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Chiral fermions and the standard model from the matrix model compactified on a torus

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Abstract

It is shown that the IIB matrix model compactified on a six-dimensional torus with a nontrivial topology can provide chiral fermions and matter content close to the standard model on our four-dimensional spacetime. In particular, generation number three is given by the Dirac index on the torus.

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1 Introduction

Matrix models are a promising candidate to formulate the superstring theory nonperturbatively [1, 2], and they indeed include quantum gravity and gauge theory. One of the important subjects in such studies is to connect these models to phenomenology. Spacetime structures can be analyzed dynamically in the IIB matrix model [3], and four dimensionality seems to be preferred [3, 4]. Assuming four-dimensional spacetime is obtained, we next want to show the standard model of particle physics on it. An important ingredient of the standard model is the chirality of fermions. Chirality also ensures existence of massless fermions, since otherwise quantum corrections would induce mass of order of the Planck scale or of the Kaluza-Klein scale in general.

A way to obtain chiral spectrum in our spacetime is to consider topologically nontrivial configurations in the extra dimensions[1]. Owing to the index theorem [7], topological charge of the background provides the index of the Dirac operator, *i.e.*, the difference of the numbers of chiral zero modes, which then produce massless chiral fermions on our spacetime. Generalizations of the index theorem to matrix models or noncommutative (NC) spaces were provided by using a Ginsparg-Wilson (GW) relation[2] developed in the lattice gauge theory [11].

In $M^4 \times S^2 \times S^2$ embeddings in the IIB matrix model, however, we could not obtain chiral spectrum on M^4, even though the IIB matrix model is chiral in ten dimensions, and topological configurations give chiral zero modes on $S^2 \times S^2$, since the remainder dimensions $M^{10}/(M^4 \times S^2 \times S^2)$ interrupt [12]. This obstacle arises generally in the cases with remainder dimensions, such as the coset space constructions. We thus have to consider the situations where topological configurations are embedded in the entire six extra dimensions[3].

We then consider torus compactifications, such as $M^4 \times T^6$ embeddings in the IIB matrix model. A matrix model formulation for gauge theories with adjoint matter in nontrivial topological sectors on a NC torus was given by using the Morita equivalence [13]. For the fundamental matter, since the Morita equivalence is not satisfied in this case, the matrix model formulation was pro-

[1] Having this mechanism in mind, we analyzed dynamics of a model on a fuzzy 2-sphere and showed that topologically nontrivial configurations are indeed realized [5]. Models of four-dimensional field theory with fuzzy extra dimensions were studied in [6].

[2] GW Dirac operators on a fuzzy 2-sphere and a NC torus were given in [8] and [9], respectively. A general formulation for constructing GW Dirac operators on general geometries and defining the corresponding index theorem was provided in [10].

[3] In the case of spheres, if we also embed topological structures in the direction for the thickness of the sphere shell, the problem is resolved.

vided in a purely algebraic way [14].

In this paper, we begin with a gauge theory with adjoint matter in the trivial topological sector, and then introduce block-diagonal matrix configurations as topologically nontrivial gauge field backgrounds. The off-diagonal blocks of the adjoint matter field, which are in the bifundamental representations of the gauge group produced by the background, thus obtain nonzero Dirac indices. We show that such configurations, when embedded in the IIB matrix model, indeed give chiral spectrum on our spacetime. We also study dynamics of these configurations by investigating their classical actions, and find that they appear in the continuum limit as in the gauge theories on the commutative spaces. We finally present an example of configuration which gives matter content close to the standard model.

In section 2, we briefly review the finite matrix formulation of gauge theory with adjoint matter on a NC torus, including the formulation of the GW Dirac operator and the index theorem. Then in section 3, we introduce block-diagonal configurations as topological backgrounds. Explicit forms of the configurations on 2-dimensional and 6-dimensional tori are given in section 4 and section 5, respectively. Dynamics of the configurations are studied in section 4.1. In section 6, we show an example of configuration which gives matter content close to the standard model. Section 7 is devoted to conclusions and discussions. In appendix A, we calculate the index of the GW Dirac operator.

2 Gauge theory with adjoint matter on a NC torus

In this section, we briefly review the finite matrix formulation of gauge theory with adjoint matter on a noncommutative (NC) torus. For details, see [13], for instance. We here consider a simple setting that gives topologically trivial sector, however.

An action for the gauge fields on a d-dimensional NC torus was given by the twisted Eguchi-Kawai model [15, 16]

$$S_b = -\mathcal{N}\beta \sum_{\mu\neq\nu} \mathcal{Z}_{\nu\mu} \mathrm{tr}\ \left(V_\mu\, V_\nu\, V_\mu^\dagger\, V_\nu^\dagger\right) + d(d-1)\beta\mathcal{N}^2\ , \tag{2.1}$$

with $\mu, \nu = 1, \ldots, d$. Here V_μ are $U(\mathcal{N})$ matrices representing the link variables on the lattice, β stands for the lattice gauge coupling constant, and $\mathcal{Z}_{\nu\mu}$ are $Z_{\mathcal{N}}$ factors which are assumed to be specified to give the topologically trivial sector. The constant term is added to make the action vanish at its minimum.

