Limit-circle invariance of non-symmetric discrete Hamiltonian systems [1]

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Abstract

In this paper, we first give the related important Lemmas, and after discusses the non-symmetric discrete Hamiltonian system, and obtain the limit-circle invariance theorem. The main results contain corresponding contents of the symmetric discrete Hamiltonian system has been discussed in much paper.
Key words difference operator, deficiency index, non-symmetric, limit-circle

1 Introduction

Discrete Hamiltonian systems originated from the discretization of continuous Hamiltonian systems and from discrete process acting in accordance with the Hamiltonian principle. They play an important role in the practical applications. In spite of the similarity between the theories of continuous and discrete Hamiltonian systems, there are many differences. For example, sometimes some results on continuous Hamiltonian systems are dissimilar to related results on discrete Hamiltonian systems; some results of continuous and discrete Hamiltonian systems are similar, but the method of the proofs are dissimilar. So, study of discrete Hamiltonian systems is more difficult and challenging in some way. In especially, the deficient index of discrete Hamiltonian systems is great significance, which determines is generated by the system self-adjoint operators to add the required number of boundary conditions, allowing the expansion of the system if the corresponding difference operator is self-adjoint.

At present, for discrete Hamiltonian system studied by many scholars of its oscillation, Lyapunov inequality, non-conjugated nature deficit index (see [1-4]). There are so many differences among symmetric systems,non-symmetric systems, high dimension systems and scalar equations that the study of singular and non-symmetric systems is very difficult. Many problems need to be solved, such as the necessary and sufficient conditions of deficient index, the deficient index and spectrum problems of non-symmetric discrete Hamiltonian systems and so on. Now, most deficient index problems discussion of discrete Hamiltonian systems focused on the symmetric form, while the result for the deficiency index of non-symmetric form is much fewer(see [5-10]).

This paper studies on the limit-circle invariance of non-symmetric discrete Hamiltonian systems. Some related lemmas and the limit-circle invariance theorem are obtained. The rest parts are arranged as following: the second part is preliminary knowledge, the third part is the proof of limit-circle invariance theorem.

2 Preliminary knowledge

Consider the singular discrete Hamiltonian system

$$J\triangle y(t) = [\lambda W(t) + Q(t)]R(y(t)), \quad t \in \mathbb{N}, \tag{2.1}$$

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where λ is a complex parameter, $\mathbb{N}=\{0,1,2,\cdots\}$, $y(t)=\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ is a $2n\times 1$ matrix, $y_1(t), y_2(t)\in\mathbb{C}^n$ for $t\in\mathbb{N}$. $\Delta y(t)$ is the forward difference operator defined by

$$\triangle y(t)=y(t+1)-y(t), \tag{2.2}$$

and J is the canonical symplectic matrix, i.e., $J=\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with I_n the identity matrix, $W^*(t)=W(t)\geq 0$ and $Q(t)$ are $2n\times 2n$ complex-valued matrices which be named with weighted function and potential function, respectively. Here W^* is the complex conjugate transpose of W and inequalities of Hermitian matrices are in the positive, non-negative sense. The discrete Hamiltonian system (2.1) called non-symmetric if $Q^*(t)\neq Q(t)$, otherwise known as symmetric discrete Hamiltonian system. Assume that the weighted function $W(t)$ is of the block diagonal form

$$W(t)=\mathrm{diag}\{W_1(t), W_2(t)\}, \tag{2.3}$$

and satisfies the following definiteness condition, where $W_j(t)$ is a $n\times n$ nonnegative Hermitian matrix, $j=1,2$.

(I) There exists $n_0\in\mathbb{N}$ such that for all nontrivial solutions $y(t)$ of (2.1), the following inequality always holds

$$\sum_{t=0}^{n} R(y)^*(t)W(t)R(y)(t)>0, \quad n\geq n_0. \tag{2.4}$$

Remark 2.1 *The definiteness assumption for the weighted function in the continuous case was first proposed by Atkinson(see* $[1, P_{253}]$*), and was used by Hinton, Shaw and Shi(see* $[10, P_{455}]$*).*

Let $Q(t)$ be blocked as

$$Q(t)=\begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}, \tag{2.5}$$

where $A(t), B(t), C(t)$ are $n\times n$ complex matrices. To ensure the existence, uniqueness of the solution of any initial value problem for (2.1), we always assume that

(II) $I_n - A(t)$ is invertible in $\mathbb{N}$.

