D-polynomials and Taylor formula in quantum calculus

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ABSTRACT

Quantum calculus based on the right invertible divided difference operator D^{τ}_{σ} is proposed here in context of algebraic analysis [9]. The linear operator D^{τ}_{σ}, specified with the help of two fixed maps $\sigma\ ,\tau\colon M\to M$, generalizes the quantum derivative operator used in h- or q-calculus [5]. In the domain of D^{τ}_{σ} there are special elements defined as D^{τ}_{σ}-polynomials and the corresponding Taylor formula is proved.

Keywords: quantum calculus; Taylor formula; difference operator; right invertible operator
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1 Introduction

The usual h- or q-calculus as well as many other types of quantum calculi are specified by a fixed right invertible linear operator D, the so-called quantum derivative. The right inverses of D allow us to define the concept of indefinite D-integrals. Then, for example by applying Jackson formula, one can define the corresponding definite integrals [4, 5]. On the other hand, the calculus of right invertible operators turns out to be a part of algebraic analysis developed by D. Przeworska-Rolewicz. For a right invertible operator D, its right inverses give rise to the corresponding indefinite D-integrals, while definite D-integrals are defined with the help of the corresponding initial operators. For the comprehensive study of the topic we recommend Ref. [9].

The general concept of D-polynomials defined for a right invertible linear operator D is analyzed in Section 2. In particular, an interesting result is that the dimension of the linear space $P_n(D)$ of all D-polynomials of degree less or equal n, $n \in \mathbb{N}$, additionally depends on the dimension of $ker D$ (the so-called space of D-constants), i.e. $dim P_n(D) = (n+1) \cdot ker D$, which is infinite if $ker D$ is of infinite dimension. Then, in Section 3 we define the so-called (σ, τ)-quantum derivative as a divided difference operator D_σ^τ based on three fixed mappings $\sigma, \tau \colon M \to M$ (shifts) and $\theta \colon M \times M \to \mathbb{R}$ (tension function), specifying the essence of quantum calculus considered. One can show that D_σ^τ is right invertible [7]. In analogy to the usual concept of polynomials, their (σ, τ)-quantum counterparts are defined in Section 4 for the assumed (σ, τ)-quantum derivative D_σ^τ. Finally, in Section 5 the corresponding (σ, τ)-quantum Taylor formula is proved (for analogy with q-calculus see [5]).

2 Polynomials in algebraic analysis

Let X be a linear space over a field $\mathbb{K}$ and $\mathcal{L}(X)$ be the family of all linear mappings $D : U \to V$, for any U, V - linear subspaces of X. We shall use the notation $dom(D) = U$, $codom(D) = V$ and $im\, D = \{Du : u \in U\}$ for the domain, codomain and image of D, correspondingly. For any operators $D_1, D_2 \in \mathcal{L}(X)$ and scalars $k_1, k_2 \in \mathbb{K}$, the linear combination $k_1 D_1 + k_2 D_2$ as well as the composition $D_1 D_2$ are the elements of $\mathcal{L}(X)$ defined on the corresponding domains

$$dom(k_1 D_1 + k_2 D_2) = dom(D_1) \cap dom(D_2)\,, \tag{2.1}$$

and

$$dom(D_1 D_2) = D_2^{-1}(dom(D_1))\,. \tag{2.2}$$

The domains of all linear mappings considered in the sequel will be understood in the sense of formulae (2.1), (2.2).

Throughout this paper we use the notation

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}\,. \tag{2.3}$$

Whenever $D_1 = \ldots = D_m = D \in \mathcal{L}(X)$, we shall write $D^m = D_1 \ldots D_m$, for $m \in \mathbb{N}$, and additionally $D^0 = I \equiv id_{dom(D)}$.

For any $D \in \mathcal{L}(X)$ and $m \in \mathbb{N}$, we assume the notation

$$Z_0(D) = \{0\} \quad \text{and} \quad Z_m(D) = ker D^m \setminus ker D^{m-1} \, . \tag{2.4}$$

Evidently, for any $D \in \mathcal{L}(X)$ there is

$$Z_i(D) \cap Z_j(D) = \emptyset, \tag{2.5}$$

whenever $i \neq j$, and

$$\bigcup_{k=0}^{m} Z_k(D) = ker\, D^m. \tag{2.6}$$

In the sequel we shall use the notation

$$Z(D) = ker\, D\,, \tag{2.7}$$

and refer to $Z(D)$ as the space of constants for $D \in \mathcal{L}(X)$.

Proposition 2.1. *Let $D \in \mathcal{L}(X)$, $m \in \mathbb{N}$, and $Z_i(D) \neq \emptyset$ for $i = 1, \dots, m$. Then, any elements $u_i \in Z_i(D)$, $i = 1, \dots, m$, are linearly independent.*

Proof. Consider a linear combination $u = \lambda_1 u_1 + \dots \lambda_m u_m$ and suppose that $u = 0$ for some coefficients $\lambda_1, \dots, \lambda_m \in \mathbb{K}$. Hence we obtain the sequence of equations: $D^k u = \lambda_{k+1} D^k u_{k+1} + \dots + \lambda_m D^k u_m = 0$, for $k = 1, \dots, m-1$. Step by step, from these equations we compute $\lambda_m = 0, \dots, \lambda_1 = 0$. □

Let us define

$$\mathcal{R}(X) = \{D \in \mathcal{L}(X) : codom(D) = im\, D\}, \tag{2.8}$$

i.e. each element $D \in \mathcal{R}(X)$ is considered to be a surjective mapping (onto its codomain). Thus, $\mathcal{R}(X)$ consists of all right invertible elements.

