Oscillation and non-oscillation of second-order half-linear differential equations[1]

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Abstract

In this paper, we consider the oscillation and non-oscillation of second order half-linear differential equation. By using some new technique, we establish new oscillation and non-oscillation criteria which extend and improve some known results of second order linear differential equation in the references.

Keywords: Half-linear differential equations, oscillation, non-oscillation.

1. Introduction

Consider the second order half-linear differential equation

$$(|u'(t)|^{\alpha-1}u'(t))' + p(t)|u(t)|^{\alpha-1}u(t) = 0, \tag{1}$$

where $\alpha > 0$ is a constant, $p \in C([0, +\infty), [0, +\infty))$ is an integrable function.

During the last three decades, investigation of oscillation and non-oscillation of second order half-linear differential equations has been attracting attention of numerous researchers. The reader is referred to the monographs by Agarwal, Grace and O'Regan [1,2], Dosly and Rehak [3], papers[5-17] and references therein.

By a solution of (1) is meant a function $u \in C^1[T_u, \infty), T_u \geq 0$, which has the property $|u'|^{\alpha-1}u' \in C^1[T_u, \infty)$ and satisfies the equation for all $t \geq T_u$. We consider only those solutions $u(t)$ of (1) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq T_u$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be non-oscillatory.

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The purpose of this paper is to establish new oscillation and non-oscillation criteria of Eq.(1) which extend and improve some criteria of linear differential equation in the references.

We can easily show that if for some $\lambda < \alpha$ the integral $\int^{+\infty} s^\lambda p(s)ds$ diverges, then equation (1) is oscillatory. Therefore, we shall always assume below that

$$\int^{+\infty} s^\lambda p(s)ds < +\infty \quad \text{for } \lambda < \alpha.$$

2. Main results

Introduce the notations

$$h_\lambda(t) = t^{\alpha-\lambda} \int_t^{\infty} s^\lambda p(s)ds \quad \text{for } t > 0 \quad \text{and} \quad \lambda < \alpha.$$

$$h_\lambda(t) = t^{\alpha-\lambda} \int_1^{t} s^\lambda p(s)ds \quad \text{for} \quad t > 0, \quad \text{and} \quad \lambda > \alpha \tag{2}$$

$$p_*(\lambda) = \liminf_{t\to+\infty} h_\lambda(t), \quad p^*(\lambda) = \limsup_{t\to+\infty} h_\lambda(t).$$

The following lemmas will be useful for establishing oscillation criteria for Eq.(1). The first one is a well-known inequality which is due to Hardy et al.[4].

Lemma 1. [4] If X and Y are nonnegative, then

$$X^q + (q-1)Y^q \geq qXY^{q-1}, \quad \text{for} \quad q > 1, \tag{3}$$

where the equality holds if and only if $X = Y$.

Lemma 2. Let equation (1) be non-oscillatory. Then there exits $t_0 > 0$ such that the equation

$$\rho' + p(t) + \alpha\rho^{1+1/\alpha} = 0 \tag{4}$$

has a solution $\rho: \ [t_0, +\infty) \to [0, +\infty)$; moreover,

$$\rho(t_0+) = +\infty, \quad (t - t_0)(\rho(t))^{\frac{1}{\alpha}} < 1 \quad \text{for} \quad t_0 < t < +\infty \tag{5}$$

$$\lim_{t\to+\infty} t^\lambda(\rho(t))^{\frac{1}{\alpha}} = 0, \quad \text{for} \quad \lambda < 1 \tag{6}$$

and

$$\liminf_{t\to+\infty} t^\alpha\rho(t) \geq A, \qquad \limsup_{t\to+\infty} t^\alpha\rho(t) \leq B, \tag{7}$$

where

$$A = \min\{r | p_*(0) - r + r^{1+1/\alpha} \leq 0\}, \ B = \max\{R | p_*(1+\alpha) - \alpha R + \alpha R^{1+1/\alpha} \leq 0\}. \tag{8}$$

