GRADED LIMITS OF MINIMAL AFFINIZATIONS AND BEYOND: THE MULTIPLICTY FREE CASE FOR TYPE E_6

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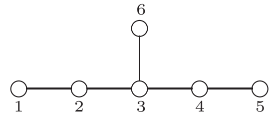
Abstract: We obtain a graded character formula for certain graded modules for the current algebra over a simple Lie algebra of type E_6. For certain values of their highest weight, these modules were conjectured to be isomorphic to the classical limit of the corresponding minimal affinizations of the associated quantum group. We prove that this is the case under further restrictions on the highest weight. Under another set of conditions on the highest weight, Chari and Greenstein have recently proved that they are projective objects of a full subcategory of the category of graded modules for the current algebra. Our formula applies to all of these projective modules.

Introduction

The problem of determining the structure of the minimal affinizations of quantum groups is one of the most studied problems in the finite-dimensional representation theory of quantum affine algebras in recent years (see [6] for a recent survey with a comprehensive list of references). In particular, determining the character of such representations when regarded as modules for the quantum group $U_q(\mathfrak{g})$ over the underlying semisimple Lie algebra $\mathfrak{g}$ is of special interest. Determining the character is theoretically equivalent to determining the multiplicity of the irreducible constituents of these representations when regarded as $U_q(\mathfrak{g})$-modules. In practice, computing the multiplicities out of a given character is a laborious task which can be performed algorithmically.

One of the methods which have been used to approach this problem is that of considering the classical limit of the given module and regard it as a representation for the current algebra $\mathfrak{g}[t] = \mathfrak{g}\otimes\mathbb{C}[t]$. This approach was first considered in [2, 7] and it was then further developed in [9, 10, 21]. In this paper, we apply this method for $\mathfrak{g}$ of type E_6 and obtain a formula for the multiplicities of the irreducible constituents of the graded pieces of these modules assuming certain conditions on the highest weight. Our formula actually holds for a larger class of $\mathfrak{g}[t]$-modules. Namely, given a dominant integral weight λ of $\mathfrak{g}$, the first author defined in [21] a $\mathfrak{g}[t]$-module denoted by $M(\lambda)$. The definition is by generator and relations which naturally generalize the relations of the classical limits of Kirillov-Reshetikhin modules obtained in [2]. It was conjectured in [21] that $M(\lambda)$ is isomorphic to the classical limit of the minimal affinizations of the irreducible $U_q(\mathfrak{g})$-module of highest-weight λ provided that there exists a unique equivalence class of minimal affinizations associated to λ. Our main results are a formula for the multiplicities of the irreducible constituents of the graded pieces of the modules $M(\lambda)$ and the proof of the conjecture of [21] assuming certain conditions on λ. To explain these conditions, let us label the nodes of the Dynkin diagram of $\mathfrak{g}$ as follows.



Let $I = \{1, 2, \ldots, 6\}$ and identify it with the set of nodes of the Dynkin diagram of $\mathfrak{g}$ following the above labeling. For an integral weight μ, the support of μ is the subset of I consisting of labels such that the value of μ on the corresponding co-root is nonzero. The connected closure of the support is the minimal connected subdiagram of the Dynkin diagram of $\mathfrak{g}$ containing the nodes in the support of μ.

We mostly focus our study on the modules $M(\lambda)$ with λ not supported in the trivalent node and prove that the character formula (3.12) below holds for all λ with support contained in one of the following subsets of I: $\{1,2,5,6\},\{1,4,5,6\},\{2,4,6\}$. Following the conjecture of [21], we conjecture that (3.12) holds for all λ not supported in the trivalent node and prove in such generality that (3.12) gives an upper bound for the multiplicities of the $\mathfrak{g}$-irreducible constituents of the graded pieces of $M(\lambda)$ (see (3.10)). In particular, it follows from (3.9) that all irreducible constituents are multiplicity free (even if the grading is not taken into account). As a byproduct of the proof of (3.12), we obtain a realization of $M(\lambda)$ as a submodule of the tensor product of the classical limits of certain Kirillov-Reshetikhin modules (Theorem 3.14(a)), thus establishing part of the conjecture of [21] for such λ.

Keeping the above conditions on λ and further assuming that the connected closure of the support of λ is of type A, we prove that $M(\lambda)$ is isomorphic to the classical limit of the corresponding minimal affinizations when regarded as $\mathfrak{g}[t]$-modules (Theorem 3.14(b)). This establishes the other part of the conjecture of [21] for these values of λ. In particular, (3.12) gives the multiplicities of the irreducible constituents of the minimal affinizations when the support of λ is contained in one of the following subsets of I: $\{1,2,5\},\{1,4,5\},\{1,2,6\},\{4,5,6\},\{2,4\}$. Moreover, we also prove that, if (3.12) indeed holds for any λ not supported in the trivalent node as conjectured, then we can include $\{1,2,4,5\}$ in this list. Dropping all the assumptions on λ except that the connected closure of its support is of type A, we prove that the classical limit of the corresponding minimal affinizations are quotients of $M(\lambda)$ (Proposition 3.15). This is a further step towards the proof of the conjecture of [21] in general. However, the graded character formula for the Kirillov-Reshetikhin modules associated to the trivalent node given in [16] implies that, if λ is supported on that node, then these modules are not multiplicity free. We remark that, in [22], Nakajima developed an algorithm for computing the t-analogue of the q-character of any finite-dimensional irreducible representation of the quantum affine algebra associated to any simply laced simple Lie algebra $\mathfrak{g}$. In particular, without any assumption on λ, the graded character of the classical limits of the minimal affinizations associated to λ can be computed using this algorithm. Theoretically, one can then compute the multiplicities from the character as mentioned in the first paragraph of this introduction. On the other hand, with the above assumptions on λ, formula (3.12) gives these multiplicities directly.

Let us explain the reasons behind the several aforementioned restrictions on λ. First we recall that, for simply laced $\mathfrak{g}$, there exists a unique equivalence class of minimal affinizations associated to λ if and only if the connected closure of its support is of type A. Let θ be the highest root of $\mathfrak{g}$ and, given $i\in I$, let $\epsilon_i(\theta)$ be its coordinate in the basis of simple roots. Given a positive integer r, let $\mathfrak{g}[t:r]$ be the quotient of $\mathfrak{g}[t]$ by the ideal $\mathfrak{g}\otimes t^r\mathbb{C}[t]$. It turns out that $M(\lambda)$ factors to a module for $\mathfrak{g}[t:r]$ where r is the maximum of $\epsilon_i(\theta)$ for i running on the support of λ. In particular, if $\mathfrak{g}$ is of type E_6, $M(\lambda)$ can be regarded as a module for $\mathfrak{g}[t:3]$. Moreover, if λ is not supported on the trivalent node, then $M(\lambda)$ factors to a module for $\mathfrak{g}[t:2]$. The category $\mathcal{G}_2$ of graded $\mathfrak{g}[t:2]$-modules with finite-dimensional graded pieces has been recently studied in [4, 5] by exploring its interplay with the theory of Koszul algebras and quiver representations. The literature on the representation theory of $\mathfrak{g}[t:r]$ for $r>2$ is more limited and results such as the ones from [4, 5] are yet to be established. Thus, we focus on the case that $M(\lambda)$ factors to a $\mathfrak{g}[t:2]$-module which, for type E_6, is equivalent to assuming that λ is not supported on the trivalent node (as mentioned above, this is also the necessary and sufficient condition for the modules $M(\lambda)$ to be multiplicity free). It follows from [5, Theorem 1] that, if λ satisfies certain conditions, then $M(\lambda)$ is a projective object of a full subcategory of $\mathcal{G}_2$ naturally attached to λ. Moreover, [5, Theorem 2] gives a graded character formula for $M(\lambda)$ provided λ satisfies the conditions of [5, Theorem 1]. We remark that [5, Theorem 2] expresses the graded character of $M(\lambda)$ in terms of an alternating sum of the graded characters of $M(\mu)$ with μ strictly smaller than λ with respect to the usual partial order on the weight lattice of $\mathfrak{g}$. Hence, the formula of [5, Theorem 2] is of recursive nature. For $\mathfrak{g}$ of type E_6, we prove that the conditions on λ required on [5, Theorem 1] is equivalent to requiring that the support of λ be contained in one of the following subsets of I: $\{1,2,5,6\},\{1,4,5,6\}$. Therefore, (3.12) holds beyond the cases covered by [5, Theorem 2]. This latter list of subsets of I also hints that it should be expected that when the support of λ contains $\{2,4\}$ the situation should be more complicated than otherwise. Indeed, the proof of (3.12) for this case is significantly more technically involved than for the others.