Actions for adjoint matter are given by using covariant forward and backward difference operators

$$
\begin{aligned}
\nabla_\mu \psi &= \frac{1}{\epsilon}\left(V_\mu\, \psi\, V_\mu^\dagger - \psi\right) \ , \\
\nabla_\mu^* \psi &= \frac{1}{\epsilon}\left(\psi - V_\mu^\dagger\, \psi\, V_\mu\right) \ ,
\end{aligned}
\tag{2.2}
$$

with $V_\mu \in U(\mathcal{N})$ introduced above. ϵ is an analog of the lattice spacing. For instance, a Wilson-Dirac operator D_{W} is defined as

$$
D_{\mathrm{W}} = \frac{1}{2}\sum_{\mu=1}^{d}\left\{\gamma_\mu\left(\nabla_\mu^* + \nabla_\mu\right) - \epsilon \nabla_\mu^* \nabla_\mu\right\} \ , \tag{2.3}
$$

where γ_μ are d-dimensional Dirac matrices.

One can also define a Ginsparg-Wilson (GW) Dirac operator as

$$
D_{\mathrm{GW}} = \frac{1}{\epsilon}(1 - \gamma\hat{\gamma}) \ , \tag{2.4}
$$

where γ is an ordinary chirality operator on the d-dimensional space, and $\hat{\gamma}$ is a modified one defined by

$$
\hat{\gamma} = \frac{H}{\sqrt{H^2}} \ , \tag{2.5}
$$

$$
H = \gamma\,(1 - \epsilon D_{\mathrm{W}}) \ , \tag{2.6}
$$

with D_{W} given in (2.3). By the definition (2.4), the Dirac operator satisfies a GW relation

$$
\gamma D_{\mathrm{GW}} + D_{\mathrm{GW}}\hat{\gamma} = 0 \ . \tag{2.7}
$$

Hence, the index, *i.e.*, the difference of the numbers of chiral zero modes, is given by the trace of the chirality operators as

$$
\mathrm{index}(D_{\mathrm{GW}}) = \frac{1}{2}\mathcal{T}r\,[\gamma + \hat{\gamma}] \ , \tag{2.8}
$$

where $\mathcal{T}r$ is the trace over the whole configuration space. Since the definition of $\hat{\gamma}$ depends on the link variables V_μ, the right-hand side (rhs) of (2.8) is a functional of the gauge field configurations. It also takes only integer values. Moreover, it is shown to become the Chern character with star product in the continuum limit for the fundamental matter [17]. It then gives a noncommutative generalization of the topological charge for the gauge field backgrounds. Thus, eq. (2.8) gives an index theorem on the NC torus.

We expect, however, that the rhs of (2.8) vanishes for any configurations V_μ that survive in the continuum limit because of the following reasons: First, the rhs of (2.8) is thought to have an appropriate continuum limit, as shown for the fundamental matter case in [17]. Since the adjoint matter is chiral anomaly free in 2 (mod 4) dimensions, it must vanish. Second, since we now begin with the matrix model (2.1) describing the trivial module, only the topologically trivial sector appears in the continuum limit, as shown in [18, 19]. We therefore need some modifications in order to have nontrivial topologies, which we will study in the next section.

3 Topological configurations

As topologically nontrivial gauge configurations, we introduce the following block-diagonal matrices:

$$V_\mu = \begin{pmatrix} V_\mu^1 & & & \\ & V_\mu^2 & & \\ & & \ddots & \\ & & & V_\mu^h \end{pmatrix} , \tag{3.1}$$

with h blocks and $\mu = 1, \dots, d$.

We also introduce the following projection operators P^a with $a = 1, \dots, h$, which pick up the space that ath block acts:

$$P^a = \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & \mathbb{1} & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} . \tag{3.2}$$

Since P^a commute with the chirality operator (2.5) and the Dirac operator (2.4), the index theorem (2.8) is satisfied in each space projected by P^a as

$$\mathrm{index}(P^{aL} P^{bR} D_{\mathrm{GW}}) = \frac{1}{2} \mathcal{T}r \left[P^{aL} P^{aR} (\gamma + \hat{\gamma}) \right] , \tag{3.3}$$

where the superscript L (R) means that the operator acts from left (right) on matrices: $\mathcal{O}^L M \equiv \mathcal{O} M$, $\mathcal{O}^R M \equiv M \mathcal{O}$. $P^{aL} P^{bR}$ picks up the following block ψ^{ab} from the matter field ψ in the adjoint representation:

$$\psi = \begin{pmatrix} \psi^{11} & \psi^{12} & \cdots & \psi^{1h} \\ \psi^{21} & \psi^{22} & \cdots & \psi^{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \psi^{h1} & \psi^{h2} & \cdots & \psi^{hh} \end{pmatrix} , \tag{3.4}$$

if we decompose ψ into blocks in the same way as (3.1). The diagonal blocks ψ^{aa} are in the adjoint representations under the gauge group, while the off-diagonal blocks ψ^{ab} with $a \neq b$ are in the bifundamental representations. As shown in the following sections, the index of each block (3.3) can have nonzero values, although the total matrix ψ has a vanishing index.