Proof. Since equation (1) is non-oscillatory, there exists $t_0 > 0$ such that the solution $u(t)$ of equation (1) under the initial conditions $u(t_0) = 0,\ u'(t_0) = 1$ satisfies the inequalities

$$u(t) > 0,\ u'(t) \geq 0 \ \text{ for } \ t_0 < t < +\infty.$$

Clearly, the function $\rho(t) = (u'(t)/u(t))^\alpha$ for $t_0 < t < +\infty$ is the solution of equation (4), and $\lim_{t\to t_0+} \rho(t) = +\infty$. From (4) we have

$$\frac{-\rho'(t)}{\alpha(\rho(t))^{1+1/\alpha}} > 1 \ \text{ for } \ t_0 < t < +\infty.$$

Integrating the above inequality from t_0 to t, we obtain $(t - t_0)(\rho(t))^{1/\alpha} < 1$ for $t_0 < t < +\infty$. In particular, equality (6) holds for any $\lambda < 1$.

We now show that inequalities (7) are valid. Assume $p_*(0) \neq 0$ and $p_*(1 + \alpha) \neq 0$ (inequalities (7) are trivial, otherwise). We introduce the notation

$$r = \liminf_{t\to+\infty} t^\alpha \rho(t), \quad R = \limsup_{t\to+\infty} t^\alpha \rho(t)$$

From (4) we easily find that for any $t_1 > t_0$

$$t^\alpha \rho(t) = t^\alpha \int_t^{+\infty} p(t)ds + t^\alpha \int_t^{\infty} (\rho(t))^{1+\frac{1}{\alpha}} ds \tag{9}$$

$$t^\alpha \rho(t) = \frac{t_1^{1+\alpha}\rho(t_1)}{t} - \frac{1}{t}\int_{t_1}^t p(s)s^{1+\frac{1}{\alpha}}ds + \frac{1}{t}\int_{t_1}^t s^\alpha \rho(s)[1 + \alpha - \alpha s(\rho(s))^{\frac{1}{\alpha}}]ds \tag{10}$$

$$\text{for } \ t_1 < t < +\infty.$$

Using Lemma 1 with $X = 1, Y = s(\rho(s))^{\frac{1}{\alpha}}$, we have that

$$(1 + \alpha)s^\alpha \rho(s) - \alpha s^{1+\alpha}(\rho(s))^{1+1/\alpha} \leq 1.$$

Hence, for $t_1 < t < +\infty$,

$$t^\alpha \rho(t) \leq \frac{t_1^{1+\alpha}\rho(t_1)}{t} - \frac{1}{t}\int_{t_1}^t p(s)s^{1+\frac{1}{\alpha}}ds + \frac{t - t_1}{t}.$$

Therefore, (9) and (10) imply that $r \geq p_*(0)$ and $R \leq 1 - p_*(1 + \alpha)$ respectively.

It is easily seen that for any $0 < \varepsilon < \min\{r, 1 - R\}$ there exists $t_\varepsilon > t_1$ such that for $t_\varepsilon < t < +\infty$,

$$r - \varepsilon < t^\alpha \rho(t) < R + \varepsilon,$$

$$t^\alpha \int_t^{+\infty} p(s)ds > p_*(0) - \varepsilon,$$

and

$$\frac{1}{t}\int_{t_1}^t s^{1+\alpha} p(s)ds > p_*(1 + \alpha) - \varepsilon.$$

Taking into account the above argument, from (9) and (10) we have that for $t_\varepsilon < t < +\infty$,

$$t^\alpha p(t) > p_*(0) - \varepsilon + (r-\varepsilon)^{1+\frac{1}{\alpha}},$$

$$t^\alpha \rho(t) < \frac{t_\varepsilon^{1+\alpha}\rho(t_\varepsilon)}{t} - p_*(1+\rho) + \varepsilon[(1+\alpha) - \alpha(R+\varepsilon)^{\frac{1}{\alpha}}].$$

Hence

$$r \geq p_*(0) + r^{1+\frac{1}{\alpha}}, \quad R \leq -p_*(1+\alpha) + R[(1+\alpha) - \alpha R^{\frac{1}{\alpha}}],$$

that is, $r \geq A$ *and* $R \leq B$, where A and B are defined by equalities (8). Hence (7) holds.