The paper is organized as follows. In Section 1, we review the basic notation on simply laced simple Lie algebras and the associated loop algebras, current algebras, quantum groups, and quantum affine algebras. In Section 2, we review the relevant facts on the finite-dimensional representation theory of these algebras. After reviewing the classification of minimal affinizations in Subsection 3.1, the main results (Theorem 3.14, Proposition 3.15, the multiplicity free property (3.10), and the character formula (3.12)) are stated in Subsection 3.2. The relation of our results with those of [5] is explained in Subsection 3.3. The proofs are given in Section 4.

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1. Quantum and classical loop algebras

Throughout the paper, let $\mathbb{C},\mathbb{R},\mathbb{Z},\mathbb{Z}_{\geq m}$ denote the sets of complex numbers, reals, integers, and integers bigger or equal m, respectively. Given a ring $\mathbb{A}$, the underlying multiplicative group of units is denoted by $\mathbb{A}^\times$. The dual of a vector space V is denoted by V^*. The symbol $\cong$ means "isomorphic to". The cardinality of a set S will be denoted by $|S|$.

1.1. **Classical algebras.** Let $I=\{1,\dots,n\}$ be the set of vertices of a finite-type simply laced Dynkin diagram and let $\mathfrak{g}$ be the associated semisimple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$. Fix a set of positive roots R^+ and let

$$\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^+}\mathfrak{g}_{\pm\alpha} \quad \text{where} \quad \mathfrak{g}_{\pm\alpha}=\{x\in\mathfrak{g}:[h,x]=\pm\alpha(h)x,\ \forall\, h\in\mathfrak{h}\}.$$

The simple roots will be denoted by α_i and the fundamental weights by ω_i, $i\in I$. Q,P,Q^+,P^+ will denote the root and weight lattices with corresponding positive cones, respectively. Let also $h_i\in\mathfrak{h}$, be the co-root associated to $\alpha_i, i\in I$. We equip $\mathfrak{h}^*$ with the partial order $\lambda\leq\mu$ iff $\mu-\lambda\in Q^+$. Let $C=(c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$, i.e., $c_{ij}=\alpha_j(h_i)$. The Weyl group is denoted by $\mathcal{W}$.

The subalgebras $\mathfrak{g}_{\pm\alpha},\alpha\in R^+$, are one-dimensional and $[\mathfrak{g}_{\pm\alpha},\mathfrak{g}_{\pm\beta}]=\mathfrak{g}_{\pm\alpha\pm\beta}$ for every $\alpha,\beta\in R^+$. We denote by $x_\alpha^\pm$ any generator of $\mathfrak{g}_{\pm\alpha}$ and, in case $\alpha=\alpha_i$ for some $i\in I$, we may also use the notation $x_i^\pm$ in place of $x_{\alpha_i}^\pm$. In particular, if $\alpha+\beta\in R^+$, $[x_\alpha^\pm,x_\beta^\pm]$ is a nonzero generator of $\mathfrak{g}_{\pm\alpha\pm\beta}$ and we simply write $[x_\alpha^\pm,x_\beta^\pm]=x_{\alpha+\beta}^\pm$. For each subset J of I let $\mathfrak{g}_J$ be the Lie subalgebra of $\mathfrak{g}$ generated by $x_{\alpha_j}^\pm, j\in J$, and define $\mathfrak{n}_J^\pm,\mathfrak{h}_J$ in the obvious way. Let also Q_J be the subgroup of Q generated by $\alpha_j, j\in J$, and $R_J^+=R^+\cap Q_J$. Given $\lambda\in P$, let λ_J be the restriction of λ to $\mathfrak{h}_J^*$ and $\lambda^J\in P$ be such that $\lambda^J(h_j)=\lambda(h_j)$ if $j\in J$ and $\lambda^J(h_j)=0$ otherwise. By abuse of language, we will refer to any subset J of I as a subdiagram of the Dynkin diagram of $\mathfrak{g}$. The support of $\mu\in P$ is defined to be the subdiagram $\operatorname{supp}(\mu)\subseteq I$ given by $\operatorname{supp}(\mu)=\{i\in I:\mu(h_i)\neq 0\}$. Let also $\overline{\operatorname{supp}}(\mu)$ be the minimal connected subdiagram of I containing $\operatorname{supp}(\mu)$.

If $\mathfrak{a}$ is a Lie algebra over $\mathbb{C}$, define its loop algebra to be $\tilde{\mathfrak{a}}=\mathfrak{a}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$ with bracket given by $[x\otimes t^r,y\otimes t^s]=[x,y]\otimes t^{r+s}$. Clearly $\mathfrak{a}\otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to $\mathfrak{a}$ and, by abuse of notation, we will continue denoting its elements by x instead of $x\otimes 1$. We also consider the current algebra $\mathfrak{a}[t]$ which is the subalgebra of $\tilde{\mathfrak{a}}$ given by $\mathfrak{a}[t]=\mathfrak{a}\otimes\mathbb{C}[t]$. Then $\tilde{\mathfrak{g}}=\tilde{\mathfrak{n}}^-\oplus\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra and similarly for $\mathfrak{g}[t]$. The elements $x_\alpha^\pm\otimes t^r, x_i^\pm\otimes t^r$, and $h_i\otimes t^r$ will be denoted by $x_{\alpha,r}^\pm, x_{i,r}^\pm$, and $h_{i,r}$, respectively. Also, Diagram subalgebras $\tilde{\mathfrak{g}}_J$ are defined in the obvious way.

Let $U(\mathfrak{a})$ denote the universal enveloping algebra of a Lie algebra $\mathfrak{a}$. Then $U(\mathfrak{a})$ is a subalgebra of $U(\tilde{\mathfrak{a}})$. Given $a\in\mathbb{C}$, let τ_a be the Lie algebra automorphism of $\mathfrak{a}[t]$ defined by $\tau_a(x\otimes f(t)) = x\otimes f(t-a)$ for every $x\in\mathfrak{a}$ and every $f(t)\in\mathbb{C}[t]$. If $a\neq 0$, let $\mathrm{ev}_a:\tilde{\mathfrak{a}}\to\mathfrak{a}$ be the evaluation map $x\otimes f(t)\mapsto f(a)x$. We also denote by τ_a and ev_a the induced maps $U(\mathfrak{a}[t])\to U(\mathfrak{a}[t])$ and $U(\tilde{\mathfrak{a}})\to U(\mathfrak{a})$, respectively. Given a nonzero $x\in\mathfrak{a}$ we shall denote by $U(x)$ the universal enveloping algebra of the one-dimensional subalgebra generated by x regarded as a subalgebra of $U(\mathfrak{a})$.

For each $i\in I$ and $r\in\mathbb{Z}$, define elements $\Lambda_{i,r}\in U(\tilde{\mathfrak{h}})$ by the following equality of formal power series in the variable u:

$$\sum_{r=0}^{\infty}\Lambda_{i,\pm r}u^r = \exp\left(-\sum_{s=1}^{\infty}\frac{h_{\alpha_i,\pm s}}{s}u^s\right). \tag{1.1}$$

1.2. **Quantum algebras.** Let $\mathbb{C}(q)$ be the ring of rational functions on an indeterminate q and $\mathbb{A}=\mathbb{C}[q,q^{-1}]$. Set

$$[m]=\frac{q^m-q^{-m}}{q-q^{-1}},\qquad [m]! = [m][m-1]\dots[2][1],\qquad \left[\begin{smallmatrix} m\\ r\end{smallmatrix}\right] = \frac{[m]!}{[r]![m-r]!},$$

for $r,m\in\mathbb{Z}_{\geq 0}$, $m\geq r$. Notice that $[m],\left[\begin{smallmatrix} m\\ r\end{smallmatrix}\right]\in\mathbb{A}$.

The quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ of $\mathfrak{g}$ is the associative $\mathbb{C}(q)$-algebra with generators $x_{i,r}^{\pm}$ ($i\in I$, $r\in\mathbb{Z}$), $k_i^{\pm 1}$ ($i\in I$), $h_{i,r}$ ($i\in I$, $r\in\mathbb{Z}\backslash\{0\}$) and the following defining relations:

$$\begin{gathered}
k_ik_i^{-1}=k_i^{-1}k_i=1,\quad k_ik_j=k_jk_i,\\
k_ih_{j,r}=h_{j,r}k_i,\\
k_ix_{j,r}^{\pm}k_i^{-1}=q^{\pm c_{ij}}x_{j,r}^{\pm},\\
[h_{i,r},h_{j,s}]=0,\ \ [h_{i,r},x_{j,s}^{\pm}]=\pm\frac{1}{r}[rc_{ij}]x_{j,r+s}^{\pm},\\
x_{i,r+1}^{\pm}x_{j,s}^{\pm}-q^{\pm c_{ij}}x_{j,s}^{\pm}x_{i,r+1}^{\pm}=q^{\pm c_{ij}}x_{i,r}^{\pm}x_{j,s+1}^{\pm}-x_{j,s+1}^{\pm}x_{i,r}^{\pm},\\
[x_{i,r}^{+},x_{j,s}^{-}]=\delta_{i,j}\frac{\psi_{i,r+s}^{+}-\psi_{i,r+s}^{-}}{q-q^{-1}},\\
\sum_{\sigma\in S_m}\sum_{k=0}^{m}(-1)^k\left[\begin{smallmatrix} m\\ k\end{smallmatrix}\right]x_{i,r_{\sigma(1)}}^{\pm}\dots x_{i,r_{\sigma(k)}}^{\pm}x_{j,s}^{\pm}x_{i,r_{\sigma(k+1)}}^{\pm}\dots x_{i,r_{\sigma(m)}}^{\pm}=0,\quad\text{if } i\neq j,
\end{gathered}$$

for all sequences of integers $r_1,\dots,r_m$, where $m=1-c_{ij}$, S_m is the symmetric group on m letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\Psi_i^{\pm}(u)=\sum_{r=0}^{\infty}\psi_{i,\pm r}^{\pm}u^r = k_i^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{s=1}^{\infty}h_{i,\pm s}u^s\right).$$

Denote by $U_q(\tilde{\mathfrak{n}}^{\pm}),U_q(\tilde{\mathfrak{h}})$ the subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by $\{x_{i,r}^{\pm}\},\{k_i^{\pm 1},h_{i,s}\}$, respectively. Let $U_q(\mathfrak{g})$ be the subalgebra generated by $x_i^{\pm}:=x_{i,0}^{\pm},k_i^{\pm 1}, i\in I$, and define $U_q(\mathfrak{n}^{\pm}),U_q(\mathfrak{h})$ in the obvious way. $U_q(\mathfrak{g})$ is a subalgebra of $U_q(\tilde{\mathfrak{g}})$ and multiplication establishes isomorphisms of $\mathbb{C}(q)$-vectors spaces:

$$U_q(\mathfrak{g})\cong U_q(\mathfrak{n}^-)\otimes U_q(\mathfrak{h})\otimes U_q(\mathfrak{n}^+)\qquad\text{and}\qquad U_q(\tilde{\mathfrak{g}})\cong U_q(\tilde{\mathfrak{n}}^-)\otimes U_q(\tilde{\mathfrak{h}})\otimes U_q(\tilde{\mathfrak{n}}^+).$$

Let $J\subseteq I$ and consider the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ generated by $k_j^{\pm 1},h_{j,r},x_{j,s}^{\pm}$ for all $j\in J, r,s\in\mathbb{Z}, r\neq 0$. If $J=\{j\}$, the algebra $U_q(\tilde{\mathfrak{g}}_j):=U_q(\tilde{\mathfrak{g}}_J)$ is isomorphic to $U_q(\tilde{\mathfrak{sl}}_2)$. Similarly we define the subalgebra $U_q(\mathfrak{g}_J)$, etc.

For $i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$, define $(x_{i,r}^{\pm})^{(k)} = \frac{(x_{i,r}^{\pm})^k}{[k]!}$. Define also elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$ by

$$\sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(-\sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]} u^s \right). \tag{1.2}$$

Although we are denoting the elements $x_{i,r}^{\pm}, h_{i,r}$, and $\Lambda_{i,r}$ above by the same symbol as their classical counterparts, this will not create confusion as it will be clear from the context.

Let $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ be the $\mathbb{A}$-subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the elements $(x_{i,r}^{\pm})^{(k)}, k_i^{\pm 1}$ for $i \in I, r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$. Define $U_{\mathbb{A}}(\mathfrak{g})$ similarly and notice that $U_{\mathbb{A}}(\mathfrak{g}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\mathfrak{g})$. Henceforth $\mathfrak{a}$ will denote a Lie algebra of the following set: $\mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{n}}^{\pm}, \tilde{\mathfrak{h}}$. For the proof of the next proposition see [2, Lemma 2.1] and the locally cited references.

Proposition 1.1. The canonical map $\mathbb{C}(q) \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{a}) \to U_q(\mathfrak{a})$ is an isomorphism. □

Regard $\mathbb{C}$ as an $\mathbb{A}$-module by letting q act as 1 and set

$$\overline{U_q(\mathfrak{a})} = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{a}). \tag{1.3}$$

Denote by $\bar{x}$ the image of $x \in U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ in $\overline{U_q(\tilde{\mathfrak{g}})}$. For a proof of the next proposition see [11, Proposition 9.2.3] and the locally cited references.

Proposition 1.2. $U(\tilde{\mathfrak{g}})$ is isomorphic to the quotient of $\overline{U_q(\tilde{\mathfrak{g}})}$ by the ideal generated by $\overline{k}_i - 1$. In particular, the category of $\overline{U_q(\tilde{\mathfrak{g}})}$-modules on which k_i act as the identity operator for all $i \in I$ is equivalent to the category of all $\tilde{\mathfrak{g}}$-modules. □

The algebra $U_q(\tilde{\mathfrak{g}})$ is a Hopf algebra and induces a Hopf algebra structure (over $\mathbb{A}$) on $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$. Moreover, the induced Hopf algebra structure on $U(\tilde{\mathfrak{g}})$ coincides with the usual one (see [11, 20]). On $U_q(\mathfrak{g})$ we have

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \qquad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \qquad \Delta(k_i) = k_i \otimes k_i \tag{1.4}$$

for all $i \in I$. The next lemma is easily established (cf. [21, Lemma 1.5]).

Lemma 1.3. Suppose $x = [x_{i_1}^-, [x_{i_2}^-, \cdots [x_{i_{l-1}}^-, x_{i_l}^-] \cdots]]$. Then $x \in U_{\mathbb{A}}(\mathfrak{n}^-)$ and

$$\Delta(x) \in x \otimes (\prod_{j=1}^{l} k_{i_j}^{-1}) + 1 \otimes x + f(q)y$$

for some $y \in U_{\mathbb{A}}(\mathfrak{g}) \otimes U_{\mathbb{A}}(\mathfrak{g})$ and some $f(q) \in \mathbb{A}$ such that $f(1) = 0$. □

An expression for the comultiplication Δ of $U_q(\tilde{\mathfrak{g}})$ in terms of the generators $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}$ is not known. The following partial information will suffice for our purposes (see [21, Lemma 1.6] and the locally cited references).