In the remainder of this section, we show that, by embedding the configurations (3.1) with $d = 6$ in the IIB matrix model, chiral fermions on our four-dimensional spacetime are obtained. See [12] for detailed arguments. For $d = 2$ (mod 4), topological charge becomes the $(d/2)$th Chern character, with $d/2$ being an odd integer. Hence, ψ^{ab} and ψ^{ba}, which are in the conjugate representations under the gauge group, have the opposite indices. We denote the corresponding chiral zero modes as ψ_R^{ab} and ψ_L^{ba}, where the subscripts R and L stand for the chirality. (Choosing ψ_L^{ab} and ψ_R^{ba} instead would give the identical results.) Taking spinors φ on our four-dimensional spacetime as well, we obtain the following possible Weyl spinors:

$$\varphi_R \otimes \psi_R^{ab} \ , \tag{3.5}$$

$$\varphi_L \otimes \psi_L^{ba} \ , \tag{3.6}$$

$$\varphi_L \otimes \psi_R^{ab} \ , \tag{3.7}$$

$$\varphi_R \otimes \psi_L^{ba} \ . \tag{3.8}$$

The spinors (3.5) and (3.6) are in the charge conjugate representations to each other. So are (3.7) and (3.8).

Since the IIB matrix model has the ten-dimensional Majorana-Weyl spinor, we now impose these conditions. By the Weyl condition, (3.5) and (3.6) are chosen. (Choosing (3.7) and (3.8) gives the identical results.) Since φ_R in (3.5) and φ_L in (3.6) are in the different representations under the gauge group, they give chiral spectrum on our spacetime, although we have a doubling of (3.5) and (3.6). Furthermore, by the Majorana condition, (3.5) and (3.6) are identified. (So are (3.7) and (3.8).) Then, the unwanted doubling of (3.5) and (3.6) is also resolved.

4 2-dimensional torus

In this section, we show explicit forms of the configurations (3.1) with $d = 2$. In the context of $M^4 \times T^6$ embeddings in the IIB matrix model, this T^2 corresponds to the one in $T^6 = T^2 \times T^2 \times T^2$.

We consider the configurations

$$V_\mu = \begin{pmatrix} \Gamma_\mu^1 \otimes \mathbb{1}_{p^1} & & & \\ & \Gamma_\mu^2 \otimes \mathbb{1}_{p^2} & & \\ & & \ddots & \\ & & & \Gamma_\mu^h \otimes \mathbb{1}_{p^h} \end{pmatrix} , \tag{4.1}$$

with $\mu = 1, 2$. Γ_μ^a with $a = 1, \ldots, h$ are the shift operators on the dual tori specified by a set of integers n^a, m^a, j^a, k'^a. See ref. [14] for detail. In fact, the configurations (4.1) are classical solutions for the action (2.1), as shown in [18].

The integers satisfy the Diophantine equation

$$m^a j^a + n^a k'^a = 1 \tag{4.2}$$

for each a. The integers specifying the original torus also satisfy the Diophantine equation

$$2rs - kN = -1 \ . \tag{4.3}$$

The dual tori and the original torus are related by integers q^a, which specify magnetic fluxes on the dual tori, as[4]

$$m^a = -s + kq^a \ , \quad n^a = N - 2rq^a \ . \tag{4.4}$$

Equations (4.4) can be inverted as

$$1 = 2rm^a + kn^a \ , \quad q^a = Nm^a + sn^a \ . \tag{4.5}$$

Explicit forms of the coordinate and the shift operators on the dual tori are given, for instance, as

$$\begin{aligned} Z_1^a = W_{n^a} \quad &, \quad Z_2^a = (V_{n^a})^{j^a} \ , \\ \Gamma_1^a = V_{n^a} \quad &, \quad \Gamma_2^a = (W_{n^a})^{-m^a} \ , \end{aligned} \tag{4.6}$$

in terms of the shift and clock matrices

$$V_n = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} , \quad W_n = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/n} & & & \\ & & \mathrm{e}^{4\pi i/n} & & \\ & & & \ddots & \\ & & & & \mathrm{e}^{2\pi i(n-1)/n} \end{pmatrix} , \tag{4.7}$$

[4] In [14], the dual torus is determined by the two integers p and q, which specify the gauge group $U(p)$ and the abelian flux. The present case corresponds to $p = p^a, q = p^a q^a$, and hence $p_0 = p^a, \tilde{p} = 1, \tilde{q} = q^a$.

which are $U(n)$ matrices obeying the commutation relations

$$V_n W_n = \mathrm{e}^{2\pi i/n} W_n V_n \ . \tag{4.8}$$

The off-diagonal block ψ^{ab} in (3.4) can be interpreted as in the fundamental representation, if we identify the bth block as an original torus. The corresponding integer q is thus given by (4.5), with N and s replaced by n^b and $-m^b$, respectively. Substituting (4.4) and using (4.3), we obtain

$$n^b m^a - m^b n^a = q^a - q^b \ . \tag{4.9}$$

Then, the index for the block ψ^{ab} (3.3) should become

$$\frac{1}{2}\mathcal{T}r\,[P^{aL}P^{aR}(\gamma+\hat{\gamma})] = p^a p^b (q^a - q^b) \ . \tag{4.10}$$

Indeed, as shown by the explicit calculations in appendix A, eq.(4.10) is satisfied in general, except for the rare cases with $|r| = 1$, $n^a = 1$, and $n^b = 2|q^a - q^b| + 1$, or the cases with n^a and n^b reversed. As far as we consider the cases with the block sizes n^a greater than one, eq.(4.10) is satisfied. The Monte Carlo results in [20] also support (4.10). Equation (4.10) means that the index of each component in the $(p^a, \bar{p^b})$ representation under the gauge group $U(p^a) \times U(p^b)$ is $q^a - q^b$. By using a relation

$$n^a - n^b = -2r(q^a - q^b) \tag{4.11}$$

given by (4.4), eq.(4.10) is rewritten as

$$\frac{1}{2}\mathcal{T}r\,[P^{aL}P^{aR}(\gamma+\hat{\gamma})] = -\frac{1}{2r} p^a p^b (n^a - n^b) \ . \tag{4.12}$$

The same equation was given for the fuzzy 2-sphere case in eq.(5.4) of [12][5], except for the factor $2r$.