Remark 2.2 *From* (2.3), (2.5), *system* (2.1) *can be rewritten into the linear discrete Hamiltonian system as*

$$\begin{cases} \Delta y_1(t)=A(t)y_1(t+1)+[B(t)+\lambda W_2(t)]y_2(t), \\ \Delta y_2(t)=[C(t)-\lambda W_1(t)]y_1(t+1)-A^*(t)y_2(t), \ t\in\mathbb{N}. \end{cases} \tag{2.6}$$

In order to show the main results of this paper clearly, we first give the related definitions and symbols. The sequential apace

$$l[0,\infty):=\{y: y=\{y(t)\}_{t=0}^{\infty}\subset\mathbb{C}^{2n}\} \tag{2.7}$$

and let

$$L_W^2[0,\infty):=\{y\in l[0,\infty): \sum_{t=0}^{\infty} R(y)^*(t)W(t)R(y)(t)<\infty\} \tag{2.8}$$

with inner product

$$\langle y,z\rangle:=\sum_{t=0}^{\infty} R(z)^*(t)W(t)R(y)(t), \tag{2.9}$$

where the weighted function $W(t)$ is a nonnegative Hermitian matrix. Let

$$L_W^2[0,n+1]:=\{y: y\in l[0,n+1]\} \tag{2.10}$$

with inner product $\langle\cdot,\cdot\rangle_n$,

$$\langle y,z\rangle_n:=\sum_{t=0}^{n} R(z)^*(t)W(t)R(y)(t). \tag{2.11}$$

Denote $\|y\|_W = (\langle y, y\rangle)^{\frac{1}{2}}$ for $y \in L^2_W[0,\infty)$ and $\|y\|_n = (\langle y, y\rangle_n)^{\frac{1}{2}}$ for $y \in L^2_W[0, n+1]$. Since $W(t)$ may be singular, the inner products for $L^2_W[0,\infty)$ and $L^2_W[0, n+1]$ may not be positive, so we have to introduce the following quotient spaces. For $y, z \in L^2_W[0,\infty)$, y is said to be equal z if $\|y - z\|_W = 0$. In this sense, $L^2_W[0,\infty)$ is an inner product space with the inner product $\langle\cdot,\cdot\rangle$. Similarly, for $y, z \in L^2_W[0, n+1]$, y is said to be equal to z if $\|y - z\| = 0$ and thus $L^2_W[0, n+1]$ is an inner product space with the inner product $\langle\cdot,\cdot\rangle_n$.

For convenience, introduce the following natural operator corresponding to the non-symmetric discrete Hamiltonian (2.1):

$$Ly(t) := J\Delta y(t) - Q(t)R(y)(t) = \lambda W(t)R(y)(t), \quad t \in \mathbb{N} \tag{2.12}$$

for $y \in l[0, n+1] := \{y = \{y(t)\}_{t=0}^{n+1} \subset \mathbb{C}^{2n}\}$.

Lemma 2.1 (Shi, lemma2.5[10]) *$L^2_W[0,\infty)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $L^2_W[0, n+1]$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_n$. Furthermore, $\dim L^2_W[0, n+1] = \sum_{t=0}^{n}\mathrm{rank}W(t)$.*

Remark 2.3 *From the proof of lemma 2.1, we see that the potential function not be used during the proofing process, so the conclusion of lemma 2.1 suit to the non-symmetric discrete Hamiltonian.*

3 Main results

In order to prove the main results of this paper, we give some lemmas firstly. By (II) and (2.6), we get

$$y(t+1) = S(t,\lambda)y(t), \tag{3.1}$$

where $E(t) := (I_n - A(t))^{-1}$,

$$S(t,\lambda) = \begin{pmatrix} E & E(B + \lambda W_2) \\ (C - \lambda W_1)E & I_n - A^* + (C - \lambda W_1)E(B + \lambda W_2) \end{pmatrix}, \tag{3.2}$$

and all the matrix-valued functions on the right-hand side of (3.2) are evaluated at t. In addition, it can be directly derived from the first relation in (2.6) that every solution $y(\cdot,\lambda)$ of non-symmetric discrete Hamiltonian (2.1) satisfies

$$R(y)(t,\lambda) = \begin{pmatrix} E(t) & E(t)(B(t) + \lambda W_2(t)) \\ 0 & I_n \end{pmatrix} y(t,\lambda), \tag{3.3}$$

which will be repeatedly used in the sequel.