Lemma 1.4. $\Delta(x_{i,1}^-) = x_{i,1}^- \otimes k_i + 1 \otimes x_{i,1}^- + x$ for some $x \in U_{\mathbb{A}}(\mathfrak{g}) \otimes U_{\mathbb{A}}(\mathfrak{g})$ such that $\bar{x} = 0$. □

1.3. **The ℓ-weight lattice.** Given a field $\mathbb{F}$ consider the multiplicative group $\mathcal{P}_{\mathbb{F}}$ of n-tuples of rational functions $\boldsymbol{\mu} = (\boldsymbol{\mu}_1(u), \cdots, \boldsymbol{\mu}_n(u))$ with values in $\mathbb{F}$ such that $\boldsymbol{\mu}_i(0) = 1$ for all $i \in I$. We shall often think of $\boldsymbol{\mu}_i(u)$ as a formal power series in u with coefficients in $\mathbb{F}$. Given $a \in \mathbb{F}^{\times}$ and $i \in I$, let $\boldsymbol{\omega}_{i,a}$ be defined by

$$(\boldsymbol{\omega}_{i,a})_j(u) = 1 - \delta_{i,j} a u.$$

Clearly, if $\mathbb{F}$ is algebraically closed, $\mathcal{P}_\mathbb{F}$ is the free abelian group generated by these elements which are called fundamental ℓ-weights. It is also convenient to introduce elements $\boldsymbol{\omega}_{\lambda,a}, \lambda \in P, a \in \mathbb{F}$, defined by

$$\boldsymbol{\omega}_{\lambda,a} = \prod_{i\in I} (\boldsymbol{\omega}_{i,a})^{\lambda(h_i)}. \tag{1.5}$$

If $\mathbb{F}$ is algebraically closed, introduce the group homomorphism (weight map) $\mathrm{wt} : \mathcal{P}_\mathbb{F} \to P$ by setting $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$. Otherwise, let $\mathbb{K}$ be an algebraically closed extension of $\mathbb{F}$ so that $\mathcal{P}_\mathbb{F}$ can be regarded as a subgroup of $\mathcal{P}_\mathbb{K}$ and define the weight map on $\mathcal{P}_\mathbb{F}$ by restricting the one on $\mathcal{P}_\mathbb{K}$.

Define the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}})$ to be $\mathcal{P}_q := \mathcal{P}_{\mathbb{C}(q)}$. The submonoid $\mathcal{P}_q^+$ of $\mathcal{P}_q$ consisting of n-tuples of polynomials is called the set of dominant ℓ-weights of $U_q(\tilde{\mathfrak{g}})$. Given $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ with $\boldsymbol{\lambda}_i(u) = \prod_j (1 - a_{i,j}u)$, where $a_{i,j}$ belongs to some algebraic closure of $\mathbb{C}(q)$, let $\boldsymbol{\lambda}^- \in \mathcal{P}_q^+$ be defined by $\boldsymbol{\lambda}_i^-(u) = \prod_j (1 - a_{i,j}^{-1}u)$. We will also use the notation $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}$. Given $\boldsymbol{\nu} \in \mathcal{P}_q$, say $\boldsymbol{\nu} = \boldsymbol{\lambda}\boldsymbol{\mu}^{-1}$ with $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_q^+$, define a $\mathbb{C}(q)$-algebra homomorphism $\boldsymbol{\Psi}_{\boldsymbol{\nu}} : U_q(\tilde{\mathfrak{h}}) \to \mathbb{C}(q)$ by setting $\boldsymbol{\Psi}_{\boldsymbol{\nu}}(k_i^{\pm 1}) = q_i^{\pm \mathrm{wt}(\boldsymbol{\nu})(h_i)}$ and

$$\sum_{r\geq 0} \boldsymbol{\Psi}_{\boldsymbol{\nu}}(\Lambda_{i,\pm r})u^r = \frac{(\boldsymbol{\lambda}^\pm)_i(u)}{(\boldsymbol{\mu}^\pm)_i(u)}. \tag{1.6}$$

One easily checks that the map $\boldsymbol{\Psi} : \mathcal{P}_q \to (U_q(\tilde{\mathfrak{h}}))^*$ given by $\boldsymbol{\nu} \mapsto \boldsymbol{\Psi}_{\boldsymbol{\nu}}$ is injective. Define the ℓ-weight lattice $\mathcal{P}$ of $\tilde{\mathfrak{g}}$ to be the subgroup of $\mathcal{P}_q$ generated by $\boldsymbol{\omega}_{i,a}$ for all $i \in I$ and all $a \in \mathbb{C}^\times$ or, equivalently, $\mathcal{P} = \mathcal{P}_\mathbb{C}$. Set also $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{P}_q^+$. From now on we will identify $\mathcal{P}_q$ with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under $\boldsymbol{\Psi}$. Similarly, $\mathcal{P}$ will be identified with a subset of $U(\tilde{\mathfrak{h}})^*$ via the homomorphism $\boldsymbol{\Psi}_{\boldsymbol{\nu}} : U(\tilde{\mathfrak{h}}) \to \mathbb{C}$ determined by (1.6) and $\boldsymbol{\Psi}_{\boldsymbol{\nu}}(h_i) = \mathrm{wt}(\boldsymbol{\nu})(h_i)$.

It will be convenient to introduce the following notation. Given $i \in I, a \in \mathbb{C}(q)^\times, r \in \mathbb{Z}_{\geq 0}$, define

$$\boldsymbol{\omega}_{i,a,r} = \prod_{j=0}^{r-1} \boldsymbol{\omega}_{i,aq^{r-1-2j}}. \tag{1.7}$$

If $J \subseteq I$ and $\boldsymbol{\lambda} \in \mathcal{P}_q$, let $\boldsymbol{\lambda}_J$ be the associated J-tuple of rational functions. Notice that, if $\boldsymbol{\lambda}_j(u) \in \mathbb{C}(q_j)(u)$ for all $j \in J$, $\boldsymbol{\lambda}_J$ can be regarded as an element of the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}}_J)$. Let also $\boldsymbol{\lambda}^J \in \mathcal{P}_q$ be such that $(\boldsymbol{\lambda}^J)_j(u) = \boldsymbol{\lambda}_j(u)$ for every $j \in J$ and $(\boldsymbol{\lambda}^J)_j(u) = 1$ otherwise.

Given $i \in I$ and $a \in \mathbb{C}(q)^\times$, define the simple ℓ-root $\boldsymbol{\alpha}_{i,a}$ by

$$\boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,aq,2} \prod_{j\neq i} \boldsymbol{\omega}_{j,aq,-c_{j,i}}^{-1}. \tag{1.8}$$

The subgroup of $\mathcal{P}_q$ generated by the simple ℓ-roots is called the ℓ-root lattice of $U_q(\tilde{\mathfrak{g}})$ and will be denoted by $\mathcal{Q}_q$. Let also $\mathcal{Q}_q^+$ be the submonoid generated by the simple ℓ-roots. Quite clearly $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$. Define a partial order on $\mathcal{P}_q$ by

$$\boldsymbol{\mu} \leq \boldsymbol{\lambda} \qquad if \qquad \boldsymbol{\lambda}\boldsymbol{\mu}^{-1} \in \mathcal{Q}_q^+.$$

Remark. The elements $\boldsymbol{\alpha}_{i,a}$ were first defined in [15] where they were denoted by $A_{i,aq}$. The term simple ℓ-root was introduced in [8] where an alternate definition in terms of an action of the braid group of $\mathfrak{g}$ on $\mathcal{P}_q$ was given. For more details on the ℓ-weight lattice see [19, Section 3] and the references therein.

2. Finite-dimensional representations

2.1. **Simple Lie algebras.** For the sake of fixing notation, we now review some basic facts about the representation theory of $\mathfrak{g}$ and $U_q(\mathfrak{g})$. For the details see [18] and [11] for instance.

Given a $U_q(\mathfrak{g})$-module V and $\mu \in P$, let

$$V_\mu = \{v \in V : k_i v = q^{\mu(h_i)} v \text{ for all } i \in I\}.$$

A nonzero vector $v \in V_\mu$ is called a weight vector of weight μ. If v is a weight vector such that $x_i^+ v = 0$ for all $i \in I$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ, then V is said to be a highest-weight module of highest weight λ. A $U_q(\mathfrak{g})$-module V is said to be a weight module if $V = \bigoplus_{\mu \in P} V_\mu$. Denote by $\mathcal{C}_q$ be the category of all finite-dimensional weight modules of $U_q(\mathfrak{g})$. Analogous concepts for $\mathfrak{g}$-modules are defined similarly after setting

$$V_\mu = \{v \in V : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Denote by $\mathcal{C}$ the category of finite-dimensional $\mathfrak{g}$-modules.