4.1 Classical actions

We now study dynamics of the configurations (4.1) by evaluating their classical actions (2.1). Similar analyses were given in [18], but the present case corresponds to the situation where all the configurations are in the topologically trivial sector in the sense of [18], where topology was defined in terms of the total matrix. Now, the nontrivial topologies arise from the blocks, as explained in section 3.

[5] The case with the fundamental matter was studied in [21, 22, 23]. The formulation was further extended to $S^2 \times S^2$ in [24].

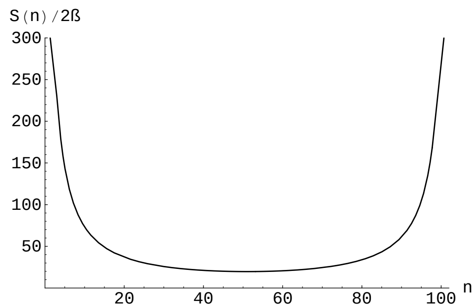

Figure 1: The classical action (4.17) as a function of n is displayed. We here take $\mathcal{N} = 153$ and $n^3 = 51$.

We take $p^1 = \cdots = p^h = 1$ without loss of generality. We also choose the integers r and k specifying the original torus to be $r = -1$, $k = -1$, which give $s = \frac{N+1}{2}$ from (4.3), following the previous works [18, 19, 20]. From (4.4), $n^a = N + 2q^a$ and $m^a = -\frac{n^a+1}{2}$ are determined. It then follows form (4.6) that

$$\Gamma_1^a \Gamma_2^a = \mathrm{e}^{2\pi i \frac{n^a+1}{2n^a}} \Gamma_2^a \Gamma_1^a \ . \tag{4.13}$$

Choosing the phase $\mathcal{Z}_{\mu\nu}$in the action (2.1) as

$$\mathcal{Z}_{12} = \mathrm{e}^{2\pi i \frac{\mathcal{N}+1}{2\mathcal{N}}} \ , \tag{4.14}$$

the actions for the configurations (4.1) become

$$S = -2\mathcal{N}\beta \sum_{a=1}^{h} n^a \cos\left(\pi\left(\frac{1}{\mathcal{N}} - \frac{1}{n^a}\right)\right) + 2\beta\mathcal{N}^2 \ . \tag{4.15}$$

For h blocks with the same sizes, $n^1 = \ldots = n^h$, (4.15) becomes

$$S^h = \beta\pi^2(h-1)^2 - \frac{1}{12}\beta\pi^4(h-1)^4\frac{1}{\mathcal{N}^2} + \mathcal{O}\left((1/\mathcal{N})^4\right) \ . \tag{4.16}$$

We now study the cases where the block sizes are different. For simplicity, we consider the cases with $h = 3$ and $\mathcal{N}$ and n^3 fixed. They correspond to the cases where we focus on the two blocks with the other $h - 2$ blocks fixed. The action (4.15) for $n = n^1$ becomes

$$S(n) = -2\mathcal{N}\beta\left[n\cos\left(\pi\left(\frac{1}{\mathcal{N}} - \frac{1}{n}\right)\right) + (\mathcal{N} - n^3 - n)\cos\left(\pi\left(\frac{1}{\mathcal{N}} - \frac{1}{\mathcal{N} - n^3 - n}\right)\right)\right] \ , \tag{4.17}$$

where we did not write the constant terms. The action $S(n)$ has its minimum at $n = \frac{\mathcal{N}-n^3}{2}$ with a flat plateau around it, as shown in figure 1. The function

$S(n)$ is in fact symmetric at $n = \frac{\mathcal{N}-n^3}{2}$ and convex downwards. Moreover, by expanding in $1/(\mathcal{N}-n^3)$, we obtain

$$S\left(\frac{\mathcal{N}-n^3}{2}+m\right)-S\left(\frac{\mathcal{N}-n^3}{2}\right) = 16\pi^2\beta\frac{m^2}{(\mathcal{N}-n^3)^2}+\mathcal{O}\left(1/(\mathcal{N}-n^3)^3\right) \ . \tag{4.18}$$

The difference of the block sizes $n^1 - n^2 = 2m$ is also given as (4.11). Thus, (4.18) becomes

$$\Delta S \simeq 16\pi^2\beta r^2\frac{(q^1-q^2)^2}{(\mathcal{N}-n^3)^2} \ . \tag{4.19}$$

Therefore, within the configurations with restricted number of blocks, the topological configurations appear in the continuum limit, since the continuum limit is taken by sending β and $\mathcal{N}$ to infinity with $\beta/\mathcal{N}$ fixed [25].

