Caputo 分数阶常微分方程变分迭代法的收敛性分析

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摘要： 本文利用变分迭代法求解 Caputo 分数阶常微分方程初值问题。证明了变分迭代方法求解这类方程初值问题是收敛的。通过数值实验，表明了变分迭代方法求解这类方程的初值问题是有效的。

关键词： 分数阶微分方程；变分迭代法；Caputo 导数；收敛性
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Convergence Analysis of Variational Iteration Method for Caputo Fractional Differential Equations

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Abstract: In this paper, the variational iteration method is applied to solve initial value problems of Caputo fractional differential equations. The convergence of the variational iteration method for solving the initial value problems of this class of equation has been proved. The numerical examples show the efficiency of the variational iteration method for solving the initial value problems of this class of equations.

Keywords: Fractional differential equation;Variational iteration method;Caputo derivatives;Convergence

0 引言

分数阶微积分发展至今已有三百多年的历史，由于分数阶微积分算子是一种非局部算子，具有记忆性，这些问题给研究工作带来了困难。近三十年来 ，分数阶微积分在许多领域上得到了广泛的应用，如粘弹性材料[1-2]、色噪声[4]、电介质极化现象[5]，电气化学过程[6]、控制理论[7]、混沌[8]等。在这些领域中，较之整数阶微积分模型，由于分数阶微积分具有描述物质记忆功能和遗传效应的特征，这使得它能更精确地模拟现实问题，其中又以分数阶微分方程的应用更为广泛。

由于许多分数阶微分方程的解析解由比较特殊的函数表示，而要具体求出这些函数的数值比较困难而且有许多方程无法求其解析解，因此，这需要展开求解分数阶微分方程数值方法研究工作。许多学者提出了很多方法，主要有：Adomian 分解法[10-14]、线性多步法[15-16]，配置方法[17]等。

变分迭代法（VIM）是由何吉欢[19-22,31]发展起来的，已经被应用到很多问题上。例如自治常微分方程[24]、延迟微分方程[23,30-31]、变系数非线性偏微分方程[25]、非线性 Volterra 积分微分方程[26]及分数阶微分方程的某些模型[13-14,27]等。

但 VIM 的收敛性的研究工作（特别是给出收敛性理论结果）却十分有限，如 VIM 关于二阶初值问题[28]、常系数线性常微分方程[29]、二类特殊的积分方程[32]、线性多项比例延迟

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微分方程[23]等。特别文[33]对一类关于分数阶、整数阶无时滞的微分方程初值问题得到了一般收敛结果。

本文利用变分迭代法求解分数阶微分方程，证明该方法的收敛性，通过数值算例来说明该算法的有效性。本文 VIM 迭代格式和收敛性证明方法均不同于[33]。

1 问题和方法

考虑分数阶微分方程初值问题

$$\begin{cases} {}_a^C D_t^{\alpha} y(t) = f(t, y(t)), & n-1 < \alpha \le n, \\ y^{(k)}(0) = y_0^k, & k = 0, \cdots, n-1, \end{cases} \tag{1.1}$$

这里$t \in [0,T]$，$y^{(k)}(t)$是y的k次导数，$f:[0,T]\times C \to C$，且满足 Lispschitz 条件

$$| f(t,u_1) - f(t,u_2) | \le L | u_1 - u_2 |, \quad t \ge 0, \quad u_1, u_2 \in C,$$

其中L为 Lispschitz 常数，定义范数$\| y \|_\infty = \max\limits_{0\le t\le T} | y(t) |$。${}_a^C D_t^{\alpha} y(t)$为 Caputo 分数阶导数，对于正实数$\alpha$, $0 \le n-1 < \alpha \le n$，定义在区间$[a,b]$上的函数$f(t)$的α阶 Caputo 分数阶导数为

$$ {}_a^C D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha-n+1}}, & if \quad 0 \le n-1 < \alpha \le n, \\ \dfrac{d^n}{dt^n f(t)}, & if \quad \alpha = n \in N. \end{cases} \tag{1.2}$$

Diethelm 等人在[1]中证明了问题(1.1)等价于 Volterra 积分方程

$$y(t) = \sum_{j=0}^{[\alpha]-1} y_0^{(j)} \frac{t^j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.3}$$

