Homoclinic solutions for a class of non-autonomous Hamiltonian systems with potential changing sign◇

Ziheng Zhang, Rong Yuan
School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,
Beijing 100875, People's Republic of China

Abstract

In this paper we consider the existence of homoclinic solutions for the following second order non-autonomous Hamiltonian system

$$\ddot{q} - L(t)q + W_q(t, q) = 0, \tag{HS}$$

where $L(t) \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W(t, q) = a(t)|q|^\gamma$ such that $a : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and may change its sign and $1 \le \gamma < 2$ is a constant. Assuming some reasonable assumptions on L and W, we obtain the existence of nontrivial homoclinic solutions of (HS) by using a standard minimizing argument in critical point theory. Recent results in the literature are generalized and significantly improved.

1 Introduction

The purpose of this work is to deal with the existence of homoclinic solutions for the following second order non-autonomous Hamiltonian system

$$\ddot{q} - L(t)q + W_q(t, q) = 0, \tag{HS}$$

where $L(t) \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W(t, q) = a(t)|q|^\gamma$ such that $a : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and may change its sign and $1 \le \gamma < 2$ is a constant. We say that a solution $q(t)$ of (HS) is homoclinic (to 0) if $q(t) \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$. If $q(t) \not\equiv 0$, $q(t)$ is called a nontrivial homoclinic solution.

The existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been already recognized from Poincaré [15]. Only more recently such a problem has been studied by using variational methods. Assuming that $L(t)$ and $W(t, q)$ (not necessarily of the kind $W(t, q) = a(t)|q|^\gamma$) are independent of t or T-periodic in t, many authors have studied the existence of homoclinic solutions for the Hamiltonian system (HS) via critical point theory and variational methods, see for instance [2, 5, 8, 14, 17] and the references therein and a more general case is considered in the recent paper [10]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t, q)$

Keyword: Homoclinic solutions; Critical point; Variational methods
2000 Mathematics Subject Classification: 34C37, 35A15, 37J45
◇ Supported by National Natural Science Foundation of China and RFDP
Email address: zhzh@mail.bnu.edu.cn (Ziheng Zhang), ryuan@bnu.edu.cn (Rong Yuan)

are neither autonomous nor periodic in t, this problem is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding, see for instance [1, 4, 6, 11, 13, 18] and the references listed therein for information on this subject. However, it is worthy of pointing out that to establish the existence of homoclinic solutions of (HS), all of the papers mentioned above assumed that the potential $W(t,q)$ is positive.

In mathematical physics, it is of frequent occurrence in (HS) that the potential $W(t,q)$ can change its sign. As far as the authors know, for this case, the existence of homoclinic solutions for (HS) was only considered in [3, 7, 9, 12]. Supposing that $W(t,q) = a(t)V(q)$ with $a(t)$ changing sign, in [7], under the conditions that $L(t)$ and $a(t)$ are T-periodic, and V is homogeneous of degree $\mu > 2$ such that $V(q) > 0$ for $q \neq 0$, i.e.,

$$V(\lambda q) = \lambda^\mu V(q), \quad \text{for all} \quad \lambda \geq 0, \quad q \in \mathbb{R}^n,$$

the authors obtained the existence of one homoclinic solutions of (HS) via the convergence of subharmonic solutions. In [3], Caldiroli and Montecchiari proved the existence of infinitely many homoclinic solutions for general periodic potentials $W(t,q)$ (not necessarily of the kind $W(t,q) = a(t)V(q)$) under the further conditions that there is $(t_0, q_0) \in \mathbb{R} \times \mathbb{R}^n$ with $q_0 \neq 0$ such that

$$W(t_0, q_0) - (L(t_0)q_0, q_0) \geq 0,$$

and there are two constants $\theta > 2$ and $\tau < \frac{\theta}{2} - 1$ such that

$$\theta W(t,q) - (W_q(t,q), q) \leq \tau (L(t)q, q), \quad \text{for all} \quad (t,q) \in \mathbb{R} \times \mathbb{R}^n.$$