For the completion of the picture we give a proposition, which is proved by Kusano, Naito and Ogata in [9].

Proposition. [9] If $p_*(0) > \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$, then every solution of equation (1) is oscillatory.

Theorem 1. If $p_*(1+\alpha) > (\frac{\alpha}{\alpha+1})^{\alpha+1}$, then every solution of equation (1) is oscillatory.

Proof. Assume that equation (1) is non-oscillatory. From the proof of Lemma 1, we have

$$p_*(1+\alpha) \leq \alpha R - \alpha R^{1+1/\alpha} \leq (\frac{\alpha}{\alpha+1})^{\alpha+1}$$

which contradicts the condition of Theorem 1 and so the proof is complete.

Theorem 2. Assume that $p_*(0) \leq \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$. If for some $\lambda < \alpha$

$$p^*(\lambda) > \frac{\lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\alpha-\lambda)} + B, \tag{11}$$

then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) be non-oscillatory. Then, according to Lemma 2, equation (4) has a solution $\rho : [t_0, +\infty) \to [0, +\infty)$, satisfying condition (5)-(7). Suppose $\lambda < \alpha$. Because of (7) we have that for any $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$t^\alpha \rho(t) < B + \varepsilon, \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Multiplying equality (4) by t^λ, integrating it from t to $+\infty$, and taking into account (5)-(7), we get

$$\begin{aligned}\int_t^{+\infty} s^\lambda p(s)ds &= -\int_t^{+\infty} s^\lambda \rho'(s)ds - \int_t^{+\infty} s^\lambda (\rho(s))^{1+\frac{1}{\alpha}}ds \\ &= t^\lambda \rho(t) + \int_t^{+\infty} \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1}ds \\ &- \int_t^{+\infty} \left(\alpha s^\lambda (\rho(s))^{1+\frac{1}{\alpha}} + \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} - \lambda s^{\lambda-1}\rho(s)\right)ds.\end{aligned}$$

Using Lemma 1 with

$$X = \frac{\lambda s^{\frac{\lambda}{1+\alpha}-1}}{1+\alpha}, \quad Y = s^{\frac{\lambda}{1+\alpha}}(\rho(s))^{\frac{1}{\alpha}},$$

we have that

$$\alpha s^{\lambda}(\rho(s))^{1+\frac{1}{\alpha}} + \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} - \lambda s^{\lambda-1}\rho(s) \geq 0.$$

Hence,

$$\begin{aligned}\int_t^{+\infty} s^{\lambda} p(s) ds &< t^{\lambda-\alpha}\left(t^{\alpha}\rho(t) + t^{\alpha-\lambda}\int_t^{+\infty} \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} ds\right) \\ &< t^{\lambda-\alpha}\left(B + \varepsilon + \frac{\lambda^{\alpha+1}}{(\alpha-\lambda)(\alpha+1)^{\alpha+1}}\right),\end{aligned}$$

hence we have $p^*(\lambda) \leq \frac{\lambda^{\alpha+1}}{(\alpha-\lambda)(\alpha+1)^{\alpha+1}} + B$, which contradicts equality (11). The proof is complete.