Definition 2.2. *An operator $R \in \mathcal{L}(X)$ is said to be a right inverse of $D \in \mathcal{R}(X)$ if $dom(R) = im(D)$ and $DR = I \equiv id_{im(D)}$. By $\mathcal{R}_D$ we denote the family of all right inverses of D.*

In fact, $\mathcal{R}_D$ is a nonempty family, since for each $y \in im(D)$ we can select an element $x \in D^{-1}(\{y\})$ and define $R \in \mathcal{R}_D$ such that $R : y \mapsto x$.

The fundamental role in the calculus of right invertible operators play the so-called initial operators, projecting the domains of linear operators onto the corresponding space of their constants.

Definition 2.3. *Any operator $F \in \mathcal{L}(X)$, such that $dom(F) = dom(D)$, $im(F) = Z(D)$ and $F^2 = F$ is said to be an initial operator induced by $D \in \mathcal{R}(X)$. We say that an initial operator F corresponds to a right inverse $R \in \mathcal{R}_D$ whenever $FR = 0$ or equivalently if*

$$F = I - RD\,. \tag{2.9}$$

The family of all initial operators induced by D will be denoted by $\mathcal{F}_D$.

The families $\mathcal{R}_D$ and $\mathcal{F}_D$ uniquely determine each other. Indeed, formula (2.9) characterizes initial operators by means of right inverses, whereas formula

$$R = R' - FR'\,, \tag{2.10}$$

which is independent of R', characterizes right inverses by means of initial operators. Both families $\mathcal{R}_D$ and $\mathcal{F}_D$ are fully characterized by formulae

$$\mathcal{R}_D = \{R + FA : dom\,A = im\,D,\ A \in \mathcal{L}(X)\}, \tag{2.11}$$

$$\mathcal{F}_D = \{F(I - AD) : dom\,A = im\,D,\ A \in \mathcal{L}(X)\}, \tag{2.12}$$

where $R \in \mathcal{R}_D$ and $F \in \mathcal{F}_D$ are fixed arbitrarily.

Let us illustrate the above concepts with two basic examples.

Example 2.4. $X = \mathbb{R}^{\mathbb{R}}$ - the linear space of all functions, $D \in \mathcal{R}(X)$ - usual derivative, i.e. $Dx(t) \equiv x'(t)$, with $dom(D) \subset X$ consisting of all differentiable functions. Then, for an arbitrarily fixed $a \in \mathbb{R}$, by formula $Rx(t) = \int\limits_a^t x(s)ds$ one can define a right inverse $R \in \mathcal{R}_D$ and the initial operator $F \in \mathcal{F}_D$ corresponding to R is given by $Fx(t) = x(a)$.

Example 2.5. $X = \mathbb{R}^{\mathbb{N}}$ - the linear space of all sequences, $D \in \mathcal{R}(X)$ - difference operator, i.e. $(Dx)_n = x_{n+1} - x_n$, for $n \in \mathbb{N}$. A right inverse $R \in \mathcal{R}_D$ is defined by the formulae $(Rx)_1 = 0$ and $(Rx)_{n+1} = \sum\limits_{i=1}^{n} x_i$ while $(Fx)_n = x_1$ defines the initial operator $F \in \mathcal{F}_D$ corresponding to R.

An immediate consequence of Definition 2.3, for an invertible operator $D \in \mathcal{R}(X)$, i.e. $ker\,D = \{0\}$, is that $\mathcal{F}_D = \{0\}$. Therefore, the nontrivial initial operators do exist only for operators which are right invertible but not invertible. The family of all such operators is denoted by

$$\mathcal{R}^+(X) = \{D \in \mathcal{R}(X) : dim\,Z(D) > 0\}. \tag{2.13}$$

Proposition 2.6 (Taylor Formula). *Suppose $D \in \mathcal{R}(X)$ and let $F \in \mathcal{F}_D$ be an initial operator corresponding to $R \in \mathcal{R}_D$. Then the operator identity*

$$I = \sum_{k=0}^{m} R^k F D^k + R^{m+1} D^{m+1}\,, \tag{2.14}$$

holds on $dom(D^{m+1})$, for $m \in \mathbb{N}_0$.

Proof. (Induction) See Ref.[9]. □

Equivalent identity, expressed as

$$x = \sum_{k=0}^{m} R^k F D^k x + R^{m+1} D^{m+1} x\,, \tag{2.15}$$

for $x \in dom(D^{m+1})$ and $m \in \mathbb{N}_0$, is an algebraic counterpart of the Taylor expansion formula, commonly known in mathematical analysis. The first component of the last formula reflects the polynomial part while the second one can be viewed as the corresponding reminder.

Example 2.7. To clearly demonstrate the resemblance of formula (2.15) with the commonly used Taylor expression, we take D, R and F as in Example 2.4. Since there are many forms of the reminders in use, it is more interesting to calculate the polynomial part, which gives the well known result $\sum\limits_{k=0}^{m} R^k F D^k x(t) = \sum\limits_{k=0}^{m} \frac{x^{(k)}(a)}{k!}(t-a)^k$, for any function $x \in dom(D^{m+1})$.

Proposition 2.8. *Let $D \in \mathcal{R}(X)$ and $R \in \mathcal{R}_D$. Then R is not a nilpotent operator.*

Proof. Suppose that $R^n \neq 0$ and $R^{n+1} = 0$, for some $n \in \mathbb{N}$. Then $0 \neq R^n = IR^n = DRR^n = DR^{n+1} = 0$, a contradiction. □

Proposition 2.9. *If $D \in \mathcal{R}^+(X)$, then $Z_m(D) \neq \emptyset$, for any $m \in \mathbb{N}$.*

Proof. The relation $Z_1(D) \neq \emptyset$ is straightforward. Let $R \in \mathcal{R}_D$ and $z \in Z_1(D)$ be arbitrarily chosen elements. Then, for any $m \in \mathbb{N}$, there is $R^{m-1}z \in Z_m(D)$. □

With right invertible operators possessing nontrivial kernels we associate the following concept of D-polynomials.