This situation agrees with the cases in gauge theories on the commutative spaces, where one has

$$\Delta S_{\mathrm{com}} = 4\pi^2\beta\left(\frac{q}{(\mathcal{N}-n^3)/2}\right)^2 \ , \tag{4.20}$$

which becomes $4\pi^2(q/gL)^2$ in the continuum limit, where $L = \epsilon(\mathcal{N}-n^3)/2$ is the physical size of the torus, and g is the gauge coupling constant. This is contrary to the cases in [18, 19], where topologies were defined by the total matrix, not by the blocks as in the present case, on the NC torus. There, studies by classical actions and Monte Carlo calculations gave $\Delta S \sim \beta(\mathcal{N}-n^3)$, or $\Delta S \sim \beta$ at best, and topologically nontrivial configurations did not survive in the continuum limit [18, 19].

5 6-dimensional torus

Extension of the configurations (4.1) to six dimensions is straightforward. They are given as

$$
\begin{aligned}
V_\mu &= \begin{pmatrix} \Gamma^1_{1,\mu}\otimes \mathbb{1}_{n^1_2}\otimes \mathbb{1}_{n^1_3}\otimes \mathbb{1}_{p^1} & & & \\ & \Gamma^2_{1,\mu}\otimes \mathbb{1}_{n^2_2}\otimes \mathbb{1}_{n^2_3}\otimes \mathbb{1}_{p^2} & & \\ & & \ddots & \\ & & & \Gamma^h_{1,\mu}\otimes \mathbb{1}_{n^h_2}\otimes \mathbb{1}_{n^h_3}\otimes \mathbb{1}_{p^h} \end{pmatrix} , \\
V_{2+\mu} &= \begin{pmatrix} \mathbb{1}_{n^1_1}\otimes \Gamma^1_{2,\mu}\otimes \mathbb{1}_{n^1_3}\otimes \mathbb{1}_{p^1} & & & \\ & \mathbb{1}_{n^2_1}\otimes \Gamma^2_{2,\mu}\otimes \mathbb{1}_{n^2_3}\otimes \mathbb{1}_{p^2} & & \\ & & \ddots & \\ & & & \mathbb{1}_{n^h_1}\otimes \Gamma^h_{2,\mu}\otimes \mathbb{1}_{n^h_3}\otimes \mathbb{1}_{p^h} \end{pmatrix} , \\
V_{4+\mu} &= \begin{pmatrix} \mathbb{1}_{n^1_1}\otimes \mathbb{1}_{n^1_2}\otimes \Gamma^1_{3,\mu}\otimes \mathbb{1}_{p^1} & & & \\ & \mathbb{1}_{n^2_1}\otimes \mathbb{1}_{n^2_2}\otimes \Gamma^2_{3,\mu}\otimes \mathbb{1}_{p^2} & & \\ & & \ddots & \\ & & & \mathbb{1}_{n^h_1}\otimes \mathbb{1}_{n^h_2}\otimes \Gamma^h_{3,\mu}\otimes \mathbb{1}_{p^h} \end{pmatrix} ,
\end{aligned}
\tag{5.1}
$$

with $\mu = 1, 2$. In $\Gamma^a_{l,\mu}$, n^a_l, and p^a, $a = 1, \ldots, h$ specifies block, and $l = 1, 2, 3$ specifies lth T^2 in $T^6 = T^2 \times T^2 \times T^2$.

The operators $\Gamma^a_{l,\mu}$ are shift operators on the dual tori specified by a set of integers $n^a_l, m^a_l, j^a_l, k'^a_l$, while the original tori are specified by N_l, s_l, r_l, k_l. The integers satisfy the Diophantine equations

$$m^a_l j^a_l + n^a_l k'^a_l = 1 \ , \tag{5.2}$$

$$2 r_l s_l - k_l N_l = -1 \ , \tag{5.3}$$

for each $a = 1, \ldots, h$ and $l = 1, 2, 3$. The dual tori and the original tori are related by integers q^a_l as

$$m^a_l = -s_l + k_l q^a_l \ , \quad n^a_l = N_l - 2 r_l q^a_l \ , \tag{5.4}$$

for each a and l. Equations (5.4) can be inverted as

$$1 = 2 r_l m^a_l + k_l n^a_l \ , \quad q^a_l = N_l m^a_l + s_l n^a_l \ . \tag{5.5}$$

Explicit forms of the coordinate and the shift operators on the dual tori are given, for instance, as

$$
\begin{aligned}
Z^a_{l,1} = W_{n^a_l} \quad &, \quad Z^a_{l,2} = (V_{n^a_l})^{j^a_l} \ , \\
\Gamma^a_{l,1} = V_{n^a_l} \quad &, \quad \Gamma^a_{l,2} = (W_{n^a_l})^{-m^a_l} \ ,
\end{aligned}
\tag{5.6}
$$

in terms of the shift and clock matrices (4.7). As shown in [18] , the configurations (5.1) are classical solutions for the action (2.1). Note also that (5.1) represents configurations with flux in each T^2, and does not exhaust all of topological configurations in T^6.

The index for the block ψ^{ab} (3.3) should become

$$\frac{1}{2}\mathcal{T}r\,[P^{aL}P^{aR}(\gamma+\hat{\gamma})] = p^a p^b \prod_{l=1}^{3}(q_l^a - q_l^b)\ . \tag{5.7}$$

This can also be checked as in appendix A. Since numerical calculations take much longer time for the six-dimensional case, we will report on it in a future publication.

