一类分数微分系统解的存在唯一性*

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摘要: 近期,一些学者通过常微分方程的研究方法和技巧研究了分数阶微分方程并且获得了相当不错的结果. 本文将常微分方程解的单调迭代法和*Nagumo*型条件引入到分数微分方程中来, 证明了一类分数微分系统解的存在唯一性. 另外, 通过分数阶微分不等式等手段推广了 *V.Lakshmikantham*等人的结果.

关 键 词: 分数微分方程, 单调迭代法, *Nagumo*型条件, 存在性, 唯一性.

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1 引言

在论述分数微分方程之前, 先说明分数微积分是必要的. 所谓分数微分或积分, 不是指一个分数或者一个分式函数的微分或积分, 而是指微分的阶数及积分的次数不是整数, 它可以是任意实数,乃至是复数. 仅仅由于习惯的原因才坚持这个名称[1]. 由于分数微分、积分有多种定义格式, 为明确起见,本文除非特别指明, 都采用$Riemann-Liouville$意义下的分数积分和微分. 我们可以从多次积分、积分变换、广义函数、常微分方程, 以及类似经典积分微分作为"和"与"差"的极限等各种途径来定义 $Riemann-Liouville$分数积分与微分[2–4]. 设 $\nu \in (0,1), a, b \in \mathbf{R}, a < t < b$, $f(t)$在(a,b)上连续, 定义 $Riemann-Liouville$分数积分与微分为

$$_aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\xi)^{\nu-1} f(\xi) d\xi,$$

其中 $\Gamma(\nu)$是第二类 $Euler$积分—— $Gamma$ 函数, 算子 $_aD_t^{-\nu}$ 中, 下标 a, t表示积分限, 上标 $-\nu$表示ν次积分, 这意味着 $\nu(>0)$将表示导数, 下面将 $Riemann-Liouville$意义下的分数导数简记为 $D^\nu, \nu \in (0,1)$.

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最近, 一些学者运用一些常微分方程中的技巧来研究分数微分方程, 并取得了相当好的结果[5–11], 实际上每个方法都需要不同的条件以及适用的环境, 而其中分数阶微分不等式的运用在解决问题时起到了很大的应用. 现在, 分数微分方程的计算和应用已经渗透到各种领域中[3, 12–19].

本文研究了一类分数微分系统的解的存在唯一性, 其中引入了常微分方程中解的单调迭代方法和 *Nagumo*型条件, 并通过一些不等式技巧等获得了解的存在唯一性.

2 主要结果

考虑如下形式的分数微分方程微分系统:

$$(P)\begin{cases} D^p(x) = f(t,y), x(t_0) = x^0 = x(t)(t-t_0)^{1-p}|_{t-t_0}; & (2.1) \\ D^q(x) = g(t,x), y(t_0) = y^0 = y(t)(t-t_0)^{1-q}|_{t-t_0}. & (2.1') \end{cases}$$

其中 $f \in C(\mathbf{R}_1, \mathbf{R})$, $g \in C(\mathbf{R}_2, \mathbf{R})$, $\mathbf{R}_1 = \{(t,y) : t_0 \le t \le t_0 + a, |y - y^0(t)| \le c\}$, $\mathbf{R}_2 = \{(t,x) : t_0 \le t \le t_0 + a, |x - x^0(t)| \le b\}$. 并且,

$$x^0(t) = \frac{x^0(t-t_0)^{p-1}}{\Gamma(p)},\ y^0(t) = \frac{y^0(t-t_0)^{q-1}}{\Gamma(q)}. \tag{2.2}$$

另外, D^p,D^q分别表示 p和 q阶的 *Riemann – Liouville*型分数阶微分, 且 $0 < p, q < 1$.

初值问题 (P)等价于如下的 *Volterra*分数积分方程:

$$\begin{cases} x(t) = x^0(t) + \dfrac{1}{\Gamma(p)} \displaystyle\int_{t_0}^{t} (t-s)^{p-1} f(s, y(s))ds; \\ y(t) = y^0(t) + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))ds. \end{cases} \tag{2.3}$$

其中 $x^0(t), y^0(t)$如 (2.2)所述, 且方程 (P)的解就是方程 (2.3)的解.