Definition 2.10. *If $D \in \mathcal{R}^+(X)$, then any element $u \in Z_{m+1}(D)$ is said to be a D-polynomial of degree m, i.e. $deg\, u = m$, for $m \in \mathbb{N}_0$. We assign no degree to the zero polynomial $u \in Z_0(D) \equiv \{0\}$.*

For the convenience' sake, one can also use the convention $deg\, 0 = -\infty$.

Proposition 2.11. *If $D \in \mathcal{R}^+(X)$ and $R \in \mathcal{R}_D$, then for any D-polynomial $u \in Z_{m+1}(D)$ there exist elements $z_0, \ldots, z_m \in Z_1(D)$ such that*

$$u = z_0 + Rz_1 + \ldots + R^m z_m. \tag{2.16}$$

Proof. By formula (2.15) we can write the identity $u = \sum_{k=0}^{m} R^k F D^k u$ since $u \in Z_{m+1}(D)$ and $R^{m+1}D^{m+1}u = 0$. Then we define elements $z_k = FD^k u$, $k = 0, \ldots, m$, which ends the proof. □

Definition 2.12. *Let $D \in \mathcal{R}^+(X)$ and $R \in \mathcal{R}_D$. Then, any element $R^k z \in Z_{k+1}(D)$, for $z \in Z_1$, is said to be an R-homogeneous D-polynomial (or R-monomial) of degree $k \in \mathbb{N}_0$.*

Thus, any D-polynomial $u \in Z_{m+1}(D)$, of degree $deg\, u = m$, is a sum of linearly independent R-homogeneous elements $R^k z_k$, $k = 0, \ldots, m$. The linear space of all D-polynomials is then

$$P(D) = \bigcup_{k=0}^{\infty} ker\, D^k \tag{2.17}$$

whereas

$$P_n(D) = \bigcup_{k=0}^{n} ker\, D^k = ker\, D^n \tag{2.18}$$

is the linear space of all D-polynomials of degree at most $n \in \mathbb{N}_0$.

Let us fix a basis

$$\{\zeta_s\}_{s \in S} \subset Z(D)\,, \tag{2.19}$$

of the linear space $Z(D)$, $D \in \mathcal{R}^+(X)$, and define

$$Z^s(D) = Lin\{\zeta_s\}\,, \tag{2.20}$$

for $s \in S$. Then

$$Z(D) = \bigoplus_{s \in S} Z^s(D)\,. \tag{2.21}$$

Proposition 2.13. *For an arbitrary right inverse $R \in \mathcal{R}_D$, the family $\{R^m\zeta_s \colon s \in S, m \in \mathbb{N}_0\}$ is the basis of the linear space $P(D)$. Naturally, $\{R^m\zeta_s \colon s \in S, m = 0, 1, \ldots n\}$ forms the basis of the linear space $P_n(D)$, for $n \in \mathbb{N}_0$.*

Proof. Let $u = \sum_{i=1}^{k} \sum_{s\in S_i} a_{is} R^{m_i}\zeta_s$, $m_1 < \ldots < m_k$ and $S_i \subset S$ be finite subsets for $i = 1, \ldots, k$. Assume $u = 0$ and calculate $D^{m_k}u = \sum_{s\in S_k} a_{ks}\zeta_s = 0$, which implies $a_{ks} = 0$, for all $s \in S_k$. Hence $u = \sum_{i=1}^{k-1} \sum_{s\in S_i} a_{is}R^{m_i}\zeta_s = 0$ and analogously we get $D^{m_{k-1}}u = \sum_{s\in S_{k-1}} a_{(k-1)s}\zeta_s = 0$, which implies $a_{(k-1)s} = 0$, for all $s \in S_{k-1}$. Similarly we prove that $a_{is} = 0$, for all $s \in S_i$, $i = k-2, \ldots, 1$. Now, let $u \in P(D)$ be a polynomial of degree $deg\, u = n \in \mathbb{N}_0\}$. Then, on the strength of Proposition 2.11, we can write $u = \sum_{k=0}^{n} R^k z_k$, for some elements $z_0, \ldots, z_n \in Z(D)$. In turn, each element z_k can be expressed as a linear combination $z_k = \sum_{s\in S_k} a_{ks}\zeta_s$, for some finite subset of indices $S_k \subset S$. Hence we obtain $u = \sum_{k=0}^{n} \sum\limits_{s\in S_k} a_{ks}R^k\zeta_s$. □

With a right inverse $R \in \mathcal{R}_D$, $s \in S$ and $n \in \mathbb{N}_0$, we shall associate the linearly independent family $\{R^m\zeta_s \colon m \in \{0, \ldots, n\}\}$ forming a basis of the linear space of s-homogeneous D-polynomials

$$V_s^n(D) = Lin\{R^m\zeta_s \colon m \in \{0, \ldots, n\}\}\,, \tag{2.22}$$

(independent of the choice of R) of dimension

$$dim\, V_s^n(D) = n + 1\,, \tag{2.23}$$

being a linear subspace of $P_n(D)$. Then, on the strength of Proposition 2.13, the linear space $P_n(D)$ is a direct sum

$$P_n(D) = \bigoplus_{s\in S} V_s^n(D)\,. \tag{2.24}$$

Corollary 2.14. *If $dim\, Z(D) < \infty$, the following formula holds*

$$dim\, P_n(D) = (n + 1) \cdot dim\, Z(D)\,, \tag{2.25}$$

for any $n \in \mathbb{N}_0$.