Here and subsequently $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is the induced norm. For the non-periodic case, in [9], on the assumption that $W(t,q) = a(t)V(q)$ with $a(t)$ changing sign such that there exist $\eta > 2$ and $b_1 > 0$ such that

$$V(q) \geq b_1 |q|^\eta, \quad \text{for all} \quad q \in \mathbb{R}^n,$$

and some other reasonable hypotheses on V, the author obtained the existence of homoclinic solution for (HS). In [12], under the conditions that there exist $\xi > 2$, $1 \leq \delta < 2$ and $b_2 \geq 0$ such that

$$|(W_q(t,), q) - \xi W(t,q)| \leq b_2 |q|^\delta, \quad \text{for all} \quad (t,q) \in \mathbb{R} \times \mathbb{R}^n,$$

and there exists a pair (t_0, q_0) such that $|q_0| = 1$ and

$$W(t_0, q_0) > \frac{b_2}{\xi - \delta},$$

the authors obtained the same result. However, we must point that all the assumptions mentioned in the above papers implies that $W(t,q)$ is superquadratic as $|q| \to \infty$.

Motivated by the works mentioned above, in this paper we deal with the case that $W(t,q)$ can change its sign and is of subquadratic growth as $|q| \to +\infty$. We are interested in the case that (HS) is not periodic and give a new criterion to guarantee that (HS) has one nontrivial homoclinic solution. Now we state the basic hypotheses on L and W. Suppose that the symmetric matrix $L(t)$ satisfies

(H1) $L(t) \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $\Big(L(t)q, q\Big) \geq \alpha(t)|q|^2$ and $\alpha(t) \to +\infty$ as $|t| \to +\infty$;

and the potential $W(t,q)$ satisfies the following condition

(H2) $W(t,q) = a(t)|q|^\gamma$ where $a : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $a(t_0) > 0$ for some $t_0 \in \mathbb{R}$ and $a(t) \in L^2(\mathbb{R},\mathbb{R}) \cap L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R})$, $1 \le \gamma < 2$ is a constant.

Remark 1.1 From (H1), we see that there is a constant $\beta > 0$ such that

$$\Big(L(t)q, q\Big) \ge \beta |q|^2, \quad \text{for all } t \in \mathbb{R} \text{ and } q \in \mathbb{R}^n, \tag{1.1}$$

and by (H2), we have W is of subquadratic growth as $|q| \to +\infty$ and

$$W_q(t,q) = \gamma a(t)|q|^{\gamma-2}q. \tag{1.2}$$

Up to now, we can state our main result.

Theorem 1.1 *Suppose that the conditions (H1) and (H2) are satisfied, then (HS) possesses at least one nontrivial homoclinic solution.*

The remainder of this paper is organized as following. In section 2, some preliminary results are presented. In section 3, we give the proof of Theorem 1.1.

2 Preliminary Results

In order to establish our result via the critical point theory, we firstly describe some properties of the space on which the variational associated with (HS) is defined. Let

$$E = \left\{ q \in H^1(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} \Big[|\dot{q}(t)|^2 + \Big(L(t)q(t), q(t)\Big) \Big] dt < +\infty \right\}.$$

Then the space E is a Hilbert space with the inner product

$$(x,y) = \int_{\mathbb{R}} \Big(\dot{x}(t), \dot{y}(t)\Big) + \Big(L(t)x(t), y(t)\Big) dt$$

and the corresponding norm $\|x\|^2 = (x,x)$. Note that

$$E \subset H^1(\mathbb{R},\mathbb{R}^n) \subset L^p(\mathbb{R},\mathbb{R}^n)$$

for all $p \in [2,+\infty]$ with the embedding being continuous. In particular, for $p = +\infty$, there exists a constant $C > 0$ such that