Let $\mathbb{Z}[P]$ be the integral group ring over P and denote by $e : P \to \mathbb{Z}[P], \lambda \mapsto e^\lambda$, the inclusion of P in $\mathbb{Z}[P]$ so that $e^\lambda e^\mu = e^{\lambda+\mu}$. The character of an object V from $\mathcal{C}_q$ or $\mathcal{C}$ is defined by

$$\mathrm{char}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu. \tag{2.1}$$

The following theorem summarizes the basic facts about the categories $\mathcal{C}_q$ and $\mathcal{C}$.

Theorem 2.1. Let V be an object either of $\mathcal{C}_q$ or of $\mathcal{C}$. Then:

(a) $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathcal{W}$.
(b) V is completely reducible.
(c) For each $\lambda \in P^+$, the $\mathfrak{g}$-module $V(\lambda)$ generated by a vector v satisfying
$$x_i^+ v = 0, \qquad h_i v = \lambda(h_i) v, \qquad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \forall\, i \in I,$$
is irreducible and finite-dimensional. If $V \in \mathcal{C}$ is irreducible, then V is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.
(d) For each $\lambda \in P^+$ the $U_q(\mathfrak{g})$-module $V_q(\lambda)$ generated by a vector v satisfying
$$x_i^+ v = 0, \qquad k_i v = q^{\lambda(h_i)} v, \qquad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \forall\, i \in I,$$
is irreducible and finite-dimensional. If $V \in \mathcal{C}_q$ is irreducible, then V is isomorphic to $V_q(\lambda)$ for some $\lambda \in P^+$.
(e) For all $\lambda \in P^+$, $\mathrm{char}(V_q(\lambda)) = \mathrm{char}(V(\lambda))$. □

If $J \subseteq I$ we shall denote by $V_q(\lambda_J)$ the simple $U_q(\mathfrak{g}_J)$-module of highest weight λ_J. Similarly $V(\lambda_J)$ denotes the corresponding irreducible $\mathfrak{g}_J$-module.

Proposition 2.2. Let $\lambda \in P^+, J \subseteq I$, and suppose $v \in V_q(\lambda)_\lambda$ (respectively $v \in V(\lambda)_\lambda$) is nonzero. Then $U_q(\mathfrak{g}_J)v \cong V_q(\lambda_J)$ (respectively $U(\mathfrak{g}_J)v \cong V(\lambda_J)$). □

Assume $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_j$ are semisimple Lie algebras. Then $P = P_1 \times P_2$ where P_j is the weight lattice of $\mathfrak{g}_j$ for $j = 1, 2$, and so on. Given $\lambda \in P_j^+$, denote by $V_j(\lambda)$ the irreducible $\mathfrak{g}_j$-module of highest-weight λ. If V_1 is a $\mathfrak{g}_1$-module and V_2 is a $\mathfrak{g}_2$-module, then $V_1 \otimes V_2$ is naturally a $\mathfrak{g}$-module.

Proposition 2.3. Let $\lambda = (\lambda_1, \lambda_2) \in P^+$ and $\mu = (\mu_1, \mu_2) \in P$. Then:

(a) $V(\lambda) \cong V_1(\lambda_1) \otimes V_2(\lambda_2)$ as $\mathfrak{g}$-modules.
(b) $V(\lambda)_\mu \cong (V_1(\lambda_1)_{\mu_1}) \otimes (V_2(\lambda_2)_{\mu_2})$ as $\mathfrak{h}$-modules. □

We will need the following elementary lemma (a proof can be found in [21, Lemma 2.3]).

Lemma 2.4. Let V be a finite-dimensional $\mathfrak{g}$-module and suppose $l \in \mathbb{Z}_{\geq 1}, \nu_k \in P, v_k \in V_{\nu_k}$, for $k = 1, \ldots, l$, are such that $V = \sum_{k=1}^{l} U(\mathfrak{n}^-)v_k$. Fix a decomposition $V = \bigoplus_{j=1}^{m} V_j$ where $m \in \mathbb{Z}_{\geq 1}, V_j \cong V(\mu_j)$ for some $\mu_j \in P^+$, and let $\pi_j : V \to V_j$ be the associated projection for $j = 1, \ldots, m$. Then, there exist distinct $k_1, \ldots, k_m \in \{1, \ldots, l\}$ such that $\nu_{k_j} = \mu_j$ and $\pi_j(v_{k_j}) \neq 0$. □

2.2. **Loop algebras.** Let V be a $U_q(\tilde{\mathfrak{g}})$-module. We say that a nonzero vector $v \in V$ is an ℓ-weight vector if there exists $\boldsymbol{\lambda} \in \mathcal{P}_q$ and $k \in \mathbb{Z}_{>0}$ such that $(\eta - \boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\eta))^k v = 0$ for all $\eta \in U_q(\tilde{\mathfrak{h}})$. In that case, $\boldsymbol{\lambda}$ is said to be the ℓ-weight of v. V is said to be an ℓ-weight module if every vector of V is a linear combination of ℓ-weight vectors. In that case, let $V_{\boldsymbol{\lambda}}$ denote the subspace spanned by all ℓ-weight vectors of ℓ-weight $\boldsymbol{\lambda}$. An ℓ-weight vector v is said to be a highest-ℓ-weight vector if $\eta v = \boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\eta)v$ for every $\eta \in U_q(\tilde{\mathfrak{h}})$ and $x_{i,r}^+ v = 0$ for all $i \in I$ and all $r \in \mathbb{Z}$. V is said to be a highest-ℓ-weight module if it is generated by a highest-ℓ-weight vector. Denote by $\widetilde{\mathcal{C}}_q$ the category of all finite-dimensional ℓ-weight modules of $U_q(\tilde{\mathfrak{g}})$. Quite clearly $\widetilde{\mathcal{C}}_q$ is an abelian category.

Observe that if $V \in \widetilde{\mathcal{C}}_q$, then $V \in \mathcal{C}_q$ and

$$V_\lambda = \bigoplus_{\boldsymbol{\lambda} : \mathrm{wt}(\boldsymbol{\lambda}) = \lambda} V_{\boldsymbol{\lambda}}. \tag{2.2}$$

Moreover, if V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda}$, then

$$\dim(V_{\mathrm{wt}(\boldsymbol{\lambda})}) = 1 \qquad \text{and} \qquad V_\mu \neq 0 \Rightarrow \mu \leq \mathrm{wt}(\boldsymbol{\lambda}). \tag{2.3}$$

Define the concepts of ℓ-weight vector, etc., for $\tilde{\mathfrak{g}}$ in a similar way and denote by $\widetilde{\mathcal{C}}$ the category of all finite-dimensional $\tilde{\mathfrak{g}}$-modules. The next proposition is easily established using (2.3).

Proposition 2.5. If V is a highest-ℓ-weight module, then it has a unique proper submodule and, hence, a unique irreducible quotient. □

Definition 2.6. Let $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. The Weyl module $W_q(\boldsymbol{\lambda})$ of highest ℓ-weight $\boldsymbol{\lambda}$ is the $U_q(\tilde{\mathfrak{g}})$-module defined by the quotient of $U_q(\tilde{\mathfrak{g}})$ by the left ideal generated by the elements $x_{i,r}^+, (x_{i,r}^-)^{\lambda(h_i)+1}$, and $\eta - \boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\eta)$ for every $i \in I, r \in \mathbb{Z}$, and $\eta \in U_q(\tilde{\mathfrak{h}})$. Denote by $V_q(\boldsymbol{\lambda})$ the irreducible quotient of $W_q(\boldsymbol{\lambda})$. The Weyl module $W(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathcal{P}^+$, of $\tilde{\mathfrak{g}}$ is defined in a similar way. Its irreducible quotient will be denoted by $V(\boldsymbol{\lambda})$.

The next theorem was proved in [13].

Theorem 2.7. For every $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ (resp. $\mathcal{P}^+$) the module $W_q(\boldsymbol{\lambda})$ (resp. $W(\boldsymbol{\lambda})$) is the universal finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module (resp. $\tilde{\mathfrak{g}}$-module) with highest ℓ-weight $\boldsymbol{\lambda}$. Every simple object of $\widetilde{\mathcal{C}}_q$ (resp. $\widetilde{\mathcal{C}}$) is highest-ℓ-weight. □

We shall need the following lemma which is a consequence of the proof of Theorem 2.7.