Naturally, one can extend formula (2.22) and define

$$V_s(D) = Lin\{R^m\zeta_s \colon m \in \mathbb{N}_0\}\,, \tag{2.26}$$

which is both D- and R-invariant subspace of $P(D)$, i.e.

$$DV_s(D) \equiv \{Du\colon u \in V_s(R)\} \subset V_s(D)\,, \tag{2.27}$$

$$RV_s(D) \equiv \{Ru\colon u \in V_s(D)\} \subset V_s(D)\,. \tag{2.28}$$

Thus, $P(D)$ turns out to be simultaneously D- and R-invariant linear subspace of X, since it can be decomposed as the following direct sum

$$P(D) = \bigoplus_{s\in S} V_s(D)\,. \tag{2.29}$$

Since $P(D)$ is a linear subspace of X, there exists (not uniquely) another linear subspace $Q(D)$ of X such that

$$X = P(D) \oplus Q(D)\,. \tag{2.30}$$

Then, every linear mapping $\phi\colon X \to X$ can be decomposed as the direct sum

$$\phi = \phi_{_P} \oplus \phi_{_Q}\,, \tag{2.31}$$

of two restrictions $\phi_{_P} = \phi_{|P(D)}$ and $\phi_{_Q} = \phi_{|Q(D)}$, i.e. for any $x' \in P(D)$ and $x'' \in Q(D)$ there is $\phi(x'+x'') = \phi_{_P}(x') + \phi_{_Q}(x'')$. In particular, the mappings $D \in \mathcal{R}^+(X)$, $R \in \mathcal{R}_D$ can be decomposed as direct sums $D = D_{_P} \oplus D_{_Q}$, $R = R_{_P} \oplus R_{_Q}$ such that

$$I = DR = D_{_P}R_{_P} \oplus D_{_Q}R_{_Q} = I_{_P} \oplus I_{_Q}\,, \tag{2.32}$$

$$RD = R_{_P}D_{_P} \oplus R_{_Q}D_{_Q}\,, \tag{2.33}$$

which allows for the decomposition of the initial operator F corresponding to R

$$\begin{aligned} F = I - RD = I_{_P} \oplus I_{_Q} - R_{_P}D_{_P} \oplus R_{_Q}D_{_Q} = \\ = (I_{_P} - R_{_P}D_{_P}) \oplus (I_{_Q} - R_{_Q}D_{_Q}) = F_{_P} \oplus F_{_Q}. \end{aligned} \tag{2.34}$$

Proposition 2.15. *Let $D \in \mathcal{R}^+(X)$, $R', R'' \in \mathcal{R}_D$ be any right inverses and $F', F'' \in \mathcal{F}_D$ be the initial operators corresponding to R' and R'', respectively. Then $R := R'_{_P} \oplus R''_{_Q} \in \mathcal{R}_D$ and $F := F'_{_P} \oplus F''_{_Q} \in \mathcal{F}_D$ corresponds to R.*

Proof.

$$DR = (D_P \oplus D_Q)(R'_P \oplus R''_Q) = D_P R'_P \oplus D_Q R''_Q = I_P \oplus I_Q = I\,,$$

$$RD = (R'_P \oplus R''_Q)(D_P \oplus D_Q) = R'_P D_P \oplus R''_Q D_Q\,,$$

$$\begin{aligned} F = F'_P \oplus F''_Q &= (I_P - R'_P D_P) \oplus (I_Q - R''_Q D_Q) = \\ &= I_P \oplus I_Q - R'_P D_P \oplus R''_Q D_Q = I - RD\,. \end{aligned}$$

□

The last results allow one to combine right inverses and initial operators as direct sums of independent components.

3 Divided difference operators in (σ, τ)-quantum calculus

Quantum calculus is based on a difference operator, called also a quantum differential, defined as

$$d^{hq}_{h'q'} f(x) = f(qx + h) - f(q'x + h'), \tag{3.1}$$

with the natural assumption that either $q \neq q'$ or $h \neq h'$. Let us denote by e the identity function, i.e. $e(x) \equiv x$, and define a divided difference operator $D^{hq}_{h'q'}$ by formula

$$D^{hq}_{h'q'} f(x) = \frac{df(x)}{de(x)} \equiv \frac{f(qx + h) - f(q'x + h')}{(q - q')x + h - h'}\,. \tag{3.2}$$

We shall refer to $D^{hq}_{h'q'}$ as the quantum derivative operator [5]. If the parameters are known from context, the simplified notation $d \equiv d^{hq}_{h'q'}$ and $D \equiv D^{hq}_{h'q'}$ is used.

The following four cases of quantum calculus are the most common ones:

1. h-calculus when $h \neq 0$, $h' = 0$ and $q = q' = 1$,
2. q-calculus when $q \neq 1$, $q' = 1$ and $h = h' = 0$,
3. h-symmetric calculus when $h = -h' \neq 0$ and $q = q' = 1$,

4. q-symmetric calculus when $q = q'^{-1} \neq 1$ and $h = h' = 0$.

The expressions $qx + h$, $q'x + h'$ can be replaced by more general ones $\tau(x)$, $\sigma(x)$ and formulae (3.1), (3.2) can be rewritten as

$$d_\sigma^\tau f(x) = f(\tau(x)) - f(\sigma(x)) \,, \tag{3.3}$$

and

$$D_\sigma^\tau f(x) = \frac{d_\sigma^\tau f(x)}{d_\sigma^\tau e(x)} \equiv \frac{f(\tau(x)) - f(\sigma(x)}{\tau(x) - \sigma(x)} \,. \tag{3.4}$$

If the mappings σ and τ are known from context, we shall use the simplified notation $d \equiv d_\sigma^\tau$ and $D \equiv D_\sigma^\tau$.

To prevent the denominator of formula (3.4) from being zero, all functions considered above are defined on the naturally restricted domain

$$M = \{x \in \mathbb{R} : \sigma(x) \neq \tau(x)\} \,. \tag{3.5}$$

The fixed mappings $\sigma, \tau : M \to M$, together with the domain $M \subseteq \mathbb{R}$, specify the type of quantum calculus considered.

In this paper we study a generalization of the quantum calculus presented in Ref. [5]. We assume M to be an arbitrary set and fix two mappings $\sigma, \tau : M \to M$ corresponding to those mentioned above.