6 A standard model embedding in IIB matrix model

We now present an example of configuration (5.1) which, when embedded in the IIB matrix model, gives matter content close to the standard model. The number of blocks is taken to be $h = 4$. The integers q_l^a are taken, for instance, as

$$q_1^{ab} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ & 0 & -1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\ ,\quad q_2^{ab} = \begin{pmatrix} 0 & 1 & 0 & 3 \\ & 0 & -1 & 2 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}\ ,\quad q_3^{ab} = \begin{pmatrix} 0 & 3 & 0 & 1 \\ & 0 & -3 & -2 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\ , \tag{6.1}$$

where we presented $q_l^{ab} = q_l^a - q_l^b$. The lower triangle part is obtained by the upper one from the relation $q_l^{ab} = -q_l^{ba}$. Hence, $q^{ab} = \prod_{l=1}^{3} q_l^{ab}$ becomes

$$q^{ab} = \begin{pmatrix} 0 & 3 & 0 & 3 \\ & 0 & -3 & 0 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}\ . \tag{6.2}$$

The generation number three is obtained, as we will explain in detail below.

We next incorporate gauge group structure by specifying the integers p^a as

$$V_\mu = \begin{pmatrix} \Gamma_\mu^1 \otimes 1\!\!1_3 & & & \\ & \Gamma_\mu^2 \otimes 1\!\!1_2 & & \\ & & \Gamma_\mu^3 & \\ & & & \Gamma_\mu^4 \otimes \sigma_3 \end{pmatrix}\ , \tag{6.3}$$

with $\mu = 1, \ldots, 6$. σ_3 is the Pauli matrix. The gauge group given by this background is $U(3) \times U(2) \times U(1)^3 \simeq SU(3) \times SU(2) \times U(1)^5$.

Fermionic matter content of the standard model is obtained from the fermionic matrix ψ as follows:

$$\psi = \begin{pmatrix} 0 & q & 0 & ud \\ & 0 & \bar{l} & 0 \\ & & 0 & \nu e \\ & & & 0 \end{pmatrix} , \tag{6.4}$$

where each block ψ^{ab} is $n_1^a n_2^a n_3^a p^a \times n_1^b n_2^b n_3^b p^b$ matrices. Here, q denotes the quark doublets, l the lepton doublets, ud the quark singlets, νe the lepton singlets. From (6.2), they all have q^{ab} three. Using (5.7), we find that they have appropriate indices which give generation number three. The other blocks in (6.4) denoted as 0 have a vanishing index and do not give massless particles on our spacetime.

The hypercharge Y is given by a linear combination of five $U(1)$ charges presented below (6.3) as

$$Y = \sum_{i=1}^{5} x^i Q^i , \tag{6.5}$$

where $Q^i = \pm 1$ with $i = 1, \ldots, 5$ is the charge of ith $U(1)$ gauge group. From the hypercharge of q, u, d, l, ν, and e, the following constraints are obtained:

$$\begin{aligned} x^1 - x^2 = 1/6 \ , \quad x^1 - x^4 = 2/3 \ , \quad x^1 - x^5 = -1/3 \ , \\ -x^2 + x^3 = -1/2 \ , \quad x^3 - x^4 = 0 \ , \quad x^3 - x^5 = -1 \ . \end{aligned} \tag{6.6}$$

Their general solutions are given by

$$x^1 = 2/3 + c \ , \quad x^2 = 1/2 + c \ , \quad x^3 = x^4 = c \ , \quad x^5 = 1 + c \ , \tag{6.7}$$

with c being an arbitrary constant.

7 Conclusions and Discussions

In this paper, we first introduced block-diagonal matrices for topologically nontrivial gauge field configurations on a NC torus, and found that the off-diagonal blocks of the adjoint matter can have nonzero Dirac indices. We then showed that, by embedding these configurations in the IIB matrix model, chiral fermions and matter content close to the standard model can be obtained on our four-dimensional spacetime. In particular, generation number three was given by the Dirac index on the torus. Lots of things remain to be clarified, some of which we list below. We will report on these issues in future publications.

Our model close to the standard model gave five $U(1)$ gauge fields. The hypercharge $U_Y(1)$ will remain massless, while the others become massive by some dynamics of the matrix model, or of the field theories which arise as low energy effective theories of the matrix model. While we did not discuss the Higgs field in the present paper, it should be introduced, and the mechanism of electroweak symmetry breaking and values of the Yukawa couplings should also be studied.

Our model is reminiscent of the intersecting D-brane models [26, 27]. There, one can obtain four-dimensional chiral fermions by the same reason as ours, that is, one has no remainder dimensions normal to all of the D-branes intersecting to one another [28]. The model [26] gives the standard model matter content. Since that setting is related to ours by the T-duality, it is interesting to compare them to each other. These studies will give progress in both string theories and matrix models.

In this paper we studied dynamics of the configurations by using the classical actions in the two-dimensional case, and found that topologically nontrivial configurations appear in the continuum limit, within the configurations with restricted number of blocks, as in the commutative theories. This is contrast to the cases in [18, 19], where topologies were defined by the total matrix, not by the blocks, and only the topologically trivial sector survives in the continuum limit. For studying higher dimensional cases, however, quantum corrections become relevant and should be taken into account. Owing to the quantum corrections with the noncommutativity of the torus, a topologically nontrivial sector may arise with higher probability than the trivial sector, as shown in [19]. Then, the generation number three might be chosen dynamically.

