \nabla\theta^c\rangle\right) + \left(\frac{3}{4} - \gamma_2\right)\alpha\|\nabla\chi_t^c\|^2 \\
& + (1-\beta)\|\chi_t^c\|^2 + \frac{\gamma_2}{2\sigma}\|\nabla\theta^c\|^2 + \|\nabla\cdot\mathbf{q}^c\|^2 + \frac{3\beta}{4}\|A(\chi^c - \chi_\infty)\|^2 \\
\le \; & C\|\chi^c - \chi_\infty\|_{H^1}^2 + C\|\chi - \chi_\infty\|_{H^1}^2 + \beta\|\theta^c - \theta_\infty\|^2 + \left(\frac{1}{2\sigma} + \frac{\alpha}{4}\right)\gamma_2\|\mathbf{q}^c\|^2. \qquad (2.35)
\end{aligned}$$

Denote

$$\begin{aligned}\Upsilon_1(t) &= \frac{1}{2}\|\nabla\theta^c\|^2+\frac{\sigma}{2}\|\nabla\cdot\mathbf{q}^c\|^2+\frac{1}{2}\|\chi_t^c\|^2+\frac{1}{2}\|A(\chi^c-\chi_\infty)\|^2+\beta\langle\chi_t^c,\chi^c-\chi_\infty\rangle\\ &\quad+\frac{\beta}{2}\|\chi^c-\chi_\infty\|^2+\frac{\alpha\beta}{2}\|\nabla(\chi^c-\chi_\infty)\|^2+\gamma_2\langle\mathbf{q}^c,\nabla\theta^c\rangle. \qquad (2.36)\end{aligned}$$

Taking $\gamma_2,\beta_1\in(0,\frac{1}{2})$, there exists positive constants C_4,C_5,C_6 such that

$$\Upsilon_1(t)+C_4(\|\mathbf{q}^c\|^2+\|\chi^c-\chi_\infty\|^2)\geq C_5(\|\nabla\theta^c\|^2+\|\nabla\cdot\mathbf{q}^c\|^2+\|\chi_t^c\|^2+\|A(\chi^c-\chi_\infty)\|^2), \quad (2.37)$$

and

$$\frac{d}{dt}\Upsilon_1(t)+C_6\Upsilon(t)\leq C(\|\chi^c-\chi_\infty\|_{H^1}^2+\|\chi-\chi_\infty\|_{H^1}^2+\|\theta^c-\theta_\infty\|^2+\|\mathbf{q}^c\|^2). \qquad (2.38)$$

We infer from Theorem 2.1 and (2.27) that

$$\begin{aligned}\|\chi^c(t)-\chi_\infty\|_{H^1} &\leq \|\chi-\chi_\infty\|_{H^1}+\|\chi^d\|_{H^1}\leq C(1+t)^{-\frac{\rho}{1-2\rho}}+Ce^{-Ct}\\ &\leq C(1+t)^{-\frac{\rho}{1-2\rho}},\end{aligned}$$

and in the same manner,

$$\|\mathbf{q}^c\|\leq\|\mathbf{q}\|+\|\mathbf{q}^d\|\leq C(1+t)^{-\frac{\rho}{1-2\rho}},$$

$$\|\theta^c-\theta_\infty\|\leq\|\theta-\theta_\infty\|+\|\theta^d\|\leq C(1+t)^{-\frac{\rho}{1-2\rho}}.$$

As a result, (2.38) yields

$$\frac{d}{dt}\Upsilon_1(t)+C_6\Upsilon_1(t)\leq C(1+t)^{-\frac{2\rho}{1-2\rho}},$$

which implies

$$\Upsilon_1(t)\leq C(1+t)^{-\frac{2\rho}{1-2\rho}}.$$

The above estimate together with (2.37) indicates that

$$\|\nabla\theta^c\|+\|\nabla\cdot\mathbf{q}^c\|+\|\chi_t^c\|+\|A(\chi^c-\chi_\infty)\|\leq C(1+t)^{-\frac{2\rho}{1-2\rho}}. \qquad (2.39)$$

The equation for $\mathbf{q}^c$ can be written in the strong form such that

$$\sigma\mathbf{q}_t^c+\mathbf{q}^c=-\nabla\theta^c,$$

and consequently

$$\sigma(\nabla\times\mathbf{q}^c)_t+\nabla\times\mathbf{q}^c=0.$$

Besides, the initial data $\nabla\times\mathbf{q}^c(0)=\mathbf{0}$. As a result, $\nabla\times\mathbf{q}^c\equiv\mathbf{0}$ for all time. It this case, $\|\nabla\cdot\mathbf{q}^c\|+\|\mathbf{q}^c\|$ defines an equivalent norm on $\mathbf{V}_0$. Combining (2.39), Theorem 2.1 and (2.27), we can conclude (2.29). The proof of Theorem 2.2 is complete. □

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