Lemma 2.8. If V is a highest-ℓ-weight module of $\tilde{\mathfrak{g}}$ and v be a highest-ℓ-weight vector. Then $V = U(\mathfrak{g}[t])v$. □

If $J \subseteq I$ we shall denote by $V_q(\boldsymbol{\lambda}_J)$ the $U_q(\tilde{\mathfrak{g}}_J)$-irreducible module of highest ℓ-weight $\boldsymbol{\lambda}_J$. Similarly $V(\boldsymbol{\lambda}_J)$ denotes the corresponding irreducible $\tilde{\mathfrak{g}}_J$-module. Similar notations for the Weyl modules are defined in the obvious way.

The next theorem was conjectured in [15] and proved in [14].

Theorem 2.9. Let V be a quotient of $W_q(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathcal{P}_q^+$. If $V_{\boldsymbol{\mu}} \neq 0$, then $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$. $\square$

Given V in $\widetilde{\mathcal{C}}_q$, let $\mathrm{wt}_\ell(V) = \{\boldsymbol{\mu} \in \mathcal{P}_q : V_{\boldsymbol{\mu}} \neq 0\}$. We will need the following proposition proved in [21, Section 4.8].

Proposition 2.10. Suppose $\mathfrak{g}$ is of type A, $\lambda \in P^+$, $\boldsymbol{\lambda} = \prod_{i \in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$, $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$, and $\boldsymbol{\lambda}\boldsymbol{\mu}^{-1} = \boldsymbol{\alpha}_{j,b_j}\boldsymbol{\alpha}_{j+1,b_{j+1}} \cdots \boldsymbol{\alpha}_{k,b_k}$ for some $j \leq k$ and some $a_i, b_l \in \mathbb{C}(q)^\times, i \in I, l = j, \ldots, k$.

(a) If $\frac{a_{i+1}}{a_i} = q^{\lambda(h_i)+\lambda(h_{i+1})+1}$ for all $i < n$, then $b_k = a_k q^{\lambda(h_k)-1}$.
(b) If $\frac{a_{i+1}}{a_i} = q^{-(\lambda(h_i)+\lambda(h_{i+1})+1)}$ for all $i < n$, then $b_j = a_j q^{\lambda(h_j)-1}$. $\square$

2.3. **Classical limits.** Denote by $\mathcal{P}_{\mathbb{A}}^+$ the subset of $\mathcal{P}_q$ consisting of n-tuples of polynomials with coefficients in $\mathbb{A}$. Let also $\mathcal{P}_{\mathbb{A}}^\times$ be the subset of $\mathcal{P}_{\mathbb{A}}^+$ consisting of n-tuples of polynomials whose leading terms are in $\mathbb{C}q^{\mathbb{Z}} \backslash \{0\} = \mathbb{A}^\times$. Given $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^+$, let $\overline{\boldsymbol{\lambda}}$ be the element of $\mathcal{P}^+$ obtained from $\boldsymbol{\lambda}$ by evaluating q at 1.

Recall that an $\mathbb{A}$-lattice (or form) of a $\mathbb{C}(q)$-vector space V is a free $\mathbb{A}$-submodule L of V such that $\mathbb{C}(q) \otimes_{\mathbb{A}} L = V$. If V is a $U_q(\tilde{\mathfrak{g}})$-module, a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of V is an $\mathbb{A}$-lattice of V which is also a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-submodule of V. Given a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of a $U_q(\tilde{\mathfrak{g}})$-module V, define

$$\bar{L} = \mathbb{C} \otimes_{\mathbb{A}} L, \tag{2.4}$$

where $\mathbb{C}$ is regarded as an $\mathbb{A}$-module by letting q act as 1. Then $\bar{L}$ is a $\tilde{\mathfrak{g}}$-module by Proposition 1.2 and $\dim(\bar{L}) = \dim(V)$. The next theorem is essentially a corollary of the proof of Theorem 2.7.

Theorem 2.11. Let V be a nontrivial quotient of $W_q(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^\times$, v a highest-ℓ-weight vector of V, and $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then, L is a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of V and $\mathrm{char}(\bar{L}) = \mathrm{char}(V)$. In particular, $\bar{L}$ is a quotient of $W(\overline{\boldsymbol{\lambda}})$. $\square$

Definition 2.12. Let $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^\times$, v be a highest-ℓ-weight vector of $V_q(\boldsymbol{\lambda})$ and $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. We denote by $\overline{V_q(\boldsymbol{\lambda})}$ the $\tilde{\mathfrak{g}}$-module $\bar{L}$.

3. Minimal affinizations and Beyond

3.1. **Classification of minimal affinizations.** We now review the notion of minimal affinizations of an irreducible $U_q(\mathfrak{g})$-module introduced in [1].

Given $\lambda \in P^+$, a $U_q(\tilde{\mathfrak{g}})$-module V is said to be an affinization of $V_q(\lambda)$ if, as a $U_q(\mathfrak{g})$-module,

$$V \cong V_q(\lambda) \oplus \bigoplus_{\mu < \lambda} V_q(\mu)^{\oplus m_\mu(V)} \tag{3.1}$$

for some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. Two affinizations of $V_q(\lambda)$ are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. If $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ is such that $\mathrm{wt}(\boldsymbol{\lambda}) = \lambda$, then $V_q(\boldsymbol{\lambda})$ is quite clearly an affinization of $V_q(\lambda)$. The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$. Namely, if V and W are affinizations of $V_q(\lambda)$, say that $V \leq W$ if one of the following conditions hold:

(a) $m_\mu(V) \leq m_\mu(W)$ for all $\mu \in P^+$;
(b) for all $\mu \in P^+$ such that $m_\mu(V) > m_\mu(W)$ there exists $\nu > \mu$ such that $m_\nu(V) < m_\nu(W)$.

A minimal element of this partial order is said to be a minimal affinization.

Theorem 3.1 ([12])**.** Let $\boldsymbol{\lambda} \in \mathcal{P}_q^+, \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $V = V_q(\boldsymbol{\lambda})$. Suppose $\mathfrak{g}$ is of type A. Then V is a minimal affinization of $V_q(\lambda)$ iff there exist $a \in \mathbb{C}(q)^\times$ and $\epsilon \in \{1,-1\}$ such that

$$\boldsymbol{\lambda} = \prod_{i=1}^{n} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \qquad \text{with} \qquad a_1 = a \qquad \text{and} \qquad \frac{a_{i+1}}{a_i} = q^{\epsilon(\lambda(h_i)+\lambda(h_{i+1})-1)}$$

for all $i \in I, i < n$. If $\mathfrak{g}$ is of type D or E, suppose the support of λ is contained in a connected subdiagram $J \subseteq I$ of type A. Then, V is a minimal affinization of $V_q(\lambda)$ iff $V_q(\boldsymbol{\lambda}_J)$ is a minimal affinization of $V_q(\lambda_J)$. □

The next corollaries are easily established (recall from §1.1 that $\overline{\mathrm{supp}}(\lambda)$ is the minimal connected subdiagram of I containing $\mathrm{supp}(\lambda)$).

Corollary 3.2. Suppose $\lambda \in P^+$ is such that $\overline{\mathrm{supp}}(\lambda)$ is of type A. Then, $V_q(\lambda)$ has a unique equivalence class of minimal affinizations. □

Corollary 3.3. Given $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, the modules $V_q(\boldsymbol{\omega}_{i,a,m}), a \in \mathbb{C}(q)^\times$, are the only minimal affinizations of $V_q(m\omega_i)$. □

The modules $V_q(\boldsymbol{\omega}_{i,a,m})$ are known as Kirillov-Reshetikhin modules.

We now state a few results which were used in the proof of Theorem 3.1 and will be useful for us as well. The proofs can be found in [12].

Lemma 3.4. Suppose $\emptyset \neq J \subseteq I$ is a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$. Let $V = V_q(\boldsymbol{\lambda})$, v a highest-ℓ-weight vector of V, and $V_J = U_q(\tilde{\mathfrak{g}}_J)v$. Then, $V_J \cong V_q(\boldsymbol{\lambda}_J)$. □

Definition 3.5. Suppose $\mathfrak{g}$ is of type A. A connected subdiagram $J \subseteq I$ is said to be an admissible subdiagram. If $\mathfrak{g}$ is of type D or E, let $i_0 \in I$ be the trivalent node. A connected subdiagram $J \subseteq I$ is said to be admissible if J is of type A and $J\backslash\{i_0\}$ is connected.