定义2.1 如果函数 $x(t) \in C_m([t_0, t_0 + a], \mathbf{R_1}) = \{u : u \in C((t_0, t_0 + a]),$ 且$(t-t_0)^m u(t) \in C([t_0, t_0 + a])\}$, 其中 $m = 1 - p$; 函数 $y(t) \in C_n([t_0, t_0 + a], \mathbf{R_2}) = \{u : u \in C((t_0, t_0 + a]),$ 且 $(t-t_0)^n u(t) \in C([t_0, t_0 + a])\}$, 其中 $n = 1 - q$. 若 $D^p x(t)$,$D^q y(t)$存在且在 $[t_0, t_0 + a]$连续, 并同时满足方程组 (P). 则称函数对 $(x(t), y(t))$是初值问题 (P)的一个解.

现在我们引入 *Nagumo*型条件来获得初值问题 (P)的解的存在唯一性.

定理2.1 假设初值问题 (P)中 f, g满足如下的 *Nagumo*型条件:

$$|f(t,x) - f(t,y)| \le \frac{(x-y)p\Gamma(p)}{(t-t_0)^p}, t \ne t_0, (t,x) \in \mathbf{R_1}, \tag{2.4}$$

$$|g(t,x) - g(t,y)| \le \frac{(x-y)q\Gamma(q)}{(t-t_0)^q}, t \ne t_0, (t,x) \in \mathbf{R_2}. \tag{2.5}$$

则初值问题 (P)的解的迭代序列可以表示为:

$$\begin{cases} x_{n+1}(t) = x^0(t) + \dfrac{1}{\Gamma(p)}\displaystyle\int_{t_0}^{t}(t-s)^{p-1}f(s,y_n(s))ds; & (2.6) \\ y_{n+1}(t) = y^0(t) + \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_0}^{t}(t-s)^{q-1}g(s,x_n(s))ds. & (2.6') \end{cases}$$

并且上述 $(x_{n+1}(t), y_{n+1}(t))$将在 $[t_0, t_0+\eta]$收敛到初值问题 (P)的一个解，其中 $\eta = \min\{a, (\frac{c\Gamma(1+p)}{M})^{\frac{1}{p}}, (\frac{b\Gamma(1+q)}{N})^{\frac{1}{q}}\}$, $|D^p x_n(t)| = |f(t, y_{n-1}(t))| \le M, |D^q y_n(t)| = |g(t, x_{n-1}(t))| \le N.$

证明 设 $(x_1(t), y_1(t))$,$(x_2(t), y_2(t))$是初值问题 (P)的任意两个解. 令 $\phi(t) = (\phi_x(t), \phi_y(t)) = (x_1(t)-x_2(t), y_1(t)-y_2(t))$, 并且可以看出:

$$\begin{aligned}\lim_{t\to t_0^+}\frac{\phi_x(t)}{t-t_0} &= \lim_{t\to t_0^+}\frac{1}{t-t_0}\frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}[f(s,y_1(s))-f(s,y_2(s))]ds \\ &= -\frac{1}{\Gamma(p)}(t-t_0)^{p-1}[f(t_0,y_1(t_0))-f(t_0,y_2(t_0))] = 0\end{aligned}$$

$$\begin{aligned}\lim_{t\to t_0^+}\frac{\phi_y(t)}{t-t_0} &= \lim_{t\to t_0^+}\frac{1}{t-t_0}\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}[g(s,x_1(s))-g(s,x_2(s))]ds \\ &= -\frac{1}{\Gamma(q)}(t-t_0)^{q-1}[g(t_0,x_1(t_0))-g(t_0,x_2(t_0))] = 0\end{aligned}$$

并且:

$$\begin{aligned}\lim_{t\to t_0^+}\frac{\phi_x(t)}{(t-t_0)^p} &= \lim_{t\to t_0^+}\frac{\phi_x(t)}{t-t_0}(t-t_0)^{1-p} = 0, \\ \lim_{t\to t_0^+}\frac{\phi_y(t)}{(t-t_0)^q} &= \lim_{t\to t_0^+}\frac{\phi_y(t)}{t-t_0}(t-t_0)^{1-q} = 0\end{aligned}$$