Lemma 3.1 (Liouville'sFormula) *Assume* (II) *holds. Let $\Psi(\cdot,\lambda)$ be a fundamental solution matrix of non-symmetric discrete Hamiltonian* (2.1)*. Then, for all $t \geq 0$,*

$$\begin{aligned} &\det\Psi(t+1,\lambda) = \det\Psi(0,\lambda)\prod_{s=0}^{t}\{(\det(I_n - A(s)))^{-1}\det(I_n - A^*(s))\}, \\ &\det(\Psi(t+1,\lambda)\Psi^*(t+1,\lambda) = \det(\Psi(0,\lambda)\Psi^*(0,\lambda)), \\ &|\det\Psi(t,\lambda)| = |\det\Psi(0,\lambda)|. \end{aligned} \tag{3.4}$$

Proof. It suffices to show the first relation in (3.4) holds. From (3.2), we see that $S(t,\lambda)$ can be rewritten as

$$S(t,\lambda) = \begin{pmatrix} E(t) & 0 \\ (C(t) - \lambda W_1(t))E(t) & I_n - A^*(t) \end{pmatrix} \begin{pmatrix} I_n & B(t) + \lambda W_2(t)) \\ 0 & I_n \end{pmatrix} \tag{3.5}$$

which implies that

$$\det S(t,\lambda) = \det E(t)\det(I_n - A^*(t)) = (\det(I_n - A^*(t)))^{-1}\det(I_n - A^*(t)). \tag{3.6}$$

In addition, it follows from (3.1) that

$$\Psi(t+1,\lambda) = S(t,\lambda)\Psi(t,\lambda) = S(t,\lambda)S(t-1,\lambda)\cdots S(0,\lambda)\Psi(0,\lambda), \tag{3.7}$$

together with (3.6), yields the first relation in (3.4). This completes the proof. □

The following results play an important role in the rest discussions of the paper.

Theorem 3.1 *For all* $y, z \in l[0, n+1]$,

$$\begin{aligned}&\sum_{t=0}^{n}\{R(y)^*(t)L(z)(t) - L(y)^*(t)R(z)(t)\}\\ =&y^*(t)Jz(t)|_{t=0}^{n+1} + i\sum_{t=0}^{n} R(y)^*(t)\left[\frac{Q^*(t)-Q(t)}{i}\right]R(z)(t).\end{aligned} \tag{3.8}$$

Proof. Note $y(t) = (y_1^T(t), y_2^T(t))^T, z(t) = (z_1^T(t), z_2^T(t))^T$, where $y_j(t), z_j(t) \in \mathbb{C}^n$. Using (2.12) we obtain

$$\begin{aligned}&R(y)^*(t)L(z)(t) - L(y)^*(t)R(z)(t)\\ =&R(y)^*(t)[J\triangle z(t) - Q(t)R(z)(t)] - [(\triangle y(t))^*(-J) - R(y)^*(t)Q^*(t)]R(z)(t)\\ =&[y_2^*(t+1)z_1(t+1) - y_1^*(t+1)z_2(t+1)] - [y_2^*(t)z_1(t) - y_1^*(t)z_2(t)]\\ &+ iR(y)^*(t)\left[\frac{Q^*(t)-Q(t)}{i}\right]R(z)(t)\\ =&y^*(t)Jz(t)|_t^{t+1} + iR(y)^*(t)\left[\frac{Q^*(t)-Q(t)}{i}\right]R(z)(t).\end{aligned} \tag{3.9}$$