Theorem 3. Assume that $p_*(1+\alpha) \leq (\frac{\alpha}{\alpha+1})^{\alpha+1}$. If for some $\lambda > \alpha$

$$p^*(\lambda) > \frac{\lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\lambda-\alpha)} - A, \tag{12}$$

then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) be non-oscillatory. Then, according to Lemma 2, equation (4) has a solution $\rho : (t_0, +\infty) \to [0, +\infty)$, satisfying condition (5) and (7). Suppose $\lambda > \alpha$. Because of (7) we have that for any $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$t^{\alpha}\rho(t) > A - \varepsilon, \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Multiplying equality (4) by t^{λ}, integrating it from t_ε to t, and taking into account (5) and (7), we get

$$\begin{aligned}\int_{t_\varepsilon}^{t} s^{\lambda} p(s) ds &< t^{\lambda-\alpha}\left(-t^{\alpha}\rho(t) + t^{\alpha-\lambda} t_\varepsilon^{\lambda}\rho(t_\varepsilon) + t^{\alpha-\lambda}\int_{t_\varepsilon}^{t} \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} ds\right) \\ &< t^{\lambda-\alpha}\left(-A + \varepsilon + \frac{\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}} + \frac{t_\varepsilon^{\lambda}\rho(t_\varepsilon)}{t^{\lambda-\alpha}}\right)\end{aligned}$$

hence we have $p^*(\lambda) \leq \frac{\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}} - A$, which contradicts equality (12).

Corollary 1. Let either

$$\lim_{\lambda \to \alpha-} (\alpha - \lambda) p^*(\lambda) > \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \tag{13}$$

$$\lim_{\lambda \to \alpha+} (\lambda - \alpha) p^*(\lambda) > \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \tag{14}$$

Then equation (1) is oscillatory.

Corollary 2. For some $\lambda \neq \alpha$ let

$$|\lambda - \alpha| p_*(\lambda) > \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \tag{15}$$

Then equation (1) is oscillatory.

To convince ourselves that corollary 1 is valid, let us note that (13)((14)) implies

$$\lim_{\lambda\to\alpha+}\left\{(\alpha-\lambda)p^*(\lambda)-\frac{\lambda^{\alpha+1}}{(1+\alpha)^{1+\alpha}}-(\alpha-\lambda)B\right\}>0$$

$$\left(\lim_{\lambda\to\alpha-}\left\{(\lambda-\alpha)p^*(\lambda)-\frac{\lambda^{\alpha+1}}{(1+\alpha)^{1+\alpha}}-(\lambda-\alpha)A\right\}>0\right).$$

Consequently, (11)((12)) is fulfilled for some $\lambda<\alpha$ $(\lambda>\alpha)$. Thus, according to Theorem 1 , equation (1) is oscillatory. As for Corollary 2, taking into account that the mapping $\lambda\ \mapsto(\alpha-\lambda)p_*$ for $\lambda<\alpha(\lambda\ \mapsto(\lambda-\alpha)p_*$ for $\lambda>\alpha)$ is non-decreasing(non-increasing), we easily find from (15) that (13)((14))is fulfilled for some λ.

Theorem 4. Assume that $p_*(0)\le\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$ and $p_*(1+\alpha)\le(\frac{\alpha}{\alpha+1})^{\alpha+1}$. Moreover, let either

$$p_*(0)>\frac{\lambda(1+\alpha-\lambda^{\frac{1}{\alpha}})}{(\alpha+1)^{\alpha+1}}\quad\text{and}\quad p_*(\lambda)>\frac{p_*(0)}{\alpha-\lambda}+B-A \tag{16}$$

for some $\lambda<\alpha$, or

$$p_*(1+\alpha)>\alpha\frac{\lambda(1+\alpha-\lambda^{\frac{1}{\alpha}})}{(\alpha+1)^{\alpha+1}}\quad\text{and}\quad p_*(\lambda)>\frac{p_*(1+\alpha)}{\lambda-\alpha}+B-A \tag{17}$$