运用 *Nagumo*型条件 (2.4), (2.5)可得:

$$\phi_x(t) \le \frac{1}{\Gamma(p)}\int_{t_0}^{t}\frac{p(t-s)^{p-1}|\phi_x(s)|ds\Gamma(p)}{(s-t_0)^p}, \tag{2.7}$$

$$\phi_y(t) \le \frac{1}{\Gamma(q)}\int_{t_0}^{t}\frac{q(t-s)^{q-1}|\phi_y(s)|ds\Gamma(q)}{(s-t_0)^q}. \tag{2.7'}$$

令: $\frac{\phi_x(t)}{(t-t_0)^p} = \psi_x(t)$, $\frac{\phi_y(t)}{(t-t_0)^q} = \psi_y(t)$,易知: $\psi_x(t_0) = 0$,$\psi_y(t_0) = 0$.
则: (2.7), (2.7′)可写为:

$$\begin{aligned}\phi_x(t) &\le \tfrac{p}{(t-t_0)^p}\textstyle\int_{t_0}^{t}(t-s)^{p-1}\psi_x(s)ds; \\ \phi_y(t) &\le \tfrac{q}{(t-t_0)^q}\textstyle\int_{t_0}^{t}(t-s)^{q-1}\psi_y(s)ds.\end{aligned}$$

为了证明解的唯一性, 只需证 $\psi_x(t) = \psi_y(t) \equiv 0$即可. 我们采用反证法. 这里我们只对 $\psi_x(t) \equiv 0$运用反证法, 对于 $\psi_y(t) \equiv 0$可同理证得.

如果 $\psi_x(t) \neq 0$,令$m = \max_{[t_0,t_0+\eta]} \psi_x(t) = \psi_x(t_1)$, $t_1 \in (t_0, t_0+\eta)$. 则:

$$m = \psi_x(t_1) \leq \frac{p}{(t_1-t_0)^p}\int_{t_0}^{t_1}(t-s)^{p-1}\psi_x(s)ds$$
$$< \frac{pm}{(t_1-t_0)^p}\frac{(t_1-s)^p}{-p}\Big|_{t_0}^{t_1} = \frac{pm(t_1-t_0)^p}{(t_1-t_0)^p p} = m$$

可见矛盾, 这就说明了 $\psi_x(t) \equiv 0$. 同理可证 $\psi_y(t) \equiv 0$.

由于 $\frac{\phi_x(t)}{(t-t_0)^p} = \psi_x(t)$, $\frac{\phi_y(t)}{(t-t_0)^q} = \psi_y(t)$,知: $\phi_x(t) = \phi_y(t) \equiv 0$. 此即证明了初值问题 (P)解的唯一性.

下面我们给出迭代序列 $\{(x_{n+1}(t), y_{n+1}(t))\}(n = 0, 1, 2, \cdots)$的几条性质:

(1) 在 $[t_0, t_0+\eta]$上均有定义且连续;

(2) 在 $[t_0, t_0+\eta]$上一致有界;

(3) 在 $[t_0, t_0+\eta]$上等度连续, 其中 $\eta = \min\{a, (\frac{c\Gamma(1+p)}{M})^{\frac{1}{p}}, (\frac{b\Gamma(1+q)}{N})^{\frac{1}{q}}\}$, 在 $\mathbf{R_1}$上 $|f(t, x(t))| \leq M$, 在 $\mathbf{R_2}$上 $|g(t, x(t))| \leq N$.