令

$$g(t) = \sum_{j=0}^{[\alpha]-1} y_0^{(j)} \frac{t^j}{j!}$$

则等式(1.3)化为如下形式

$$y(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.4}$$

根据 Xu [2]和 Ghorbani 等人[3]的思想，将变分迭代法应用到积分方程(1.4)中，得到如下的迭代格式

$$y_{n+1}(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \quad n = 1,2,\cdots. \tag{1.5}$$

我们用初始值$y_0(t) = y_0^{(0)} + y_0^{(1)} t + \cdots + y_0^{(n-1)} t^{n-1}$开始迭代，用第$N$次迭代值$y_N(t)$来近似问题(1.1)的真解

$$y(t) = \lim_{n\to\infty} y_n(t).$$

2 收敛性分析

定理 2.1 若$y(t), y_i(t) \in C[0,T], i = 1,2,\cdots$, 则由(1.5)得到的解序列收敛于问题(1.1)

的真解 $y(t)$。

证明 首先根据文献[9]，我们知道解 $y(t)$ 唯一，记 $E_i(t)=y_i(t)-y(t), i=1,2,\cdots$，显然，

$$y(t)=g(t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,y(\tau))d\tau. \tag{2.1}$$

由(1.5)和(2.1)，我们有

$$E_{n+1}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[f(\tau,y_n(\tau))-f(\tau,y(\tau))]d\tau, \tag{2.2}$$

下面我们分 $0<\alpha<1$ 及 $\alpha\ge 1$ 两种情况来讨论。

当 $\alpha\ge 1$ 时，对于 $\forall t\in[0,T]$，当 $\tau\in[0,t]$ 时，$(t-\tau)^{\alpha-1}$ 有界，若记

$$M=\max_{0\le\tau\le t,0\le t\le T}|(t-\tau)^{\alpha-1}|,$$

则利用 Lispschitz 条件，可得

$$\begin{aligned}
|E_{n+1}(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t |(t-\tau)^{\alpha-1}|\cdot|f(\tau,y_n(\tau))-f(\tau,y(\tau))]|d\tau\\
&\le \frac{M}{\Gamma(\alpha)}\int_0^t |f(\tau,y_n(\tau))-f(\tau,y(\tau))]|d\tau\\
&\le \frac{M\cdot L}{\Gamma(\alpha)}\int_0^t |y_n(\tau)-y(\tau)|d\tau\\
&\le \frac{M\cdot L}{\Gamma(\alpha)}\int_0^t |E_n(\tau)|d\tau,
\end{aligned}$$

利用递推关系可得

$$\begin{aligned}
|E_{n+1}(t)| &\le \frac{M^2\cdot L^2}{[\Gamma(\alpha)]^2}\int_0^t\int_0^{\tau_1}|E_{n-1}(\tau_2)|d\tau_2 d\tau_1\\
&\le \frac{M^3\cdot L^3}{[\Gamma(\alpha)]^3}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}|E_{n-2}(\tau_3)|d\tau_3 d\tau_2 d\tau_1\\
&\vdots\\
&\le \frac{M^{n+1}\cdot L^{n+1}}{[\Gamma(\alpha)]^{n+1}}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}\cdots\int_0^{\tau_n}|E_0(\tau_{n+1})|d\tau_{n+1}\cdots d\tau_3 d\tau_2 d\tau_1,
\end{aligned}$$

由此可得

$$\begin{aligned}
|E_{n+1}(t)|_\infty &\le [\frac{M\cdot L}{[\Gamma(\alpha)]}]^{n+1}\max_{0\le t\le T}[\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}\cdots\int_0^{\tau_n}|E_0(\tau_{n+1})|d\tau_{n+1}\cdots d\tau_3 d\tau_2 d\tau_1]\\
&\le [\frac{M\cdot L}{[\Gamma(\alpha)]}]^{n+1}\frac{T^{n+1}}{(n+1)!}\|E_0(t)\|_\infty.
\end{aligned}$$

因为 $M,L,T,\Gamma(\alpha),\|E_0(t)\|_\infty$ 都是常数故有

$$\lim_{n\to\infty}|E_{n+1}(t)|_\infty\le\lim_{n\to\infty}[\frac{M\cdot L\cdot T}{[\Gamma(\alpha)]}]^{n+1}\frac{\|E_0(t)\|_\infty}{(n+1)!}]=0.$$