We hope to study dynamics in wider regions of the configuration space, including various compactifications, in the IIB matrix model. From these studies, we might be able to find that the standard model or its extension is obtained as a unique solution from the IIB matrix model or its variants. Or, more complicated structures of the vacuum, such as the landscape [29], might be found. Even in this case, since the matrix model has the definite measure as well as the action, we can define probabilities taking account of the measure, and discuss entropy on the landscape. The matrix models make these studies possible.

A Calculations of the index

In this appendix, we calculate the index of the Dirac operator for the backgrounds (4.1) and confirm that eq. (4.10) is indeed satisfied. It is sufficient to consider the case with $h = 2$ and $p^1 = p^2 = 1$. For the off-diagonal block ψ^{12} of the matter field ψ, the operation $V_\mu \psi V_\mu^\dagger$ becomes $\Gamma_\mu^1 \psi^{12} \Gamma_\mu^{2\dagger}$. We hereafter will write ψ^{12} simply as ψ. By using the explicit forms of Γ^a_μ in (4.6), we obtain

$$\begin{aligned} (\Gamma^1_1\psi\Gamma^{2\dagger}_1)_{i,j} &= \psi_{i+1,j+1} \ , \\ (\Gamma^1_2\psi\Gamma^{2\dagger}_2)_{i,j} &= (\omega_{n^1})^{-m^1(i-1)}(\omega_{n^2})^{m^2(j-1)}\psi_{i,j} \ , \end{aligned} \tag{A.1}$$

with $\omega_n = \mathrm{e}^{2\pi i/n}$. Here, ψ_{ij} represent ij components of the matrix ψ.

The matrix ψ is $n^1 \times n^2$, and (A.1) is invariant under identifications $i \sim i+n^1$ and $j \sim j+n^2$. When n^1 and n^2 are coprime, $\psi_{i,j}$ with $i=1,\dots,n^1$ and $j=1,\dots,n^2$ are mapped one-to-one by the above identifications to $\psi_{i,i}$ with $i = 1,\dots,n^1n^2$, which we denote as ψ_i:

$$\psi_{i,j} \sim \psi_{i,i} \equiv \psi_i \ . \tag{A.2}$$

Then, (A.1) is rewritten as

$$\begin{aligned} (\Gamma^1_1\psi\Gamma^{2\dagger}_1)_i &= \psi_{i+1} \ , \\ (\Gamma^1_2\psi\Gamma^{2\dagger}_2)_i &= (\omega_{n^1n^2})^{-q^{12}(i-1)} \ , \end{aligned} \tag{A.3}$$

with $q^{12} = q^1 - q^2$. In the second equation, we used the relation (4.9). $\Gamma^{1\dagger}_1\psi\Gamma^2_1$ and $\Gamma^{1\dagger}_2\psi\Gamma^2_2$ are similarly estimated. It then follows from (2.2) that

$$\begin{aligned} \epsilon((\nabla^*_1+\nabla_1)\psi)_i &= \psi_{i+1}-\psi_{i-1} \ , \\ \epsilon((\nabla^*_2+\nabla_2)\psi)_i &= -2i\sin(\frac{2\pi}{n^1n^2}q^{12}(i-1))\psi_i \ , \\ \epsilon^2(\nabla^*_1\nabla_1\psi)_i &= \psi_{i+1}-2\psi_i+\psi_{i-1} \ , \\ \epsilon^2(\nabla^*_2\nabla_2\psi)_i &= 2[\cos(\frac{2\pi}{n^1n^2}q^{12}(i-1))-1]\psi_i \ . \end{aligned} \tag{A.4}$$

The operator H in (2.6) is written as

$$H = \begin{pmatrix} 1+\frac{\epsilon^2}{2}(\nabla^*_1\nabla_1+\nabla^*_2\nabla_2) & -\frac{\epsilon}{2}(\nabla^*_1+\nabla_1)+i\frac{\epsilon}{2}(\nabla^*_2+\nabla_2) \\ \frac{\epsilon}{2}(\nabla^*_1+\nabla_1)+i\frac{\epsilon}{2}(\nabla^*_2+\nabla_2) & -1-\frac{\epsilon^2}{2}(\nabla^*_1\nabla_1+\nabla^*_2\nabla_2) \end{pmatrix} \tag{A.5}$$

by taking $\gamma_\mu = \sigma_\mu$ for $\mu = 1,2$ and $\gamma = \sigma_3$. Equations (A.4) and (A.5) give the explicit operation of H on $\psi_{i,\alpha}$ where $\alpha = 1,2$ is spinor index. In particular, the operator H depends only on the two integers n^1n^2 and q^{12}.