for some $\lambda>\alpha$. Then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) be non-oscillatory. Then according to Lemma 2, equation (4) has the solution $\rho:(t_0,+\infty)\to[0,+\infty)$ satisfying conditions (5)-(7). Suppose $\lambda>\alpha$ $(\alpha>\lambda)$. By the conditions of the theorem, $p_*(0)>\frac{\lambda(1+\alpha-\lambda^{\frac{1}{\alpha}})}{(\alpha+1)^{\alpha+1}}$ $(p_*(1+\alpha)>\alpha\frac{\lambda(1+\alpha-\lambda^{\frac{1}{\alpha}})}{(\alpha+1)^{\alpha+1}})$, which implies that $A>\frac{\lambda}{(1+\alpha)^\alpha}$ $(B<\frac{\lambda}{(1+\alpha)^\alpha})$. On account of (7), for any $0<\varepsilon<A-\frac{\lambda}{(1+\alpha)^\alpha}$ $(0<\varepsilon<\frac{\lambda}{(1+\alpha)^\alpha}-B)$ there exists $t_\varepsilon>t_0$ such that

$$A-\varepsilon<t^\alpha\rho(t)<B+\varepsilon\qquad\text{for}\quad t_\varepsilon<t<+\infty$$

Multiplying equality (4) by t^λ, integrating it from t to $+\infty$ (from t_ε to t), and taking into account (5)-(7), we easily find that

$$\begin{aligned} t^{\alpha-\lambda}\int_t^{+\infty}s^\lambda p(s)ds &= t^\alpha\rho(t)+t^{\alpha-\lambda}\int_t^{+\infty}s^{\lambda-\alpha-1}s^\alpha\rho(s)(\lambda-(\rho(s))^{\frac{1}{\alpha}})ds\\ &< B+\varepsilon+\frac{1}{\alpha-\lambda}(A-\varepsilon)(\lambda-(A-\varepsilon)^{\frac{1}{\alpha}})\qquad\text{for } t_\varepsilon<t<+\infty\end{aligned}$$

$$\left(t^{\alpha-\lambda}\int_{t_\varepsilon}^{t}s^\lambda p(s)ds<\varepsilon-A+\frac{1}{\lambda-\alpha}(B+\varepsilon)(\lambda-(B+\varepsilon)^{\frac{1}{\alpha}})+t^{\alpha-\lambda}t_\varepsilon^\alpha\rho(t_\varepsilon)\ \text{ for } t_\varepsilon<t<+\infty\right)$$

This implies

$$p^*(\lambda)\le\frac{p_*(0)}{\alpha-\lambda}+B-A$$

$$\left(p^*(\lambda) \leq \frac{p_*(0)}{\lambda-\alpha} + B - A\right)$$

which contradicts condition (16)((17)).

Theorem 5. Assume that $0 < p_*(0) \leq \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$ and $p_*(1+\alpha) \leq (\frac{\alpha}{1+\alpha})^{1+\alpha}$. Moreover, for some $0 < \lambda < \alpha$ let $p_*(\lambda) < \frac{\alpha^\alpha - \lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}}$ and either

$$p_*(\lambda) > \frac{p_*(0)}{\alpha-\lambda} + \frac{\lambda}{\alpha-\lambda}(B-A)$$

and

$$p^*(\lambda) > p_*(\lambda) + B - r_1,$$

or

$$p_*(\lambda) < \frac{p_*(0)}{\alpha-\lambda} + \frac{\lambda}{\alpha-\lambda}(B-A)$$

and

$$p^*(\lambda) > \frac{p_*(0)}{\alpha-\lambda} + \frac{1}{\alpha-\lambda}(B-A), \tag{18}$$

where r_1 is the least root of the equation $\frac{1}{\alpha-\lambda}x^{1+\frac{1}{\alpha}} - x + p_*(\lambda) - \frac{\lambda}{\alpha-\lambda}B = 0$. Then equation (1) is oscillatory.

Theorem 5'. Assume that $p_*(0) \leq \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$ and $p_*(1+\alpha) \leq (\frac{\alpha}{1+\alpha})^{1+\alpha}$. Moreover, for some $0 < \lambda < \alpha$ let condition (18) be fulfilled, and let $p_*(0) > \frac{\alpha^\alpha - \lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}}$. Then equation (1) is oscillatory.