Proposition 3.6. Suppose $J \subseteq I$ is admissible and that $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ is such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$ where $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Then $V_q(\boldsymbol{\lambda}_J)$ is a minimal affinization of $V_q(\lambda_J)$. □

The next proposition was proved in [21, Proposition 3.7].

Proposition 3.7. Let $\boldsymbol{\lambda} \in P_q^+$ and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. If $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$, then there exist $a_i \in \mathbb{C}(q)^\times, i \in I$, such that $\boldsymbol{\lambda} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$ and $\frac{a_i}{a_j} \in q^{\mathbb{Z}}$ for all $i,j \in I$. □

Corollary 3.8. For every $\lambda \in P^+$ there exist $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^\times$ such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$. Moreover, $\overline{\boldsymbol{\lambda}} = \boldsymbol{\omega}_{\lambda,a}$ for some $a \in \mathbb{C}^\times$. □

3.2. **Graded characters.** Recall the definition of the maps $\tau_a : \mathfrak{g}[t] \to \mathfrak{g}[t]$ from subsection 1.1.

Definition 3.9. Let $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^\times, \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $a \in \mathbb{C}^\times$ be such that $\overline{\boldsymbol{\lambda}} = \boldsymbol{\omega}_{\lambda,a}$. The $\mathfrak{g}[t]$-module $L(\boldsymbol{\lambda})$ is defined to be the pullback of $\overline{V_q(\boldsymbol{\lambda})}$ by τ_a.

It is immediate from Theorem 2.11 that

$$\mathrm{char}(L(\boldsymbol{\lambda})) = \mathrm{char}(V_q(\boldsymbol{\lambda})). \tag{3.2}$$

Let V be a $\mathbb{Z}_{\geq 0}$-graded vector space and denote its r-th graded piece by $V[r]$. A $\mathfrak{g}[t]$-module V is said to be $\mathbb{Z}_{\geq 0}$-graded if V is a $\mathbb{Z}_{\geq 0}$-graded vector space and $x \otimes t^s v \in V[r+s]$ for every $v \in V[r], x \in \mathfrak{g}, r,s \in \mathbb{Z}_{\geq 0}$. Observe that if V is a $\mathbb{Z}_{\geq 0}$-graded $\mathfrak{g}[t]$-module, then each graded peace is a $\mathfrak{g}$-module. Given $s \in \mathbb{Z}_{\geq 0}$, denote by $V(s)$ the quotient of V by its $\mathfrak{g}[t]$-submodule $\bigoplus_{r>s} V[r]$. We shall refer to $V(s)$ as the truncation of V at degree s. If V is a finite-dimensional $\mathbb{Z}_{\geq 0}$-graded $\mathfrak{g}[t]$-module, define the graded character of V by

$$\mathrm{char}_t(V) = \sum_{r\geq 0} \mathrm{char}(V[r])\ t^r \ \in\ \mathbb{Z}[P][t].$$

Let also $m_{\mu,r}(V)$ be the multiplicity of $V(\mu)$ as an irreducible constituent of $V[r]$.

Definition 3.10. Let $m \in \mathbb{Z}_{\geq 0}$ and $i \in I$. The $\mathfrak{g}[t]$-module $M(m\omega_i)$ is the quotient of $U(\mathfrak{g}[t])$ by the left ideal generated by

$$\mathfrak{n}^+[t], \qquad \mathfrak{h}\otimes t\mathbb{C}[t], \qquad h_j, \qquad h_i - m, \qquad x^-_{\alpha_j}, \qquad (x^-_{\alpha_i})^{m+1}, \qquad x^-_{\alpha_i,1} \qquad \text{for all } j\neq i. \tag{3.3}$$

Define $T(m\omega_i)$ to be the $\mathfrak{g}[t]$-submodule of $M(\omega_i)^{\otimes m}$ generated by the top weight space.

Quite clearly $M(m\omega_i)$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathfrak{g}[t]$-module. Given $\lambda \in P^+$ one can consider the modules $A(\lambda)$ defined in [21]. These are graded $\mathfrak{g}[t]$-modules which were proved to be finite-dimensional in [21, Proposition 3.15]. One can proceed similarly to prove that the modules $M(m\omega_i)$ are finite-dimensional. Moreover, it was proved in [23, Proposition 5.2.5] that $A(m\omega_i) \cong M(m\omega_i)$ (for a general simple Lie algebra $\mathfrak{g}$). We shall not need the modules $A(\lambda)$ in this paper.

Given $i \in I, m, r \in \mathbb{Z}_{\geq 0}$, let $v_{i,m}$ be the image of 1 in $M(m\omega_i)$ and set

$$R(i,m,r) = \{\alpha \in R^+ : x^-_{\alpha,r}v_{i,m} = 0\}. \tag{3.4}$$

Since $(\mathfrak{h}\otimes t\mathbb{C}[t])v_{i,m} = 0$, it follows that

$$R(i,m,r) \subseteq R(i,m,s) \qquad \text{for all } s \geq r. \tag{3.5}$$

In particular, it follows that $M(0)$ is the trivial representation and $R^+(i,0,s) = R^+$ for all $i \in I, s \in \mathbb{Z}_{\geq 0}$. Now, given $\lambda \in P^+$ and $r \in \mathbb{Z}_{\geq 0}$, set

$$R(\lambda,r) = \bigcap_{i\in I} R(i,\lambda(h_i),r). \tag{3.6}$$

Notice $R(m\omega_i, r) = R(i,m,r)$ for all $i \in I$ and $m, r \in \mathbb{Z}_{\geq 0}$.

Definition 3.11. Let $\lambda \in P^+$. The $\mathfrak{g}[t]$-module $M(\lambda)$ is the quotient of $U(\mathfrak{g}[t])$ by the left ideal generated by

$$\mathfrak{n}^+[t], \qquad \mathfrak{h}\otimes t\mathbb{C}[t], \qquad h_i - \lambda(h_i), \qquad (x^-_{\alpha_i})^{\lambda(h_i)+1}, \qquad x^-_{\alpha,r} \tag{3.7}$$

for all $i \in I, r \in \mathbb{Z}_{\geq 0}$, and $\alpha \in R(\lambda, r)$. Define $T(\lambda)$ to be the $\mathfrak{g}[t]$-submodule of $\bigotimes_{i\,\in\, I} M(\lambda(h_i)\omega_i)$ generated by the top weight space.

Definitions 3.10 and 3.11 of $M(m\omega_i)$ coincide since $R(m\omega_i, r) = R(i,m,r)$ for all $i \in I, m, r \in \mathbb{Z}_{\geq 0}$. The modules $M(\lambda)$ are clearly $\mathbb{Z}_{\geq 0}$-graded. It follows from [21, Proposition 3.13] that $M(\lambda)$ is a quotient of the module $A(\lambda)$ of [21] and, hence, finite-dimensional. Moreover, one easily sees that $T(\lambda)$ is a graded quotient of $M(\lambda)$ for all $\lambda \in P^+$ (the details can be found in [23, Proposition 5.2.10]).

Proposition 3.12 ([21, Proposition 3.21])**.** Let $\boldsymbol{\lambda} \in \mathcal{P}^\times_{\mathbb{A}}$ be such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$ where $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Then, $T(\lambda)$ is a quotient of $L(\boldsymbol{\lambda})$. □

The following is the main conjecture of [21].

Conjecture 3.13. Let $\lambda \in P^+$. Then, $M(\lambda) \cong T(\lambda)$. Moreover, if $\overline{\mathrm{supp}}(\lambda)$ is of type A and $\boldsymbol{\lambda} \in \mathcal{P}^\times_{\mathbb{A}}$ is such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$, then, $M(\lambda) \cong L(\boldsymbol{\lambda})$.

For the rest of the subsection assume that $\mathfrak{g}$ is of type E_6 and that the nodes of the Dynkin diagram are labeled as in the introduction. We now state our main results.