However, σ and τ are not real-valued maps, in general. Therefore, to adapt formula (3.4) we need a numeric expression that will play the role of a corresponding denominator. For that purpose we endow M with a tension structure, defined with the help of one or more tension functions [7].

Definition 3.1. *A function $\theta : M \times M \to \mathbb{R}$ is said to be a tension function if*

$$\theta(p_1, p_2) + \theta(p_2, p_3) = \theta(p_1, p_3), \tag{3.6}$$

for any $p_1, p_2, p_3 \in M$.

Directly from definition it follows that a linear combination of tension functions defined on $M \times M$ is a tension function again. Thus, any family of tension functions defined on $M \times M$ generates the corresponding linear space.

Definition 3.2. *Any linear space T of tension functions defined on $M \times M$ is said to be a tension structure on M and the pair (M, T) is called the tension space of dimension $\dim T$.*

Directly from the above definition, one can show that any tension function $\theta \in T$ is skew symmetric, i.e.

$$\theta(p_1, p_2) = -\,\theta(p_2, p_1)\,, \tag{3.7}$$

for any $p_1, p_2 \in M$. With any $\theta \in T$ we associate functions $\theta_q : M \to \mathbb{R}$ defined by

$$\theta_q(p) = \theta(p, q), \tag{3.8}$$

for any $p, q \in M$. Intuitively, θ_q plays the role of a potential function defined on M, associating with any point $p \in M$ the scalar potential $\theta_q(p)$ such that $\theta_q(q) = 0$.

Let (M, T) be a one-dimensional tension space with a tension structure T generated by a single tension function θ and assume that $\theta(\tau(p), \sigma(p)) \neq 0$, for any $p \in M$. Then we define the (σ, τ)-quantum difference operator $d \equiv d_\sigma^\tau$

$$df(p) = f(\tau(p)) - f(\sigma(p)), \tag{3.9}$$

and the (σ, τ)-quantum derivative operator $D \equiv D_\sigma^\tau$

$$Df(p) = \frac{df(p)}{d\theta_q(p)} \equiv \frac{f(\tau(p)) - f(\sigma(p))}{\theta(\tau(p), \sigma(p))}\,, \tag{3.10}$$

for any $p \in M$ (independently of $q \in M$). The following Leibniz rule can be checked easily

$$D(f \cdot g)(p) = f(\tau(p)) \cdot D(g)(p) + D(f)(p) \cdot g(\sigma(p))\,, \tag{3.11}$$

for any $f, g : M \to \mathbb{R}$.

4 D-polynomials

Definition 4.1. *A mapping $\rho : M \to M$ is said to be rightward θ-directed if*

$$\theta(p, \rho(p)) < 0\,, \tag{4.1}$$

and it is said to be leftward θ-directed if

$$\theta(p, \rho(p)) > 0\,, \tag{4.2}$$

for any $p \in M$. We say that ρ is a θ-directed mapping if one of the above conditions holds.

Assume the notation: $\rho^0 = id_M$ and $\rho^n = \rho \circ \rho^{n-1}$, for any $n \in \mathbb{N}$.

Proposition 4.2. *For any θ-directed mapping $\rho : M \to M$, $n \in \mathbb{N}$, the composition ρ^n has no fixed points, i.e.*

$$\rho^n(p) \neq p\,, \tag{4.3}$$

for $p \in M$.

Proof. Let ρ be a rightward θ-directed mapping. Then we have inequalities $\theta(\rho(p), p) > 0$, ... , $\theta(\rho^n(p), \rho^{n-1}(p)) > 0$, for any $n \in \mathbb{N}$ and $p \in M$. Consequently,

$$\theta(\rho^n(p), p) = \theta(\rho^n(p), \rho^{n-1}(p)) + \ldots + \theta(\rho(p), p) > 0\,.$$

Analogously, for a leftward θ-directed mapping we show that $\theta(\rho^n(p), p) < 0$, for any $n \in \mathbb{N}$ and $p \in M$. □

Let us notice that condition (4.3) is not a consequence of the weaker assumption that $\theta(\rho(p), p) \neq 0$, for any $p \in M$. In that case there would be $\rho(p) \neq p$ but not necessarily $\rho^n(p) \neq p$, for any $n \in \mathbb{N}$ and $p \in M$.

Definition 4.3. *We say that θ is homogeneous with respect to ρ (shortly, ρ-homogeneous) if there exists $r \in \mathbb{R}$, the so-called ρ-homogeneity coefficient, such that*

$$\theta(\rho(p_1), \rho(p_2)) = r \cdot \theta(p_1, p_2), \tag{4.4}$$

for any $p_1, p_2 \in M$.

Proposition 4.4. *Let $\rho : M \to M$ be a θ-directed mapping and θ be a ρ-homogeneous tension function. Then, for the ρ-homogeneity coefficient we get $r > 0$.*

Proof. Directly from Definition (4.1) we get $r \neq 0$. Since r is a ρ-homogeneity coefficient for θ and ρ is a θ-directed mapping, $\theta(\rho^2(p), \rho(p))$ and $\theta(\rho(p), p)$ are of common sign and $\theta(\rho^2(p), \rho(p)) = r \cdot \theta(\rho(p), p)$. Hence we conclude that $r > 0$. □

Let $c \in \mathbb{R}$ and define the constant function $\hat{c} : M \to \mathbb{R}$, $\hat{c}(p) := c$. Evidently, $\hat{c} \in ker\, D$, for any $c \in \mathbb{R}$, which means that $dim\, ker\, D > 0$. Hence, by formula (2.13) we can write $D \in \mathcal{R}^+(X)$, for $X = \mathbb{R}^M$.