AS a direct conclusion of Lemma 3.2, we have

Lemma 3.2 *Assume* (II) *holds. For all* $\lambda, \mu \in \mathbb{C}$, *let* $y(\cdot,\lambda), z(\cdot,\mu)$ *be any solutions of non-symmetric discrete Hamiltonian systems* (2.1_λ) *and* (2.1_μ), *respectively. Then, for* $n \geq 1$,

$$\begin{aligned}&(\mu - \overline{\lambda})\sum_{t=0}^{n} R(y)^*(t,\lambda)W(t)R(z)(t,\mu)\\ =&y^*(t,\lambda)Jz(t,\mu)|_{t=0}^{n+1} + i\sum_{t=0}^{n} R(y)^*(t,\lambda)\left[\frac{Q^*(t)-Q(t)}{i}\right]R(z)(t,\mu).\end{aligned} \tag{3.10}$$

Lemma 3.3 *Let* A *be an* $m \times n$ *matrix and* B *be an* $n \times d$ *matrix. Then*

(1) $\| AB \| \leq \| A \| \| B \|$ *and* $\| AB \|_1 \leq \| A \|_1 \| B \|_1$;

(2) $\| A \|_1 \leq \| A \| \leq n^{\frac{1}{2}} \| A \|_1$ *in the case of* $m = n$, *where*

$$\| A \| = \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right\}^{\frac{1}{2}}, \quad \| A \|_1 = \sup_{\|\xi\|=1} \| A\xi \|, \tag{3.11}$$

$$and \quad \| \xi \| = \left(\sum_{j=1}^{n}|\xi_j|^2\right)^{\frac{1}{2}} \quad for\ \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{C}^n.$$

Theorem 3.2 (The Limit − circle Invariance Theorem). *Assume the non-symmetric discrete Hamiltonian system* (2.1) *satisfies*

$$-KW(t) \leq \frac{Q(t)-Q^*(t)}{i} \leq KW(t), \quad K > 0. \tag{3.12}$$

If there exists $\lambda_0 \in \mathbb{C}$ *such that the solutions of* (2.1_{λ_0}) *are in* $L_W^2[0,\infty)$, *then this is true for all* $\lambda \in \mathbb{C}$.

Proof. Suppose that all the solutions of (2.1_{λ_0}) are in $L_W^2[0,\infty)$ for some $\lambda_0 \in \mathbb{C}$. For any given $\lambda \in \mathbb{C}$, let $\Phi(\cdot,\lambda)$ be the normal fundamental solution matrix of (2.1_λ) (*i.e.*, $\Phi(0,\lambda) = I_{2n}$). Since $\Phi(t,\lambda), \Phi(t,\lambda_0)$ are invertible, there exists an invertible matrix $X(t,\lambda) \in \mathbb{C}^{2n\times 2n}$ such that

$$\Phi(t,\lambda) = \Phi(t,\lambda_0)X(t,\lambda). \tag{3.13}$$

As the theorem proving process is relatively complex, we divide it into two major steps as following.

The first part: $X(t,\lambda)$ is bounded on $[0,\infty)$.

(1) : From (3.13) we have that

$$\begin{aligned}&\triangle\Phi(t,\lambda)=\Phi(t+1,\lambda_0)\triangle X(t,\lambda)+(\triangle\Phi(t,\lambda_0))X(t,\lambda),\\&R(\Phi)(t,\lambda)=R(\Phi)(t,\lambda_0)X(t,\lambda)+\mathrm{diag}\{I_n,0\}\Phi(t+1,\lambda_0)\triangle X(t,\lambda).\end{aligned}\tag{3.14}$$

Using the fact $\Phi(t,\lambda)$ and $\Phi(t,\lambda_0)$ are the fundamental solution matrices of (2.1_λ) and (2.1_{λ_0}), respectively. Together with (3.14) we get

$$\begin{aligned}&[J-(Q(t)+\lambda W(t))\mathrm{diag}\{I_n,0\}]\Phi(t+1,\lambda_0)\triangle X(t,\lambda)\\=&(\lambda-\lambda_0)W(t)R(\Phi)(t,\lambda_0)X(t,\lambda),\end{aligned}\tag{3.15}$$

i.e.,

$$\begin{aligned}&\begin{pmatrix}C(t)-\lambda W_1(t) & -I_n\\ E^{-1}(t) & 0\end{pmatrix}\Phi(t+1,\lambda_0)\triangle X(t,\lambda)\\=&(\lambda-\lambda_0)W(t)R(\Phi)(t,\lambda_0)X(t,\lambda),\end{aligned}\tag{3.16}$$