当 $0<\alpha<1$ 时，

$$|E_{n+1}(t)| \le \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|y_n(\tau)-y(\tau)|d\tau$$
$$= \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|E_n(\tau)|d\tau,$$

记

$$J^{\alpha}f(t) = \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,$$

我们有

$$|E_{n+1}(t)| \le L\cdot J^{\alpha}|E_n(t)|.$$

根据[9]，算子 J^{α} 满足如下的复合关系

$$J^{\alpha}J^{\alpha}f(t) = J^{\alpha+\alpha}f(t) = J^{2\alpha},$$

因此，有

$$
\begin{aligned}
|E_{n+1}(t)| &\le L^2\cdot J^{2\alpha}|E_{n-1}(t)| \\
&\ \ \vdots \\
&\le L^{n+1}\cdot J^{(n+1)\alpha}|E_0(t)| \\
&= L^{n+1}\cdot\frac{1}{\Gamma(n\alpha+\alpha)}\int_0^t (t-\tau)^{n\alpha+\alpha-1}|E_0(\tau)|d\tau \\
&\le \frac{L^{n+1}\cdot\|E_0(t)\|_{\infty}}{\Gamma(n\alpha+\alpha)}\int_0^t (t-\tau)^{n\alpha+\alpha-1}d\tau \\
&= \frac{L^{n+1}\cdot\|E_0(t)\|_{\infty}}{\Gamma(n\alpha+\alpha)}\cdot\frac{t^{n\alpha+\alpha}}{(n\alpha+\alpha)}.
\end{aligned}
$$

根据[18]，我们有

$$\Gamma(n\alpha+\alpha) \sim \sqrt{2\pi}e^{-n\alpha}(n\alpha)^{n\alpha+\alpha-\frac{1}{2}},$$

那么，

$$\frac{L^{n+1}\cdot t^{n\alpha+\alpha}}{\Gamma(n\alpha+\alpha)(n\alpha+\alpha)} \sim \frac{L^{n+1}\cdot t^{n\alpha+\alpha}}{\sqrt{2\pi}e^{-n\alpha}(n\alpha)^{n\alpha+\alpha-\frac{1}{2}}(n\alpha+\alpha)},$$

对于 Lipschitz 常数 L，可以找到有界实数 L_1，使得

$$L_1^{\alpha} = L,$$

因此

$$\frac{L^{n+1}\cdot t^{n\alpha+\alpha}}{\sqrt{2\pi}e^{-n\alpha}(n\alpha)^{n\alpha+\alpha-\frac{1}{2}}(n\alpha+\alpha)}$$
$$= \frac{1}{\sqrt{2\pi}e^{\alpha}}\cdot\frac{L_1^{n\alpha+\alpha}\cdot t^{n\alpha+\alpha}\cdot e^{n\alpha+\alpha}}{(n\alpha)^{n\alpha+\alpha}(n\alpha+\alpha)}\cdot\frac{(n\alpha)^{\frac{1}{2}}}{(n\alpha+\alpha)}$$

$$=\frac{1}{\sqrt{2\pi e^{\alpha}}}\cdot(\frac{L_1 te}{n\alpha})^{n\alpha+\alpha}\cdot\frac{(n\alpha)^{\frac{1}{2}}}{(n\alpha+\alpha)}.$$

当$0\le t\le T$时，有

$$|E_{n+1}(t)|_{\infty}\le\frac{\|E_0(t)\|_{\infty}}{\sqrt{2\pi e^{\alpha}}}\cdot(\frac{L_1 Te}{n\alpha})^{n\alpha+\alpha}\cdot\frac{(n\alpha)^{\frac{1}{2}}}{(n\alpha+\alpha)},$$

因为$\|E_0(t)\|_{\infty},L_1,T$都为常数，故有

$$\lim_{n\to\infty}|E_{n+1}(t)|_{\infty}\le\lim_{n\to\infty}(\frac{\|E_0(t)\|_{\infty}}{\sqrt{2\pi e^{\alpha}}}\cdot(\frac{L_1 Te}{n\alpha})^{n\alpha+\alpha}\cdot\frac{(n\alpha)^{\frac{1}{2}}}{(n\alpha+\alpha)})$$

$$\le\frac{\|E_0(t)\|_{\infty}}{\sqrt{2\pi\cdot e^{\alpha}}}\cdot\lim_{n\to\infty}(\frac{L_1 Te}{n\alpha})^{n\alpha+\alpha}=0.$$