Now, by using Definition 2.10, let us analyze the elements of $Z_{m+1}(D)$, i.e. D-polynomials of degree $m \in \mathbb{N} \cup \{0\}$. Since $D(\hat{1}) = 0$, the unitary constant function $\hat{1} : M \to \mathbb{R}$ is a D-polynomial of degree $deg\,\hat{1} = 0$, i.e. $\hat{1} \in Z_1(D)$.

On the strength of Proposition 2.11, the explicit form of D-polynomials can be obtained from formula (2.16), provided a right inverse $R \in \mathcal{R}_{\mathcal{D}}$ is known explicitely. If σ and τ are two commuting bijections, a concrete example of a right inverse of D is shown in [7].

Then, let us fix a point $q \in M$ and define the following sequence of functions $\theta_q^{(m)} : M \to \mathbb{R}$, $m \in \mathbb{N}_0$,

$$\theta_q^{(0)} = \hat{1} \quad \text{and} \quad \theta_q^{(m)} = \prod_{k=1}^{m} \theta_{\tau^{m-k}\sigma^{k-1}(q)}\,, \tag{4.5}$$

e.g. for $p \in M$, $\theta_q^{(1)}(p) = \theta(p,q)\,$, $\theta_q^{(2)}(p) = \theta(p,\tau(q))\theta(p,\sigma(q))\,$, $\theta_q^{(3)}(p) = \theta(p,\tau^2(q))\theta(p,\tau\sigma(q))\theta(p,\sigma^2(q))\,$, etc.

Proposition 4.5. *Let $\sigma, \tau : M \to M$ be commuting maps and θ be a tension function homogeneous with respect to σ and τ with the homogeneity coefficients s and t, respectively, such that $\theta(\tau(p),\sigma(p)) \neq 0$ for any $p \in M$. Then, for any $n \in \mathbb{N}$, the following formula is true*

$$D\theta_q^{(n)} = [n]_\sigma^\tau \cdot \theta_q^{(n-1)}\,, \tag{4.6}$$

where

$$[n]_\sigma^\tau = \sum_{k=1}^{n} t^{n-k} s^{k-1}\,. \tag{4.7}$$

Proof.

$$d\theta_q^{(n)}(p) = \theta_q^{(n)}(\tau(p)) - \theta_q^{(n)}(\sigma(p)) =$$

$$= \prod_{k=1}^{n} \theta(\tau(p), \tau^{n-k}\sigma^{k-1}(q)) - \prod_{k=1}^{n} \theta(\sigma(p), \tau^{n-k}\sigma^{k-1}(q)) =$$

$$= \theta(\tau(p), \sigma^{n-1}(q)) \prod_{k=1}^{n-1} \theta(\tau(p), \tau^{n-k}\sigma^{k-1}(q)) -$$

$$-\theta(\sigma(p), \tau^{n-1}(q)) \prod_{k=2}^{n} \theta(\sigma(p), \tau^{n-k}\sigma^{k-1}(q)) =$$

$$= \left(t^{n-1}\theta(\tau(p),\sigma^{n-1}(q)) - s^{n-1}\theta(\sigma(p),\tau^{n-1}(q))\right)\prod_{k=1}^{n-1}\theta(p,\tau^{n-1-k}\sigma^{k-1}(q))$$

$$= \left(t^{n-1}\theta(\tau(p),\sigma^{n-1}(q)) - s^{n-1}\theta(\sigma(p),\tau^{n-1}(q))\right)\theta_q^{(n-1)}(p)\,.$$

In turn,

$$t^{n-1}\theta(\tau(p),\sigma^{n-1}(q)) = t^{n-1}\theta(\tau(p),\sigma(p)) + t^{n-1}\theta(\sigma(p),\sigma^{n-1}(q)) =$$

$$= t^{n-1}\theta(\tau(p),\sigma(p)) + t^{n-2}s\theta(\tau(p),\sigma^{n-2}\tau(q)) =$$

$$= (t^{n-1}+t^{n-2}s)\theta(\tau(p),\sigma(p)) + t^{n-3}s^2\theta(\tau(p),\sigma^{n-3}\tau^2(q)) = \ldots$$

$$\ldots = (t^{n-1}+\ldots+t^{n-i}s^{i-1})\theta(\tau(p),\sigma(p)) + t^{n-i}s^{i-1}\theta(\tau(p),\sigma^{n-i}\tau^{i-1}(q)) = \ldots$$

$$\ldots = \sum_{i=1}^{n} t^{n-i}s^{i-1}\theta(\tau(p),\sigma(p)) + s^{n-1}\theta(\tau(p),\tau^{n-1}(q))\,.$$

Finally,

$$D\theta_q^{(n)}(p) = \frac{d\theta_q^{(n)}(p)}{\theta(\tau(p),\sigma(p))} = \sum_{i=1}^{n} t^{n-i}s^{i-1}\cdot\theta_q^{(n-1)}(p) = [n]_\sigma^\tau\cdot\theta_q^{(n-1)}(p)\,.$$

□

Since there is $D\theta_q^{(0)} = \hat{0}$ (zero constant function), formula (4.6) can be extended to the case $n=0$, if we define $[0]_\sigma^\tau = 0$. Since σ and τ are fixed mappings giving rise to a concrete type of a quantum calculus considered, the indices σ and τ will be omitted, i.e. the notation $[n] \equiv [n]_\sigma^\tau$ will be used in the sequel. With the symbol $[n]$ we associate the (σ,τ)-quantum factorial defined as

$$[n]! = \begin{cases} 1 & if\;\; n=0 \\ [n]\cdot[n-1]! & if\;\; n=1,2,\ldots\,. \end{cases} \tag{4.8}$$

An immediate consequence of the last proposition is that each function $\theta_q^{(n)}\colon M\to\mathbb{R}$ is a representative element of $Z_{n+1}(D)$, i.e. it is a D-polynomial of degree $deg\,\theta_q^{(n)} = n$, for $n\in\mathbb{N}_0$.