$$\|q\|_\infty \le C\|q\|, \quad \forall q \in E. \tag{2.1}$$

Here $L^p(\mathbb{R},\mathbb{R}^n)$ $(2 \le p < +\infty)$ and $H^1(\mathbb{R},\mathbb{R}^n)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|q\|_p := \Big(\int_{\mathbb{R}} |q(t)|^p dt \Big)^{1/p} \quad \text{and} \quad \|q\|_{H^1} := \Big(\|q\|_2^2 + \|\dot{q}\|_2^2 \Big)^{1/2}$$

respectively. $L^\infty(\mathbb{R},\mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_\infty := \operatorname{ess\,sup}\{|q(t)| : t \in \mathbb{R}\}.$$

Lemma 2.1 *([13], Lemma 1) Suppose that L satisfies (H1). Then the embedding of E in $L^2(\mathbb{R}, \mathbb{R}^n)$ is compact.*

Hereafter, we denote $W(t,q)$ by $a(t)|q|^\gamma$ unless otherwise is specified, i.e., $W(t,q) = a(t)|q|^\gamma$. Similar to Lemma 2 of [13], we can get the following result.

Lemma 2.2 *Suppose that (H1) and (H2) are satisfied. If $q_k \rightharpoonup q$ (weakly) in E, then $W_q(t,q_k) \to W_q(t,q)$ in $L^2(\mathbb{R}, \mathbb{R}^n)$.*

Proof Assume that $q_k \rightharpoonup q$ in E. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus Theorem and (2.1),

$$\sup_{k\in\mathbb{N}} \|q_k\|_\infty \le d_1, \quad \|q\|_\infty \le d_1.$$

Since $1 \le \gamma < 2$, by (1.2), there exists a constant $d_2 > 0$ such that

$$|W_q(t, q_k(t))| \le d_2|a(t)|, \quad |W_q(t, q(t))| \le d_2|a(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|W_q(t, q_k(t)) - W_q(t, q(t))| \le 2d_2|a(t)|.$$

On the other hand, by Lemma 2.1, $q_k \to q$ in L^2, passing to subsequence if necessary, which implies $q_k(t) \to q(t)$ for almost every $t \in \mathbb{R}$. Then, using the Lebesgue's Convergence Theorem, the lemma is proved. □

Now we introduce more notations and some necessary definitions. Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on E. Recall that $I \in C^1(E, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset E$, for which $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in E.

We obtain the existence of homoclinic solution of (HS) by using a standard minimizing argument, see Theorem 2.7 in [16].

Lemma 2.3 *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the (PS) condition. If I is bounded from below, then*

$$c \equiv \inf_E I$$

is a critical point of I.

3 Proof of Theorem 1.1

Now we are going to establish the corresponding variational framework to obtain homoclinic solutions of (HS). Define the functional $I : E \to \mathbb{R}$ by

$$\begin{aligned} I(q) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}\Big(L(t)q(t), q(t)\Big) - W(t, q(t))\Big] dt \\ &= \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, q(t)) dt. \end{aligned} \tag{3.1}$$

Lemma 3.1 *Under the conditions of Theorem 1.1, we have*

$$I'(q)v = \int_{\mathbb{R}} \Big[\Big(\dot{q}(t), \dot{v}(t)\Big) + \Big(L(t)q(t), v(t)\Big) - \Big(W_q(t, q(t)), v(t)\Big)\Big] dt \tag{3.2}$$

for all q, $v \in E$, which yields that

$$I'(q)q = \|q\|^2 - \int_{\mathbb{R}} \Big(W_q(t, q(t)), q(t)\Big) dt. \tag{3.3}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on E, i.e., $I \in C^1(E, \mathbb{R})$ and any critical point of I on E is a classical solution of (HS) with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$.