so we have

$$\triangle X(t,\lambda)=P(t)X(t,\lambda),\tag{3.17}$$

where

$$\begin{aligned}&P(t,\lambda)=(\lambda-\lambda_0)Z^{-1}(t,\lambda)R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0),\\&Z(t,\lambda)=R(\Phi)^*(t,\lambda_0)\begin{pmatrix}C(t)-\lambda W_1(t) & -I_n\\ E^{-1}(t) & 0\end{pmatrix}\Phi(t+1,\lambda_0).\end{aligned}\tag{3.18}$$

We remark that the multiplier $R(\Phi)^*(t,\lambda_0)$ is added to (3.18) in order to study the property of $P(t,\lambda)$ more conveniently later on.

(2) : Consider $Z(t,\lambda)$. First, we want to show that $Z(t,\lambda)$ converges as $t\to\infty$. From (2.1_{λ_0}) that

$$\Phi(t+1,\lambda_0)=\begin{pmatrix}I_n & 0\\ C(t)-\lambda_0 W_1(t) & I_n-A^*(t)\end{pmatrix}R(\Phi)(t,\lambda_0).\tag{3.19}$$

Inserting (3.19) into (3.18), we get

$$\begin{aligned}Z(t,\lambda)&=R(\Phi)^*(t,\lambda_0)\begin{pmatrix}(\lambda_0-\lambda)W_1(t) & -I_n+A^*(t)\\ E^{-1}(t) & 0\end{pmatrix}R(\Phi)(t,\lambda_0)\\&=R(\Phi)^*(t,\lambda_0)\begin{pmatrix}0 & -I_n+A^*(t)\\ E^{-1}(t) & 0\end{pmatrix}R(\Phi)(t,\lambda_0)\\&\quad+(\lambda_0-\lambda)R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{W_1(t),0\}R(\Phi)(t,\lambda_0).\end{aligned}\tag{3.20}$$

Note

$$L(t)=R(\Phi)^*(t,\lambda_0)\begin{pmatrix}0 & -I_n+A^*(t)\\ E^{-1}(t) & 0\end{pmatrix}R(\Phi)(t,\lambda_0).\tag{3.21}$$

Since all the solutions of (2.1_{λ_0}) are in $L^2_W[0,\infty)$, $\Phi(\cdot,\lambda_0)\in L^2_W[0,\infty)$, that is

$$V(\lambda_0):=\sum_{t=0}^{\infty}R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0)<+\infty.\tag{3.22}$$

From (3.19), we get

$$\Phi^*(t+1,\lambda_0)=R(\Phi)^*(t,\lambda_0)\begin{pmatrix}I_n & C^*-\overline{\lambda_0}W_1(t)\\ 0 & I_n-A(t)\end{pmatrix},\tag{3.23}$$

hence

$$\begin{aligned}&\Phi^*(t+1,\lambda_0)J\Phi(t+1,\lambda_0)\\=&R(\Phi)^*(t,\lambda_0)\begin{pmatrix}(C^*-C)(t)+2i\mathrm{Im}\lambda_0W_1(t) & -I_n+A^*(t)\\ I_n-A(t) & 0\end{pmatrix}R(\Phi)(t,\lambda_0)\\=&R(\Phi)^*(t,\lambda_0)\begin{pmatrix}(C^*-C)(t)+2i\mathrm{Im}\lambda_0W_1(t) & -I_n+A^*(t)\\ E^{-1}(t) & 0\end{pmatrix}R(\Phi)(t,\lambda_0)\\=&L(t)+R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{(C^*-C)(t)+2i\mathrm{Im}\lambda_0W_1(t),0\}R(\Phi)(t,\lambda_0),\end{aligned}\tag{3.24}$$

so we have

$$L(t) = \Phi^*(t+1,\lambda_0)J\Phi(t+1,\lambda_0) - 2i\mathrm{Im}\lambda_0 R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{W_1(t),0\}R(\Phi)(t,\lambda_0) \\ - R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{(C^*-C)(t),0\}R(\Phi)(t,\lambda_0). \tag{3.25}$$