Proof of Theorems 5 and 5'. Assume the contrary. Let equation (1) be non-oscillatory. Then according to Lemma 2, equation (4) has the solution $\rho : (t_0, +\infty) \to [0, +\infty)$ satisfying conditions (5)-(7), Multiplying equation (4) by t^λ, integrating it from t to $+\infty$, and taking into account (6), we easily obtain

$$t^\alpha \rho(t) = h_\lambda(t) - \lambda t^{\alpha-\lambda} \int_t^{+\infty} s^{\lambda-1}\rho(s)ds + t^{\alpha-\lambda} \int_t^{+\infty} s^\lambda (\rho(s))^{1+\frac{1}{\alpha}} ds \quad \text{for } t_0 < t < +\infty \tag{19}$$

where $h(\lambda)$ is the function defined by (2).

Introduce the notation

$$r = \lim_{t\to+\infty} t^\alpha \rho(t)$$

On account of (7) we have $r > 0$. Therefore for any $0 < \varepsilon < \max\{r,\ p_*(\lambda)\}$ there exists $t_\varepsilon > t_0$ such that

$$r - \varepsilon < t^\alpha \rho(t) < B + \varepsilon,\ h_\lambda > p_*(\lambda) - \varepsilon \quad \text{for } t_\varepsilon < t < +\infty.$$

Owing to the above arguments, we find from (19) that

$$(\alpha-\lambda)h_\lambda < B - \varepsilon - (r-\varepsilon)^{1+\frac{1}{\alpha}} \quad \text{for } t_\varepsilon < t < +\infty.$$

$$t^{\alpha}\rho(t) > p_*(\lambda) - \varepsilon - \frac{\lambda}{\alpha-\lambda}(B+\varepsilon) + \frac{1}{\alpha-\lambda}(r-\varepsilon)^{1+\frac{1}{\alpha}} \quad \text{for } t_\varepsilon < t < +\infty$$

which implies

$$p^*(\lambda) \le \frac{B - r^{1+\frac{1}{\alpha}}}{\alpha-\lambda}, \qquad r \ge p_*(\lambda) - \frac{\lambda}{\alpha-\lambda}B + \frac{r^{1+\frac{1}{\alpha}}}{\alpha-\lambda} \tag{20}$$

The latter inequality results in $r \ge r_1$, where r_1 is the least root of the equation

$$\frac{1}{\alpha-\lambda}x^{1+\frac{1}{\alpha}} - x + p_*(\lambda) - \frac{\lambda}{\alpha-\lambda}B = 0$$

Thus $r \ge \max\{A, x_1\}$, From (20) we have that if $A < r_1$, then

$$p^*(\lambda) \le B + p_*(\lambda) - r_1, \tag{21}$$

but if $A \ge x_1$, then

$$p^*(\lambda) \le \frac{1}{\alpha-\lambda}B - \frac{1}{\alpha-\lambda}A^{1+\frac{1}{\alpha}},$$

which implies

$$p^*(\lambda) \le \frac{B}{\alpha-\lambda} - \frac{A}{\alpha-\lambda} + \frac{p_*(0)}{\alpha-\lambda} \tag{22}$$

inequalities (21), (22) contradicts the conditions of the theorem.

Corollary 3. Assume that $p_*(0) \le \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$, $p_*(1+\alpha) \le (\frac{\alpha}{1+\alpha})^{1+\alpha}$ and

$$p^*(0) > p_*(0) + B - A.$$

Then equation (1) is oscillatory.

Corollary 4. Assume that for some $\lambda \in [0, \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}})$,

$$\frac{\lambda}{\alpha-\lambda} < p_*(\lambda) < \frac{\alpha^\alpha}{(\alpha-\lambda)(\alpha+1)^{\alpha+1}}$$

and

$$p^* > 1 + p_*(\lambda) - r_1',$$

where r_1' is the least root of the equation $\frac{1}{\alpha-\lambda}x^{1+\frac{1}{\alpha}} - x + p_*(\lambda) - \frac{\lambda}{\alpha-\lambda} = 0$. Then equation (1) is oscillatory.

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