Proposition 4.6. *For any function $\zeta\in Z(D)$ and any mapping $\chi\colon M\to M$ commuting with σ and τ, there is also $\zeta\circ\chi\in Z(D)$.*

Proof. Since $\zeta \in Z(D)$, there is $\zeta \circ \tau = \zeta \circ \sigma$ and therefore $\zeta \circ \tau \circ \chi = \zeta \circ \sigma \circ \chi$. The commutativity of σ, τ with χ implies that $\zeta \circ \chi \circ \tau = \zeta \circ \chi \circ \sigma$ or equivalently $\zeta \circ \chi \in Z(D)$. $\square$

With any basis $\{\zeta_s \colon s \in S\}$ of the linear space $Z(D)$ one can always associate a family $\{q_s \in M \colon s \in S\}$ of points such that $\zeta_s(q_s) \neq 0$ and $\zeta_{s_1}(q_{s_2}) = 0$ whenever $s_1 \neq s_2$, for $s, s_1, s_2 \in S$. Then, one can always assume that functions ζ_s are normalized in such a way that

$$\zeta_s(q_t) = \begin{cases} 1 & if \;\; s = t \\ 0 & if \;\; s \neq t\,, \end{cases} \tag{4.9}$$

for any $s, t \in S$.

Let us define functions

$$\zeta^{(k)}_{q_s s} = \frac{1}{[k]!}(\zeta_s \circ \sigma^{-k}) \cdot \theta^{(k)}_{q_s}\,, \tag{4.10}$$

for $s \in S$, $k \in \mathbb{N}_0$. Then by formula (3.11) and Proposition 4.5 we obtain

$$D\zeta^{(k)}_{q_s s} = \frac{1}{[k]!}(\zeta_s \circ \sigma^{-k} \circ \sigma) \cdot D\theta^{(k)}_{q_s} = \zeta^{(k-1)}_{q_s s}\,. \tag{4.11}$$

The consequence of the last formula is that $\zeta^{(k)}_{q_s s} \in Z_{k+1}(D)$, i.e. $\zeta^{(k)}_{q_s s}$ is a D-polynomial of degree $deg\, \zeta^{(k)}_{q_s s} = k$, for any $s \in S$ and $k \in \mathbb{N}_0$.

Proposition 4.7. *The family $\{\zeta^{(k)}_{q_s s} \colon s \in S, k \in \mathbb{N}_0\}$ is a basis of the linear space $P(D)$ and $\{\zeta^{(k)}_{q_s s} \colon s \in S,\; k = 0, \ldots, n\}$ is a basis of the linear space $P_n(D)$.*

Proof. Assume that $\sum\limits_{i=0}^{k} \sum\limits_{s \in S_i} a_{is}\zeta^{(i)}_{q_s s} = \hat{0}$, for some coefficients a_{is}, $S_i \subset S$ - finite subsets, $i = 0, \ldots, k$. Then we calculate $D^k \sum\limits_{i=0}^{k} \sum\limits_{s \in S_i} a_{is}\zeta^{(i)}_{q_s s} = \sum\limits_{s \in S_k} a_{ks}\zeta_s = \hat{0}$ which implies that $a_{ks} = 0$ for $s \in S_k$. Hence we can write $\sum\limits_{i=0}^{k-1} \sum\limits_{s \in S_i} a_{is}\zeta^{(i)}_{q_s s} = \hat{0}$. Similarly, we calculate $D^{k-1} \sum\limits_{i=0}^{k-1} \sum\limits_{s \in S_i} a_{is}\zeta^{(i)}_{q_s s} = \sum\limits_{s \in S_{k-1}} a_{(k-1)s}\zeta_s = \hat{0}$ and show that $a_{(k-1)s} = 0$ for $s \in S_{k-1}$. Analogously, step by step we prove that

$a_{ms} = 0$ for $s \in S_m$ and $m = k-2, ..., 1$. Now, let $u \in P(D)$ be of degree $deg\, u = k$. We will show that $u = \sum_{i=0}^{k} \sum_{s \in S_i} a_{is} \zeta^{(i)}_{q_s s}$. Let us calculate:

$$D^k u = \sum_{i=0}^{k} \sum_{s \in S_i} a_{is} D^k \zeta^{(i)}_{q_s s} = \sum_{s \in S_k} a_{ks} \zeta_s .$$

Then, using formula (4.9) we obtain

$$a_{ks} = D^k u(q_s)\,, \tag{4.12}$$

for $s \in S_k$. Other coefficients a_{ms}, for $0 \leq m < k$, we can calculate from the recursive formula

$$a_{ms} = D^m u(q_s) - \sum_{j=1}^{k-m} \sum_{t \in S_{m+j}} a_{(m+j)t} \zeta^{(j)}_{q_t t}(q_s)\,. \tag{4.13}$$

$\square$

One can select (not uniquely) a linear subspace $Q(D)$ in $X = \mathbb{R}^M$ such that

$$X = P(D) \oplus Q(D)\,. \tag{4.14}$$

According to formula (2.31), any right inverse $R \in \mathcal{R}_D$ can be decomposed as a direct sum

$$R = R_P \oplus R_Q \tag{4.15}$$

of its restrictions $R_P = R_{|P(D)}$ and $R_Q = R_{|Q(D)}$. The component R_P can be defined on the basis $\{\zeta^{(k)}_{q_s s} \colon s \in S, k \in \mathbb{N}_0\}$ by formula

$$R_P \zeta^{(k)}_{q_s s} = \zeta^{(k+1)}_{q_s s}\,, \tag{4.16}$$

for any $s \in S$, $k \in \mathbb{N}_0$. Concerning the component R_Q, we can assume any definition. Then, on the strength of formula (2.26), the linear space $P(D)$ is a direct sum of D- and R-invariant linear subspaces $V_s(R)$, $s \in S$. The initial operator F corresponding to the above R is also a direct sum

$$F = F_P \oplus F_Q\,, \tag{4.17}$$

with the components $F_P = F_{|P(D)}$ and $F_Q = F_{|Q(D)}$ given by formula (2.9). For F_P we obtain the explicit formula

$$F_P \zeta^{(k)}_{q_s s} = (I - R_P D^{\tau}_{\sigma}) \zeta^{(k)}_{q_s s} = \begin{cases} \zeta_s & if \;\; k = 0 \\ \hat{0} & if \;\; k = 1, 2, \ldots\,, \end{cases} \tag{4.18}$$

where $s \in S$.