Proof We firstly show that $I : E \to \mathbb{R}$. Letting $q \in E$, by (H2) and the Hölder inequality, we have

$$0 \le \int_{\mathbb{R}} |W(t, q(t))| dt \le \int_{\mathbb{R}} |a(t)||q(t)|^{\gamma} dt \le \beta^{\frac{-\gamma}{2}} \|a\|_{\frac{2}{2-\gamma}} \|q\|^{\gamma} < +\infty. \tag{3.4}$$

Combining (3.1) with (3.4), we show that $I : E \to \mathbb{R}$.

Next we prove that $I \in C^1(E, \mathbb{R})$. Rewrite I as following

$$I = I_1 - I_2,$$

where

$$I_1 := \int_{\mathbb{R}} \Big[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}\Big(L(t)q(t), q(t)\Big)\Big] dt, \quad I_2 := \int_{\mathbb{R}} W(t, q(t)) dt.$$

It is easy to check that $I_1 \in C^1(E, \mathbb{R})$ and

$$I_1'(q)v = \int_{\mathbb{R}} \Big[\Big(\dot{q}(t), \dot{v}(t)\Big) + \Big(L(t)q(t), v(t)\Big)\Big] dt. \tag{3.5}$$

Thus it is sufficient to show that this is the case for I_2. In the process we will see that

$$I_2'(q)v = \int_{\mathbb{R}} \Big(W_q(t, q(t)), v(t)\Big) dt, \tag{3.6}$$

which is defined for all q, $v \in E$. For any given $q \in E$, let us define $J(q) : E \to \mathbb{R}$ as following

$$J(q)v = \int_{\mathbb{R}} \Big(W_q(t, q(t)), v(t)\Big) dt, \quad v \in E.$$

It is obvious that $J(q)$ is linear. Now we show that $J(q)$ is bounded. Indeed, for any given $q \in E$, by (2.1) and (1.2), there exists a constant $d_3 > 0$ (dependent on q) such that

$$|W_q(t, q(t))| \le d_3 |a(t)|, \quad \text{for all} \quad t \in \mathbb{R},$$

which yields that, by (1.1) and the Hölder inequality,

$$\begin{aligned} |J(q)v| &= \Big|\int_{\mathbb{R}} \Big(W_q(t, q(t)), v(t)\Big) dt\Big| \le d_3 \int_{\mathbb{R}} |a(t)||v(t)| dt \\ &\le d_3 \|a\|_2 \|v\|_2 \le \frac{d_3}{\sqrt{\beta}} \|a\|_2 \|v\|. \end{aligned} \tag{3.7}$$

Moreover, for q and $v \in E$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t, q(t) + v(t)) dt - \int_{\mathbb{R}} W(t, q(t)) dt = \int_{\mathbb{R}} \Big(W_q(t, q(t) + h(t)v(t)), v(t)\Big) dt,$$

where $h(t) \in (0,1)$. Therefore, by Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}}\Big(W_q(t,q(t)+h(t)v(t)),v(t)\Big)dt-\int_{\mathbb{R}}\Big(W_q(t,q(t)),v(t)\Big)dt\\ &=\int_{\mathbb{R}}\Big(W_q(t,q(t)+h(t)v(t))-W_q(t,q(t)),v(t)\Big)dt\to 0\end{aligned} \tag{3.8}$$

as $v \to 0$ in E. Combining (3.7) and (3.8), we see that (3.6) holds. It remains to prove that I_2' is continuous. Suppose that $q \to q_0$ in E and note that

$$I_2'(q)v - I_2'(q_0)v = \int_{\mathbb{R}}\Big(W_q(t,q(t)) - W_q(t,q_0(t)),v(t)\Big)dt.$$

By Lemma 2.2 and the Hölder inequality, we obtain that

$$I_2'(q)v - I_2'(q_0)v \to 0, \quad \text{as} \quad q \to q_0,$$

which implies the continuity of I_2' and $I \in C^1(E,\mathbb{R})$.