定理得证。

3 数值试验

在这一节中，我们给出一些例子，证明方法的有效性。

例 3.1 考虑如下齐次线性方程

$$ {}_a^C D_t^{\alpha} y(t)=-y(t),\qquad 0<\alpha<2, \tag{3.1}$$

初始条件为

$$y(0)=1,\ \ y'(0)=0, \tag{3.2}$$

其真解为

$$y(t)=E_{\alpha}(-t)^{\alpha},$$

这里$E_{\alpha}(z)$是单系数的 Mittag-Leffler 函数

$$E_{\alpha}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)},\quad \alpha>0.$$

其中变量z可以是复数。

首先根据(1.4)，问题$(3.1)-(3.2)$可化为如下 Volterrra 积分方程

$$y(t)=1-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}y(\tau)d\tau,$$

再根据迭代公式(1.5)，有如下变分迭代公式

$$y_{n+1}(t)=1-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}y_n(\tau)d\tau,\quad n=1,2,3,\cdots.$$

我们用初始值$y_0(t)=1$开始迭代，得到迭代值如下

$$y_1(t)=1-\frac{1}{\Gamma(1+\alpha)}t^{\alpha},$$

$$y_2(t)=1-\frac{1}{\Gamma(1+\alpha)}t^{\alpha}+\frac{1}{\Gamma(1+2\alpha)}t^{2\alpha},$$

$$y_3(t)=1-\frac{1}{\Gamma(1+\alpha)}t^{\alpha}+\frac{1}{\Gamma(1+2\alpha)}t^{2\alpha}-\frac{1}{\Gamma(1+3\alpha)}t^{3\alpha},$$

$$\vdots$$

表 1 和表 2 分别显示了当$\alpha=0.25,\alpha=1.25$和$t=0.1$时解析解和数值解的误差，表中的n代表迭代次数。从表中我们可以得到数值解收敛于真解，验证了定理 2.1 的结论。表 3 和表 4 分别显示了当$\alpha=0.25,\alpha=1.25$时解析解和数值解及他们的误差，从表中我们可以看出数值结果较准确。

表 1　解析解、数值解和误差($\alpha=0.25$)

Tab. 1　Extra solution, numerical solution and error

n	Extra solution	VIM	误差
3	0.6094871084	0.5429262386	-6.656E-02
6	0.6094871084	0.6170817456	7.595E-03
9	0.6094871084	0.6088193625	-6.677E-04
12	0.6094871084	0.6095355083	4.840E-05
15	0.6094871084	0.6094840997	-3.009E-06
18	0.6094871084	0.6094872735	1.651E-07
21	0.6094871084	0.6094871007	-7.700E-09
24	0.6094871084	0.6094871088	-4.000E-10

表 2　解析解和数值解及误差($\alpha=1.25$)

Tab. 2　Extra solution, numerical solution and error

n	Extra solution	VIM	误差
1	0.9513080798	0.9503671855	-9.409E-04
2	0.9513080798	0.9513187184	1.064E-05
3	0.9513080798	0.9513079970	-8.280E-08
4	0.9513080798	0.9513080803	5.000E-10
5	0.9513080798	0.9513080798	0
6	0.9513080798	0.9513080798	0

表 3　解析解和数值解及误差($\alpha=0.25$)

Tab. 3　Extra solution, numerical solution and error

t	Extra solution	VIM	误差
0.1	0.6094871084	0.6094871088	-4.000E-10
0.2	0.5665511819	0.5665511824	-4.000E-10
0.3	0.5408913015	0.5408913044	-2.900E-09
0.4	0.5225338734	0.5225338973	-2.380E-08
0.5	0.5082446751	0.5082448063	-1.312E-07
0.6	0.4965553359	0.4965558667	-5.307E-07
0.7	0.4866725890	0.4866743184	-1.729E-06
0.8	0.4781185882	0.4781233943	-4.806E-06
0.9	0.4705829898	0.4705948377	-1.185E-05
1.0	0.4638527608	0.4638793120	-2.655E-05

表 4　解析解和数值解及误差($\alpha=1.25$)