5 Taylor formula in (τ, σ)-quantum calculus

Let R be a right inverse of $D = D_\sigma^\tau$ and F be the initial operator corresponding to R. Then, according to formula (2.14), we have the Taylor formula

$$I = \sum_{j=0}^{n} R^j F D^j + R^{n+1} D^{n+1}\,, \tag{5.1}$$

which holds on $dom(D)^{n+1}$, for $n \in \mathbb{N}_0$. Now, suppose that $W \in Z_{n+1}(D)$, i.e. $W \in P(D)$ is a D-polynomial of degree $degW = n$. Then $(D)^{n+1}W = \hat{0}$ and

$$W = \sum_{k=0}^{n} R^k F D^k W\,. \tag{5.2}$$

Since $F(D)^k W \in Z(D)$, there exists a finite subset $S_W^k \subset S$ and coefficients $\lambda_{ks} \in \mathbb{R}$, $s \in S_W^k$, such that

$$F(D)^k W = \sum_{s \in S_W^k} \lambda_{ks} \zeta_s\,. \tag{5.3}$$

Thus, we obtain the formula

$$W = \sum_{k=0}^{n} \sum_{s \in S_W^k} \lambda_{ks} R^k \zeta_s\,. \tag{5.4}$$

The coefficients λ_{ks} are given by

$$\lambda_{ks} = (F D^k W)(q_s)\,, \tag{5.5}$$

which allows us to write

$$W = \sum_{k=0}^{n} \sum_{s \in S_W^k} (F D^k W)(q_s) R^k \zeta_s\,. \tag{5.6}$$

Define functions $\Lambda_{q_s s}^{(m)} \colon M \to \mathbb{R}$ recursively as $\Lambda_{q_s s}^{(0)} = \zeta_s$ and

$$\Lambda_{q_s s}^{(m)} = \zeta_{q_s s}^{(m)} - \sum_{i=0}^{m-1} \zeta_{q_s s}^{(m-i)}(q_s) \Lambda_{q_s s}^{(i)}\,, \tag{5.7}$$

for $q_s \in M$, $s \in S$ and $m \in \mathbb{N}$. One can easily check that

$$\Lambda^{(m)}_{q_s s}(q_s) = \hat{0}\,, \tag{5.8}$$

$$D\Lambda^{(m)}_{q_s s} = \Lambda^{(m-1)}_{q_s s}\,, \tag{5.9}$$

for any $s \in S$, $m \in \mathbb{N}$. Hence, for any $s \in S$, the family $\{\Lambda^{(m)}_{q_s s} \colon m \in \mathbb{N}_0\}$ is linearly independent and forms a basis of the linear space of s-homogeneous D-polynomials

$$V_s(D) = Lin\{\Lambda^{(m)}_{q_s s} \colon m \in \mathbb{N}_0\}\,. \tag{5.10}$$

On the strength of formula (2.29) there is $P(D) = \bigoplus\limits_{s\in S} V_s(D)$ and there exists another subspace $Q(D)$ such that $X = P(D) \oplus Q(D)$. Below we shall use the projection mappings

$$\pi_s \colon P(D) \to V_s(D)\,, \tag{5.11}$$

for $s \in S$.

Let us take a right inverse $R \in \mathcal{R}_D$ defined on $P(D)$ by

$$R\Lambda^{(m)}_{q_s s} = \Lambda^{(m+1)}_{q_s s}\,, \tag{5.12}$$

for any $s \in S$, $m \in \mathbb{N}_0$, while its definition on $Q(D)$ can be any. Then we can write

$$\Lambda^{(m)}_{q_s s} = R^m \zeta_s\,, \tag{5.13}$$

for any $s \in S$, $m \in \mathbb{N}_0$. Therefore formula (5.4) can be written as

$$W = \sum_{k=0}^{n} \sum_{s\in S^k_W} \lambda_{ks}\Lambda^{(k)}_{q_s s}\,. \tag{5.14}$$

By using projections (5.11), we obtain the components W_s of W defined as

$$W_s \equiv \pi_s(W) = \sum_{k=0}^{n} \lambda_{ks}\Lambda^{(k)}_{q_s s}, \tag{5.15}$$

for any $s \in S_W \equiv \bigcup\limits_{k=0}^{n} S^k_W$. Naturally, there is

$$W = \sum_{s\in S_W} W_s\,. \tag{5.16}$$

The coefficients λ_{ks} in formula (5.15) can be computed as

$$\lambda_{ks} = D^k W_s(q_s)\,, \tag{5.17}$$

for any $k \in \mathbb{N}_0$ and $s \in S_W$. Finally we obtain the Taylor formulae

$$W_s = \sum_{k=0}^{n} D^k W_s(q_s) \Lambda^{(k)}_{q_s s}\,, \tag{5.18}$$

for $s \in S_W$, and

$$W = \sum_{s \in S_W} \sum_{k=0}^{n} D^k W_s(q_s) \Lambda^{(k)}_{q_s s}\,. \tag{5.19}$$

Naturally, if $W \in V_s(D)$, for some $s \in S$, there is

$$W = \sum_{k=0}^{n} D^k W(q_s) \Lambda^{(k)}_{q_s s}\,. \tag{5.20}$$

In the particular case, for q-calculus or its symmetric version the corresponding results one can find in Ref.[5].

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