As a direct conclusion of Lemma 3.2, we have

$$\begin{aligned} &2i\mathrm{Im}\lambda_0 \sum_{s=0}^{t} R(\Phi)^*(s,\lambda_0)W(s)R(\Phi)(s,\lambda_0) \\ =&\Phi^*(s,\lambda_0)J\Phi(s,\lambda_0)|_0^{t+1} + i\sum_{t=0}^{t}\left\{R(\Phi)^*(s,\lambda_0)\left[\frac{Q^*(s)-Q(s)}{i}\right]R(\Phi)(s,\lambda_0)\right\}. \end{aligned} \tag{3.26}$$

Together with $\Phi(0,\lambda_0) = I_{2n}$, then

$$\begin{aligned} &\Phi^*(t+1,\lambda_0)J\Phi(t+1,\lambda_0) \\ =&J + 2i\mathrm{Im}\lambda_0 \sum_{s=0}^{t} R(\Phi)^*(s,\lambda_0)W(s)R(\Phi)(s,\lambda_0) \\ &- i\sum_{s=0}^{t}\left\{R(\Phi)^*(t,\lambda_0)\left[\frac{Q^*(s)-Q(s)}{i}\right]R(\Phi)(s,\lambda_0)\right\}. \end{aligned} \tag{3.27}$$

(3) : Determine the limitation of $Z(t,\lambda)$ as $t\to\infty$.

From (3.12) and (3.22), we know that the second term and the third term of (3.25) tend to zero as $t\to\infty$. Hence

$$\lim_{t\to\infty} L(t) = J + 2i\mathrm{Im}\lambda_0 V(\lambda_0) - i\sum_{s=0}^{\infty}\left\{R(\Phi)^*(t,\lambda_0)\left[\frac{Q^*(s)-Q(s)}{i}\right]R(\Phi)(s,\lambda_0)\right\}, \tag{3.28}$$

and with (3.20), so $\lim_{t\to\infty} Z(t,\lambda) = \lim_{t\to\infty} L(t)$.

(4) : We show that $Z^{-1}(t,\lambda)$ is bounded on $[0,\infty)$.

First, consider $\mathrm{det}Z(t,\lambda)$. From (3.19), we have $\mathrm{det}R(\Phi)(t,\lambda_0) = \mathrm{det}E^*(t)\mathrm{det}\Phi(t+1,\lambda_0)$. Again from (3.18) and the above relation as well as Liouville's formula, i.e., Lemma 3.1, it follows that for $t\in[0,\infty)$,

$$\begin{aligned} \mathrm{det}Z(t,\lambda) &= \mathrm{det}R(\Phi)^*(t,\lambda_0)\mathrm{det}(I_n - A(t))\mathrm{det}\Phi(t+1,\lambda_0) \\ &= \mathrm{det}\Phi^*(t+1,\lambda_0)\mathrm{det}\Phi(t+1,\lambda_0) \\ &= \mathrm{det}\Phi^*(0,\lambda_0)\mathrm{det}\Phi(0,\lambda_0) \\ &= 1. \end{aligned} \tag{3.29}$$

in which $\Phi(0,\lambda) = I_{2n}$ is used. Hence

$$Z^{-1}(t,\lambda) = [\mathrm{det}Z(t,\lambda)]^{-1}\mathrm{Adj}Z(t,\lambda) = \mathrm{Adj}Z(t,\lambda),\ t\in[0,\infty), \tag{3.30}$$

where $\mathrm{Adj}Z(t,\lambda)$ is the adjoint matrix of $Z(t,\lambda)$. In addition, from (3.12), $Z(t,\lambda)$ is bounded on $[0,\infty)$, conclude that $\mathrm{Adj}Z(t,\lambda)$ is bounded on $[0,\infty)$, so is it for $Z^{-1}(t,\lambda)$ by (3.30); that is, there exists a constant $M>0$ such that

$$\|Z^{-1}(t,\lambda)\| \le M, \quad t\in[0,\infty), \tag{3.31}$$

where the norm $\|\cdot\|$ is defined as in Lemma 3.3.