通过 (2.6), (2.6′), 我们有:

$$|x_{n+1}(t) - x^0(t)| \leq \frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}|f(s, y_n(s))|ds;$$
$$|y_{n+1}(t) - y^0(t)| \leq \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|g(s, x_n(s))|ds.$$

当 $n = 0$时我们易得:

$$|x_1(t) - x^0(t)| \leq \frac{M}{\Gamma(p)}\frac{(t-s)^p}{-p}\Big|_{t_0}^{t} = \frac{M(t-t_0)^p}{p\Gamma(p)} \leq \frac{Ma^p}{\Gamma(1+p)} \leq c$$
$$|y_1(t) - y^0(t)| \leq \frac{N}{\Gamma(q)}\frac{(t-s)^q}{-q}\Big|_{t_0}^{t} = \frac{N(t-t_0)^q}{q\Gamma(q)} \leq \frac{Na^q}{\Gamma(1+q)} \leq b$$

通过归纳演绎法知迭代序列在 $[t_0, t_0+\eta]$上是一致有界的.

通过文献[20]引理 2.3.2可知迭代序列 $\{(x_{n+1}(t), y_{n+1}(t))\}$在 $[t_0, t_0+\eta]$上是等度连续的. 则存在一个一致收敛子列 $\{(x_{n_k}(t), y_{n_k}(t))\}$. 假设 $\{x_n(t) - x_{n-1}(t)\} \to 0$, $\{y_n(t) - y_{n-1}(t)\} \to 0$ $(n \to \infty)$. 那么通过 (2.6), (2.6′)知: 任何这样的子列的极限都是方程组 (P)的唯一解 $(x(t), y(t))$, 且它与子列的选取无关, 同时 $\{(x_{n+1}(t), y_{n+1}(t))\}$一致收敛到 $x(t)$[21]. 因此为了证明定理 2.1, 我们只需证明:

$$\phi_x(t) = \lim_{n\to\infty}\sup(|x_{n+1}(t) - x_n(t)|) \equiv 0; \tag{2.8}$$
$$\phi_y(t) = \lim_{n\to\infty}\sup(|y_{n+1}(t) - y_n(t)|) \equiv 0. \tag{2.9}$$

对于 $t_0 \le t_1 \le t_2$,有:

$$\begin{aligned}
&|x_n(t_1) - x_{n-1}(t_1)| - |x_n(t_2) - x_{n-1}(t_2)| \le |x_n(t_1) - x_{n-1}(t) - x_n(t_2) + x_{n-1}(t_2)| \\
\le\ & \frac{1}{\Gamma(p)}\Big[\int_{t_0}^{t_1}(t_1-s)^{p-1}D(s)ds - \int_{t_0}^{t_2}(t_2-s)^{p-1}D(s)ds\Big] \\
=\ & \frac{1}{\Gamma(p)}\Big[\int_{t_0}^{t_1}((t_1-s)^{p-1}-(t_2-s)^{p-1})D(s)ds - \int_{t_1}^{t_2}(t_2-s)^{p-1}D(s)ds\Big] \\
\le\ & \frac{2M}{\Gamma(p)}\Big[\int_{t_0}^{t_1}((t_1-s)^{p-1}-(t_2-s)^{p-1})ds - \int_{t_1}^{t_2}(t_2-s)^{p-1}ds\Big] \\
=\ & \frac{2M}{p\Gamma(p)}[(t_1-t_0)^p - (t_2-t_0)^p + 2(t_2-t_1)^p] \\
\le\ & \frac{2M}{p\Gamma(p)}\cdot 2(t_2-t_1)^p = \frac{4M}{\Gamma(1+p)}(t_2-t_1)^p.
\end{aligned} \tag{2.10}$$

其中在 $\mathbf{R_1}$上 $|D(s)| = |f(s, y_{n-1}(s)) - f(s, y_{n-2}(s))| \le 2M$, 且 $x_n(t) = x^0(t) + \frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}f(s, y_{n-1}(s))ds$.

要使 (2.10)式 $< \varepsilon$, 只要 $|t_2-t_1| < \delta = (\frac{\varepsilon\Gamma(1+p)}{4M})^{\frac{1}{p}}$, 即 $\phi_x(t_1) \le \phi_x(t_2) + \frac{\Gamma(1+p)}{4M}(t_2-t_1)^p + \varepsilon$. 对于充分大的 n, 如果 $\varepsilon > 0$, 且 t_1, t_2可交换, 则对于任意的 $\varepsilon > 0$, 我们有:

$$|\phi_x(t_1) - \phi_x(t_2)| \le \frac{\Gamma(1+p)}{4M}(t_2-t_1)^p.$$

这就证明了 $\phi_x(t)$的连续性. 同理可证得 $\phi_y(t)$的连续性.