The index of the GW Dirac operator is given by the difference of the numbers of the positive and negative eigenvalues of the operator H. We thus diagonalized it numerically. In figure 2, we plot the indices for various values of q^{12} with n^1n^2 fixed. The result is periodic in q^{12} with periodicity n^1n^2, and asymmetric under an exchange of q^{12} to $-q^{12}$. The graphs have similar forms irrespective of the values of n^1n^2. For $n^1n^2 = 399$, which is presented in the left figure, we find that the index takes the identical value with q^{12}, and thus eq.(4.10) is

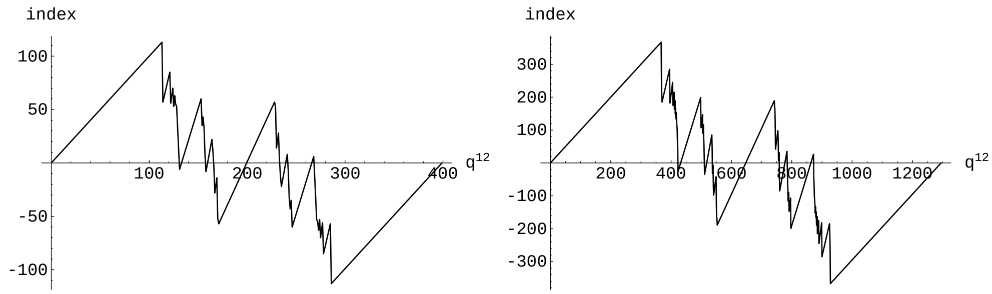

Figure 2: The indices are plotted for various values of q^{12} with n^1n^2 fixed. In the left, we take $n^1n^2 = 399$, while in the right, we take $n^1n^2 = 1295$.

satisfied, in the region $|q^{12}| \leq 113$. For $n^1n^2 = 1295$, it is satisfied in the region $|q^{12}| \leq 367$.

In figure 3, we plot the values of n^1n^2 and q^{12}, where eq.(4.10) is not satisfied. Because of the periodicity in q^{12}, it is enough to survey the region $-(n^1n^2 - 1)/2 \leq q^{12} \leq (n^1n^2 - 1)/2$ for odd n^1n^2, and $-n^1n^2/2 + 1 \leq q^{12} \leq n^1n^2/2$ for even n^1n^2. From the left figure, we find that, within $n^1n^2 \leq 21$, eq.(4.10) is satisfied at least in the region $|q^{12}| < (2/7)n^1n^2$. For $n^1n^2 \leq 101$, which is presented in the right figure, such safety region that ensures (4.10) becomes $|q^{12}| < (23/81)n^1n^2$. For $n^1n^2 \leq 201$, it becomes $|q^{12}| < (44/155)n^1n^2$. For $n^1n^2 \leq 501$, it becomes $|q^{12}| < (128/451)n^1n^2$. The coefficients $2/7, 23/81, 44/155, 128/451$ slightly decrease as we increase n^1n^2. They actually take

$$\frac{(22+1)l + (20+1)m}{(77+4)l + (70+4)m} \tag{A.6}$$

with $l = 1$ and $m = 0, 1, \ldots, 24$ up to $n^1n^2 = 1857$ [6], and thus they are bounded from below by 21/74. We then conclude that, for any values of n^1n^2, eq.(4.10) is satisfied at least in the region $|q^{12}| < (1/3.53)n^1n^2$.

In fact, from the constraint (4.11), n^1n^2 and q^{12} are required to satisfy

$$n^1n^2 = 2|rq^{12}|n + (n)^2 \ , \tag{A.7}$$

for some positive integer n. Then, only the cases with $|r| = 1$ and $n = 1$, which give $n^1n^2 = 2|q^{12}| + 1$, are really allowed in the dotted region in figure 3, where eq.(4.10) is not satisfied. They correspond to the highest and lowest points for odd n^1n^2 in figure 3. We therefore find that eq.(4.10) is satisfied in general, except for the rare cases with $|r| = 1$, $n^1 = 1$, and $n^2 = 2|q^{12}| + 1$, or the cases with n^1 and n^2 reversed.

[6] The pattern (A.6) further continues as with $l = 2$ and $m = 24, 25, \ldots$, though the safety region does not change unless m goes beyond 48. We have checked this pattern until $m = 45$, that is, $n^1n^2 = 3492$.

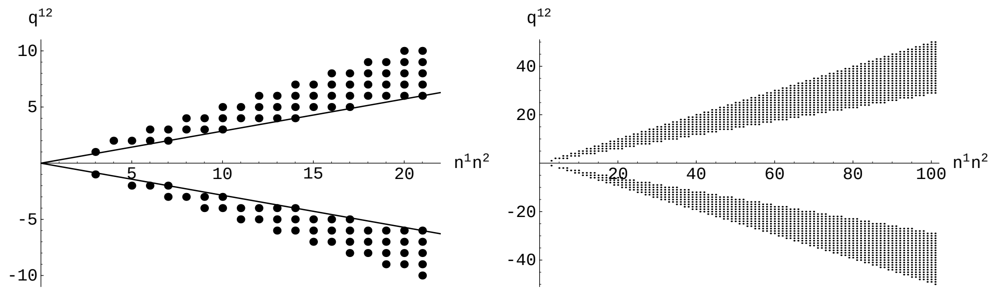

Figure 3: The values of n^1n^2 and q^{12}, where eq.(4.10) is not satisfied, are plotted. Because of the periodicity in q^{12}, we survey the region $-(n^1n^2-1)/2 \leq q^{12} \leq (n^1n^2-1)/2$ for odd n^1n^2, and $-n^1n^2/2+1 \leq q^{12} \leq n^1n^2/2$ for even n^1n^2. In the left, the region $3 \leq n^1n^2 \leq 21$ is shown, while in the right, the region $3 \leq n^1n^2 \leq 101$ is shown. The lines in the left figure represent $q^{12} = \pm(2/7)n^1n^2$.

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