Tab. 4　Extra solution, numerical solution and error

t	Extra solution	VIM	误差
0.1	0.9513080798	0.9513080798	0
0.2	0.8871936706	0.8871936703	3.000E-10
0.3	0.8182314798	0.8182314731	6.700E-09
0.4	0.7478252531	0.7478251968	5.630E-08
0.5	0.6778687280	0.677868439	2.882E-07
0.6	0.6095662530	0.6095651647	1.088E-06
0.7	0.5437236057	0.5437202754	3.330E-06
0.8	0.4808857365	0.4808769974	8.739E-06
0.9	0.4214139751	0.4213935827	2.039E-05
1.0	0.3655344400	0.365491064	4.338E-05

例 2.2 考虑如下齐次线性方程

$$\begin{cases} {}_{a}^{C}D_{t}^{\alpha} y(t) + Ay(t) = f(t), & 0 < \alpha \le 2, \\ y^{(k)}(0) = 0, & (k = 0,1), \end{cases} \tag{3.3}$$

这里 $A = 1, f(t) = 1$，方程的真解为

$$y(t) = \int_0^t G(t-\tau)d\tau, \quad G(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}),$$

这里 $E_{\alpha}, \beta(z)$ 是 Mittag-Leffler 函数

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \beta > 0.$$

其中变量 z 可以是复数。

首先根据 (1.4)，问题 (3.3) 可化为如下 Volterra 积分方程

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}(1 - y(\tau))d\tau,$$

再根据迭代公式 (1.5)，有如下变分迭代公式

$$y_{n+1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}(1 - y_n(\tau))d\tau, \quad n = 1,2,3,\cdots.$$

我们用初始值 $y_0(t) = 0$ 开始迭代。

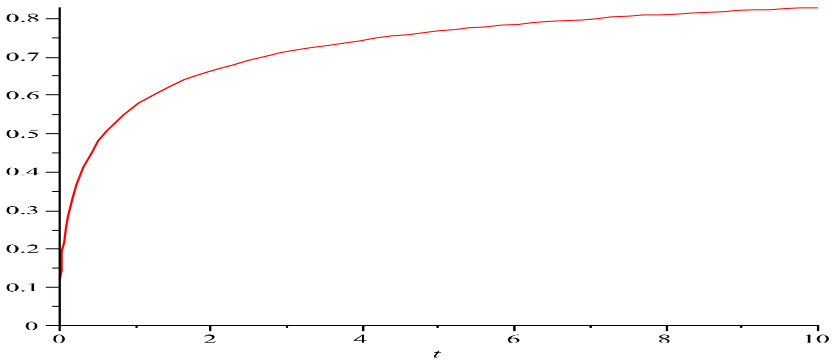

图 1　数值解($\alpha = 0.5$)

Fig. 1　Numerical solution ($\alpha = 0.5$)

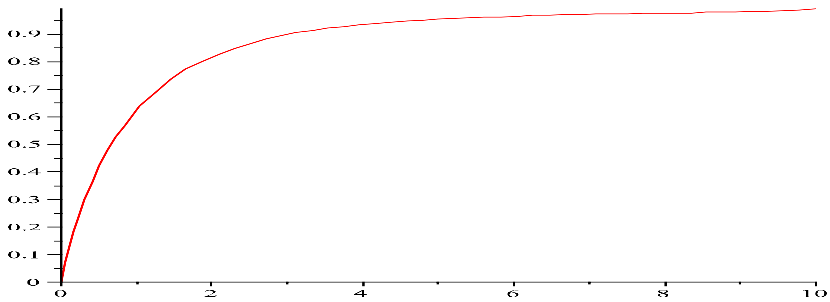

图 2　数值解($\alpha = 0.9$)

Fig. 2　Numerical solution ($\alpha = 0.9$)

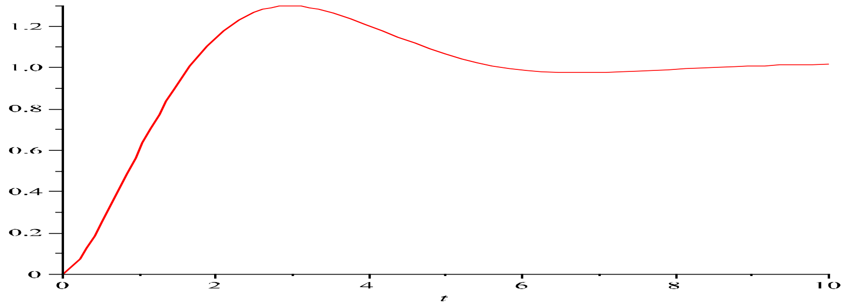

图 3　数值解($\alpha=1.5$)