Lastly, we check that critical points of I are classical solutions of (HS) satisfying $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $|t| \to +\infty$. We have known that $E \subset H^1(\mathbb{R},\mathbb{R}^n) \subset C^0(\mathbb{R},\mathbb{R}^n)$, the space of continuous functions q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to +\infty$ (see, e.g., [17]). Moreover, if q is one critical point of I, by (3.2), we have that $L(t)q(t) - W_q(t,q(t))$ is the weak derivative of $\dot{q}(t)$. Note that $L(t) \in C(\mathbb{R},\mathbb{R}^{n^2})$ and $W_q(t,q) = \gamma a(t)|q|^{\gamma-2}q$, we obtain that $q \in C^2(\mathbb{R},\mathbb{R}^n)$, i.e., q is a classical solution of (HS). Hence, it is easy to check that q satisfies $\dot{q}(t) \to 0$ as $|t| \to +\infty$. □

Lemma 3.2 *Under the conditions of (H1) and (H2), I satisfies the (PS) condition.*

Proof In fact, assume that $\{u_j\}_{j\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I^{'}(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_j)| \leq C_1, \qquad \|I^{'}(u_j)\|_{E^*} \leq C_1 \tag{3.9}$$

for every $j \in \mathbb{N}$.

We firstly prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By (3.1), (3.9) and (H2), we have

$$\begin{aligned}\frac{1}{2}\|u_j\|^2 &= I(u_j) + \int_{\mathbb{R}} W(t,u_j(t))dt\\ &= I(u_j) + \int_{\mathbb{R}} a(t)|u_j(t)|^{\gamma}dt\\ &\leq \beta^{\frac{-\gamma}{2}}\|a\|_{\frac{2}{2-\gamma}}\|u_j\|^{\gamma} + C_1.\end{aligned} \tag{3.10}$$

Since $1 \leq \gamma < 2$, the inequality (3.10) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By Lemma 2.1, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u \in E$ such that

$$u_j \rightharpoonup u, \text{ weakly in } E,$$

$$u_j \to u, \text{ strongly in } L^2(\mathbb{R},\mathbb{R}^n).$$

Hence

$$\Big(I^{'}(u_j) - I^{'}(u)\Big)(u_j - u) \to 0,$$

and by the Hölder inequality and Lemma 2.2, we have

$$\int_{\mathbb{R}}\Big(W_q(t,u_j(t))-W_q(t,u(t)),u_j(t)-u(t)\Big)dt\to 0$$

as $j\to+\infty$. On the other hand, an easy computation shows that

$$\begin{aligned}\Big(I^{'}(u_j)-I^{'}(u),u_j-u\Big) &= \|u_j-u\|^2\\ &- \int_{\mathbb{R}}\Big(W_q(t,u_j(t))-W_q(t,u(t)),u_j(t)-u(t)\Big)dt.\end{aligned}$$

Consequently, $\|u_j-u\|\to 0$ as $j\to+\infty$, i.e., I satisfies the (PS) condition. □

Up to now, we can give the proof of Theorem 1.1

Proof of Theorem 1.1 By (3.1) we have, for every $m\in\mathbb{R}\setminus\{0\}$ and $q\in E\setminus\{0\}$,

$$\begin{aligned}I(m\,q) &= \frac{m^2}{2}\|q\|^2-\int_{\mathbb{R}}W(t,m\,q(t))dt\\ &= \frac{m^2}{2}\|q\|^2-|m|^{\gamma}\int_{\mathbb{R}}a(t)q^{\gamma}(t)dt\\ &\geq \frac{m^2}{2}\|q\|^2-\beta^{\frac{-\gamma}{2}}|m|^{\gamma}\|a\|_{\frac{2-\gamma}{2}}\|q\|^{\gamma}.\end{aligned}\tag{3.11}$$