(5) : Here, we want to show that

$$\sum_{t=0}^{\infty}\|P(t,\lambda)\|_1 < \infty, \tag{3.32}$$

where the norm $\|\cdot\|_1$ is defined as in Lemma 3.3.

It follows from (3.22) that all the diagonal entries of $R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0)$ are nonnegative and absolutely summable over $[0,\infty)$, and $|\mathrm{tr}(R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0))| < +\infty$. Meanwhile, the diagonal entries and the non-diagonal entries satisfy $|a_{ij}| \le a_{ii}+a_{jj}, i \ne j$. Hence, each non-diagonal entry of $R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0)$ is also absolutely summable over $[0,\infty)$ and consequently, it follows that

$$\sum_{t=0}^{\infty} \|R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0)\| < +\infty. \tag{3.33}$$

so we get from the first term of (3.18)

$$\begin{aligned}\sum_{t=0}^{\infty} \| P(t,\lambda) \| &= |\lambda-\lambda_0| \sum_{t=0}^{\infty} \| Z^{-1}(t,\lambda)R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0) \| \\ &\le \sum_{t=0}^{\infty} \| Z^{-1}(t,\lambda) \| \| R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0) \| \\ &\le \mathrm{M} \sum_{t=0}^{\infty} \| R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0) \| \\ &< +\infty.\end{aligned} \tag{3.34}$$

By (2) in Lemma 3.3, (3.32) follows.

(6) : We show that $X(t,\lambda)$ is bounded on $[0,\infty)$. From (3.17) we have$X(t+1,\lambda) = [I_{2n} + P(t,\lambda)]X(t,\lambda)$, which implies that

$$\begin{aligned}\| X(t+1,\lambda) \|_1 &\le (1+\| P(t,\lambda) \|_1) \| X(t,\lambda) \|_1 \\ &\le (1+\| P(t,\lambda) \|_1)(1+\| P(t-1,\lambda) \|_1)\cdots(1+\| P(0,\lambda) \|_1) \| X(0,\lambda) \|_1 \\ &\le \exp\left(\sum_{s=0}^{t} \| P(s,\lambda) \|_1\right) \| X(0,\lambda) \|_1 .\end{aligned} \tag{3.35}$$

Together with (3.32), implies that $\| X(t,\lambda) \|_1$ is bounded on $[0,\infty)$.

The second part: We show that all solutions of non-symmetric discrete Hamiltonian system (2.1_λ) are in $L^2_W[0,\infty)$. From the second relation in (3.14) and (3.19), we get

$$\begin{aligned}&R(\Phi)^*(t,\lambda)W(t)R(\Phi)(t,\lambda) \\ &\quad = X^*(t,\lambda)R(\Phi)^*(t,\lambda_0)W(t)R(\Phi)(t,\lambda_0)X(t,\lambda) \\ &\qquad + X^*(t,\lambda)R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{W_1(t),0\}R(\Phi)(t,\lambda_0)\triangle X(t,\lambda) \\ &\qquad + \triangle X^*(t,\lambda)R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{W_1(t),0\}R(\Phi)(t,\lambda_0)X(t,\lambda) \\ &\qquad + \triangle X^*(t,\lambda)R(\Phi)^*(t,\lambda_0)\mathrm{diag}\{W_1(t),0\}R(\Phi)(t,\lambda_0)\triangle X(t,\lambda).\end{aligned} \tag{3.36}$$

So, using the boundedness of $\| X(t,\lambda) \|_1$ and (3.33), we have

$$\sum_{t=0}^{\infty} \| R(\Phi)^*(t,\lambda)W(t)R(\Phi)(t,\lambda) \| < +\infty, \tag{3.37}$$

consequently, $\Phi(\cdot,\lambda) \in L^2_W[0,\infty)$. Hence, all the solutions of (2.1_λ) are in $L^2_W[0,\infty)$. This completes the proof. □

Until now, no one is given the limit-circle definition of non-symmetric discrete Hamiltonian systems. So, we give the definition from the results of this paper as following:

Definition 3.1 *Let* (3.12) *holds in non-symmetric discrete Hamiltonian* (2.1)*. If all the solutions of* (2.1) *are in* $L^2_W[0,\infty)$ *(in the sense of linear independence), then this system called in limit-circle case, otherwise called in non-limit-circle case.*

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