Fig. 3　Numerical solution ($\alpha=1.5$)

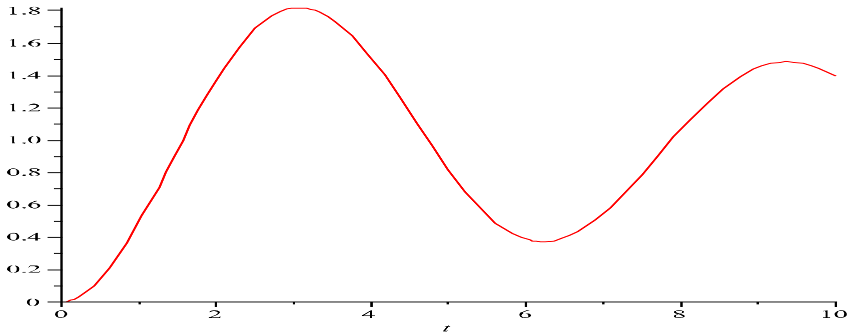

图 4　数值解($\alpha=1.9$)

Fig. 4　Numerical solution ($\alpha=1.9$)

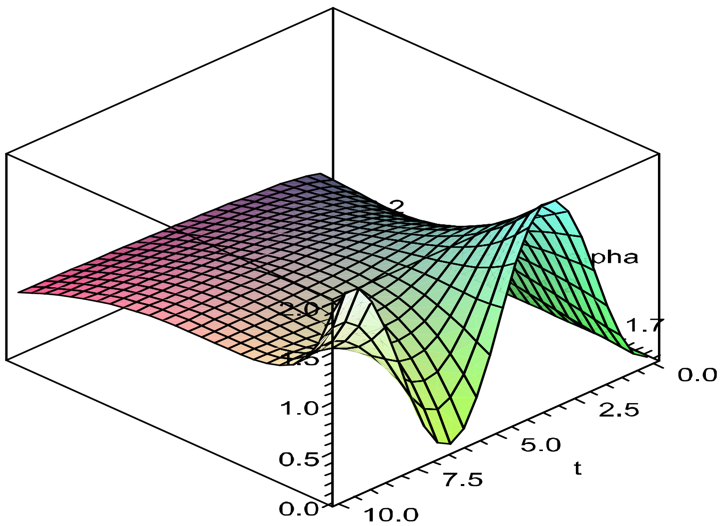

图 5　数值解($0<\alpha<2$)

Fig. 5　Numerical solution ($0<\alpha<2$)

图 1-图 4 分别为方程在 $\alpha=0.5,0.9,1.5,1.9$ 时数值解的图像。图 5 则表示 α 从 0 变化到 2 $(0<\alpha<2)$ 的图像。该图像显示了解从松弛状态逐渐变化到震荡状态的过程。

例 2.3 考虑如下非线性分数阶 predator-prey 系统

$$\begin{cases} {}_a^C D_t^{\alpha_1} x(t) = x(t) - x(t)y(t), \\ {}_a^C D_t^{\alpha_2} y(t) = -y(t) + x(t)y(t), \end{cases} \tag{3.4}$$

初始条件为

$$x(0)=1,\quad y(0)=0.5. \tag{3.5}$$

根据(1.4)，问题(3.4)–(3.5)可化为如下 Volterrra 积分方程组

$$\begin{cases} x(t) = \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-\tau)^{\alpha_1-1}x(\tau)d\tau - \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-\tau)^{\alpha_1-1}x(\tau)y(\tau)d\tau, \\ y(t) = 0.5 - \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-\tau)^{\alpha_2-1}y(\tau)d\tau - \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-\tau)^{\alpha_3-1}x(\tau)y(\tau)d\tau, \end{cases}$$