Since $1\leq\gamma<2$, (3.11) implies that $I(m\,q)\to+\infty$ as $|m|\to+\infty$. Consequently, I is a functional bounded from below. By Lemmas 3.2 and 2.3, I possesses a critical value $c=\inf_{q\in E}I(q)$, i.e., there is one critical point $q\in E$ such that

$$I(q)=c,\quad I^{'}(q)=0.$$

On the other hand, by (H2), there exist t_1 and t_2 such that $t_1<t_0<t_2$ and $a(t)>0$ for any $t\in[t_1,t_2]$. Take $c_0\in\mathbb{R}^n$ with $|c_0|\neq 0$, and let $\varphi\in E$ be given by

$$\varphi(t)=\begin{cases} c_0\sin\Big[\frac{2\pi}{t_2-t_1}(t-t_1)\Big], & \text{if } t\in[t_1,t_2],\\ 0, & \text{if } t\in\mathbb{R}\setminus[t_1,t_2],\end{cases}$$

where $-\infty<t_1<t_2<+\infty$. Then we obtain that

$$I(m\,\varphi)=\frac{m^2}{2}\|\varphi\|^2-|m|^{\gamma}\int_{t_1}^{t_2}a(t)|\varphi(t)|^{\gamma}dt,$$

which yields that $I(m\,\varphi)<0$ for $|m|$ small enough since $1\leq\gamma<2$, i.e., the critical point $q\in E$ obtained above is nontrivial.

□

References

[1] C. O. Alves, P. C . Carrião and O. H. Miyagaki, *Existence of homoclinic orbits for asymptotically periodic systems involving Duffing-like equation*, Appl. Math. Lett., **16** (2003), no. 5, 639-642.

[2] A. Ambrosetti and V. Coti Zelati, *Multiple homoclinic orbits for a class of conservative systems*, Rend. Sem. Mat. Univ. Padova., **89** (1993), 177-194.

[3] P. Caldiroli and P. Montecchiari, *Homoclinic orbits for second order Hamiltonian systems with potential changing sign*, Comm. Appl. Nonlinear Anal., **1** (1994), no. 2, 97-129.

[4] P. C. Carrião and O. H. Miyagaki, *Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems*, J. Math. Anal. Appl., **230** (1999), no. 1, 157-172.

[5] V. Coti Zelati and P. H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., **4** (1991), no. 4, 693-727.

[6] Y. Ding, *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal., **25** (1995), no. 11, 1095-1113.

[7] Y. Ding and M. Girardi, *Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign*, Dynam. Systems Appl., **2** (1993), no. 1, 131-145.

[8] A. Flavia, *Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign*, Boll. Un. Mat. Ital., B (7) **10** (1996), no. 2, 303-324.

[9] G. Fei, *The existence of homoclinic orbits for Hamiltonian systems with the potential changing sign,* Chin. Ann. Math., **17 B** (1996), no. 4, 403-410.

[10] M. Izydorek and J. Janczewska, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations, **219** (2005), no. 2, 375-389.

[11] P. Korman and A. C. Lazer, *Homoclinic orbits for a class of symmetric Hamiltonian systems*, Electron. J. Differential Equations, (1994), no. 01, 1-10.

[12] Y. Liu, C. Li, S. Zhong and W. Zhang, Homoclinic orbits for a class of Hamiltonian systems with potentials changing sign, Ann. Differential Equations, **21** (2005), no. 3, 370-372.

[13] W. Omana and M. Willem, *Homoclinic orbits for a class of Hamiltonian systems*, Differential Integral Equations, **5** (1992), no. 5, 1115-1120.

[14] E. Paturel, *Multiple homoclinic orbits for a class of Hamiltonian systems*, Calc. Val. Partial Differential Equations, **12** (2001), no. 2, 117-143.

[15] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste,* Gauthier-Villars, Pairs, 1897-1899.

[16] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS Reg. Conf. Ser. in. Math., vol. 65, American Mathematical Society, Provodence, RI, 1986.

[17] P. H. Rabinowitz, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A., **114** (1990), no. 1-2, 33-38.

[18] P. H. Rabinowitz and K. Tanaka, *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z., **206** (1991), no. 3, 473-499.