运用 *Nagumo*型条件 (2.4), (2.5)以及 (2.6), (2.6′), 有:

$$|x_{n+1}(t) - x_n(t)| \le p\int_{t_0}^{t}(t-s)^{p-1}\frac{|y_n(s) - y_{n-1}(s)|}{(s-t_0)^p}ds;$$
$$|y_{n+1}(t) - y_n(t)| \le q\int_{t_0}^{t}(t-s)^{q-1}\frac{|x_n(s) - x_{n-1}(s)|}{(s-t_0)^q}ds.$$

对于固定的 $t \in [t_0, t_0+\eta]$, 存在 $n_1, n_2, \cdots \in \mathbf{N}^+$, 使得当 $n = n_k \to \infty$时 $|x_{n+1}(t) - x_n(t)| \to \phi_x(t)$, 且$\phi_x^*(t) = \lim_{n=n_k\to\infty}|x_n(t) - x_{n-1}(t)|$在 $[t_0, t_0+\eta]$上一致存在.

因此, $\phi_x(t) \le p\int_{t_0}^{t}(t-s)^{p-1}\frac{\phi_x^*(s)}{(s-t_0)^p}ds$.又由于 $\phi_x^*(s) \le \phi_x(s)$, 我们有: $\phi_x(t) \le p\int_{t_0}^{t}(t-s)^{p-1}\frac{\phi_x(s)}{(s-t_0)^p}ds$.由前面证明知, 当 $t \to t_0^+$时有 $\frac{\phi_x(t)}{t-t_0} \to 0$, $\frac{\phi_x(t)}{(t-t_0)^p} \to 0$.

现令: $\psi_x(t) = \frac{\phi_x(t)}{(t-t_0)^p}$, 由前面知 $\psi_x(t_0) = 0$. 此时我们要证明 $\{x_{n+1}(t)\}$的收敛性, 只需证明 $\psi_x(t) \equiv 0$. 利用反证法.

如果 $\psi_x(t_0) \neq 0$, 令 $\psi_x(t_1) = \max_{[t_0, t_0+\eta]} \psi_x(t) = m$, 那么:

$$
\begin{aligned}
m = \psi_x(t_1) &= \frac{\phi_x(t_1)}{(t_1 - t_0)^p} \leq \frac{p}{(t - t_0)^p} \int_{t_0}^{t_1} (t_1 - s)^{p-1} \frac{\phi_x(s)}{(s - t_0)^p} ds \\
&< \frac{p}{(t_1 - t_0)^p} \int_{t_0}^{t_1} (t_1 - s)^{p-1} \psi_x(s) ds \\
&< \frac{pm}{(t_1 - t_0)^p} \int_{t_0}^{t_1} (t_1 - s)^{p-1} ds = \frac{pm}{(t_1 - t_0)^p} \frac{(t_1 - t_0)^p}{p} = m.
\end{aligned}
$$

由此可得矛盾. 可见 $\psi_x(t) \equiv 0$, 即 $\{x_{n+1}(t)\}$收敛. 同理可证得 $\{y_{n+1}(t)\}$收敛.

所以, (2.6), (2.6′)收敛到初值问题 (P)的唯一解.

注 当 $p = q = 1$时，定理 2.1 中的 *Nagumo*型条件就减弱为常微分方程中的 *Nagumo*条件.

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On the Existence and Uniqueness of Solutions to a Class of Fractional Differential System

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Abstract: Recently, some researchers obtained some good results of the fractional differential equations by the methods and techniques of the ordinary differential equations (ODE). This paper applied the monotone iteration method and the *Nagumo*-type condition of ODE into the fractional differential equations ,and also proved the the existence and uniqueness of solution to a couple system of fractional differential equations. In addition, through the fractional differential inequalities and other methods we obtained the results of this paper which generalizes the results of *V.Lakshmikantham* and others.

Keywords: fractional differential system, monotone iteration method, *Nagumo*-type condition, existence, uniqueness.