再根据迭代公式(1.5)，我们有如下变分迭代公式

$$\begin{cases} x_{n+1}(t) = \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-\tau)^{\alpha_1-1}x_n(\tau)d\tau - \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-\tau)^{\alpha_1-1}x_n(\tau)y_n(\tau)d\tau, \\ y_{n+1}(t) = 0.5 - \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-\tau)^{\alpha_2-1}y_n(\tau)d\tau - \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-\tau)^{\alpha_3-1}x_n(\tau)y_n(\tau)d\tau, \end{cases}$$

用初始值 $x(0)=1,\ y(0)=0.5$ 开始迭代，得到如下迭代值

$$x_1(t) = 1 + \frac{1}{\Gamma(1+\alpha_1)}t^{\alpha_1},$$

$$y_1(t) = 0.5,$$

$$x_2(t) = 1 + \frac{0.5}{\Gamma(1+\alpha_1)}t^{\alpha_1} + \frac{0.25}{\Gamma(1+2\alpha_1)}t^{2\alpha_1},$$

$$y_2(t) = 0.5 + \frac{0.5}{\Gamma(1+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2},$$

图 6 和图 7 分别显示了当 $\alpha_1 = 0.5, \alpha_2 = 0.6$ 与 $\alpha_1 = 1, \alpha_2 = 1$ 时，数值解的图像。从图中我们可以看出计算结果与[14]中得到的结果是一致的。

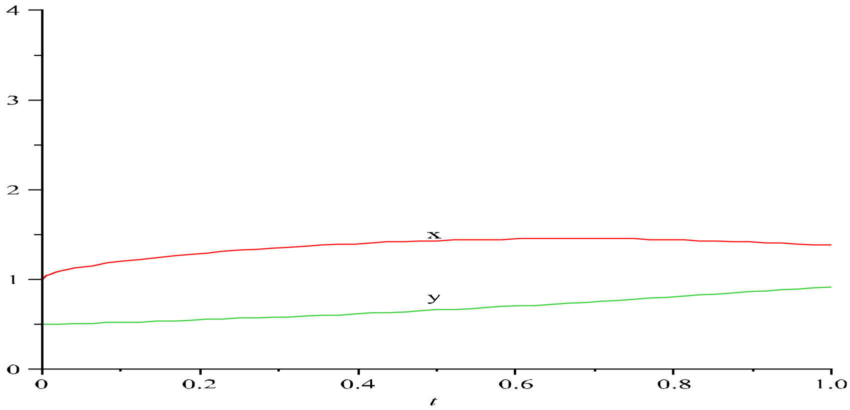

图 6　数值解($\alpha_1 = 0.5, \alpha_2 = 0.6$)

Fig. 6　Numerical solution ($\alpha_1 = 0.5, \alpha_2 = 0.6$)

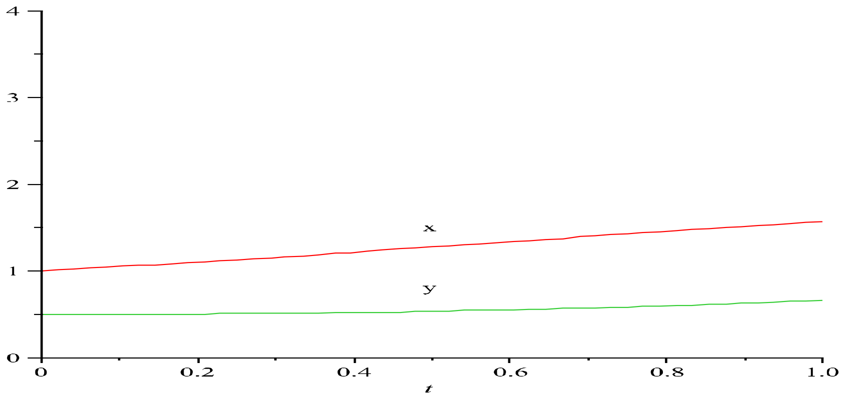

图 7　数值解($\alpha_1=1,\alpha_2=1$)

Fig. 7　Numerical solution ($\alpha_1=1,\alpha_2=1$)

4　结论

从上面的几个例子中，我们可以看出变分迭代法适用于线性和非线性的分数阶微分方程。由于分数阶算子的记忆性，这导致很多计算方法的时间长，计算繁琐。在本文中，我们用 Maple 软件计算，过程简洁，计算时间短，计算结果精度高，对于有些问题收敛于真解的速度快，这些优点都是其它数值方法无法比拟的。从这章中我们可以看出变分迭代法的高效性。

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