Theorem 3.14. Let $\lambda \in P^+$ be such that $\lambda(h_3) = 0$. Suppose that either $\{2,4\} \not\subseteq \mathrm{supp}(\lambda)$ or $\mathrm{supp}(\lambda) \subseteq \{2,4,6\}$. Then:

(a) The first isomorphism in Conjecture 3.13 holds.
(b) The second isomorphism in Conjecture 3.13 holds provided that $\overline{\mathrm{supp}}(\lambda)$ is of type A.

Notice that part (a) of Theorem 3.14 and Proposition 3.12 together with the following proposition which will be proved in Subsection 4.4 imply part (b) of Theorem 3.14.

Proposition 3.15. Let $\lambda \in P^+$ be such that is of type A. Then, $L(\boldsymbol{\lambda})$ is a quotient of $M(\lambda)$.

As a byproduct of the proof of Theorem 3.14 we are able to compute $\mathrm{char}_t(M(\lambda))$ for λ as in the theorem. In particular, we compute $\mathrm{char}(V_q(\boldsymbol{\lambda}))$ for all $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ such that $\mathrm{wt}(\boldsymbol{\lambda})$ satisfies the hypothesis of part (b) of the theorem. Let us now present these formulas and, along the way, explain the strategy of the proof of Theorem 3.14(a).

Fix $\lambda \in P^+$ and, given $\mu \in P$ and $r \in \mathbb{Z}_{\geq 0}$, set

$$m_{\mu,r} = m_{\mu,r}(M(\lambda)) \qquad \text{and} \qquad t_{\mu,r} = m_{\mu,r}(T(\lambda)).$$

We have already seen that $t_{\mu,r} \leq m_{\mu,r}$. Therefore, in order to prove the first isomorphism of Conjecture 3.13, it suffices to show that

$$m_{\mu,r} \leq t_{\mu,r} \qquad \text{for all} \qquad \mu \in P^+, r \in \mathbb{Z}_{\geq 0}. \tag{3.8}$$

For $\mathbf{r} \in \mathbb{Z}^6$, set

$$\mathrm{wt}(\mathbf{r}) = \lambda - r_1(\omega_2 - \omega_5) - r_2(\omega_4 - \omega_1) - r_3(\omega_2 - \omega_4 + \omega_5) - r_4(\omega_1 - \omega_2 + \omega_4) - r_5(\omega_2 - \omega_3 + \omega_4) - r_6\omega_6$$

and

$$\mathrm{gr}(\mathbf{r}) = r_1 + r_2 + r_3 + r_4 + r_5 + r_6.$$

Let also

$$\mathcal{A} = \{\mathbf{r} \in \mathbb{Z}_{\geq 0}^6 : r_6 \leq m_6, r_3 \leq m_5, r_4 \leq m_1, r_1 + r_3 + r_5 \leq m_2, r_2 + r_4 + r_5 \leq m_4\},$$

$$\mathcal{A}_\mu = \{\mathbf{r} \in \mathcal{A} : \mathrm{wt}(\mathbf{r}) = \mu\}, \qquad \mathcal{A}_r = \{\mathbf{r} \in \mathcal{A} : \mathrm{gr}(\mathbf{r}) = r\}, \quad \text{and} \quad \mathcal{A}_{\mu,r} = \mathcal{A}_\mu \cap \mathcal{A}_r.$$

The omission of the dependence of wt and $\mathcal{A}$ on λ in the notation will not create confusion. One easily checks that the function $\mathrm{wt} : \mathbb{Z}^6 \to P$ is injective and, if $\mathbf{r} \in \mathcal{A}$, then $\mathrm{wt}(\mathbf{r}) \in P^+$. In particular,

$$|\mathcal{A}_\mu| \leq 1 \qquad \text{for all} \qquad \mu \in P^+. \tag{3.9}$$

The basic idea for proving (3.8) is the same one used in [9, 10, 21]. Namely, in Subsection 4.5, we will use the defining relations of $M(\lambda)$ to show that,

$$\text{if} \qquad \lambda(h_3) = 0, \qquad \text{then} \qquad m_{\mu,r} \leq |\mathcal{A}_{\mu,r}|. \tag{3.10}$$

Moreover, for λ as in Theorem 3.14, by performing some explicit computations in $T(\lambda)$, we show in Subsection 4.7 that

$$t_{\mu,r} \geq |\mathcal{A}_{\mu,r}|. \tag{3.11}$$

Clearly (3.10) and (3.11) together imply (3.8). Moreover,

$$\mathrm{char}_t(M(\lambda)) = \sum_{\mathbf{r} \in \mathcal{A}} \mathrm{char}(V(\mathrm{wt}(\mathbf{r})))t^{\mathrm{gr}(\mathbf{r})} \tag{3.12}$$

for all λ as in Theorem 3.14. In particular, for λ as in Theorem 3.14(b) and $\boldsymbol{\lambda} \in \mathcal{P}_q^+$ such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$, we have

$$\mathrm{char}(V_q(\boldsymbol{\lambda})) = \sum_{\mathbf{r} \in \mathcal{A}} \mathrm{char}(V(\mathrm{wt}(\mathbf{r}))). \tag{3.13}$$

Remark. Similar results in the case that $\mathfrak{g}$ is of classical type or G_2 were obtained in [9, 10, 21] (however, the definition of the modules $T(m\omega_i)$ requires some extra care in the non simply laced case). Equation (3.12) (and similar ones for general $\mathfrak{g}$) was predicted in [16] in the case that $\lambda = m\omega_i$ for some $i \in I, m \in \mathbb{Z}_{\geq 0}$. However, the meaning of the gradation in [16] is related to the quantum context, whereas here it appears by computing the classical limit. It is not clear to us why these two gradations coincide. The formulas in [16] were obtained by assuming the Kirillov-Reshetikhin conjecture whose proof was later completed in [17]. Our results give an alternate proof of these formulas for $\mathfrak{g}$ of type E_6 and $i \neq 3$. As mentioned in the introduction, $M(m\omega_3)$ is not multiplicity free in general. Using the methods of this paper, we are able to prove that the isotypical components of $M(m\omega_3)[r]$ are exactly as given by [16]. However, so far we could only obtain an upper bound for $m_{\mu,r}$ which is most often larger than the actual value of $m_{\mu,r}$.

We end this subsection by reviewing a construction used in [9, §2.6] which will be useful for us as well. Let V_r, $0 \leq r \leq k$, be $\mathfrak{g}$-modules such that

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V_r, V_{r+1}) \neq 0, \quad \mathrm{Hom}_{\mathfrak{g}}(\wedge^2(\mathfrak{g}) \otimes V_r, V_{r+2}) = 0, \quad 0 \leq r \leq k-1, \tag{3.14}$$

where we assume that $V_{k+1} = 0$. Fix non-zero elements $p_r \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V_r, V_{r+1})$, $0 \leq r \leq k-1$, and set $p_k = 0$. It is easily checked that the following formulas extend the canonical $\mathfrak{g}$-module structure to a graded $\mathfrak{g}[t]$-module structure on $V = \oplus_{r=1}^k V_r$:

$$(x \otimes t)w = p_r(x \otimes w), \quad (x \otimes t^s)w = 0, \quad \text{for all} \quad x \in \mathfrak{g}, w \in V_r, 1 \leq r \leq k, s \geq 2. \tag{3.15}$$

Moreover, $V[r] \cong V_r$ for all $0 \leq r \leq k$. Also, if $V_0 = U(\mathfrak{g})w_0$ and the maps p_r for $r < k$ are all surjective, then $V = U(\mathfrak{n}^-[t])w_0$.

3.3. **Projectivity.** If $\overline{\mathrm{supp}}(\lambda)$ is not of type A, then Proposition 3.15 is probably false. In fact, most likely, $M(\lambda)$ is then a proper quotient of $L(\boldsymbol{\lambda})$. We now explain the motivation for studying the modules $M(\lambda)$ beyond the cases associated to minimal affinizations from the perspective of [5]. We begin with following straightforward lemma which has been implicitly used in [5].

Lemma 3.16. Let $r \in \mathbb{Z}_{>0}$ and V be a $\mathfrak{g}[t]$-module generated by a vector v satisfying $(\mathfrak{g}\otimes t^r\mathbb{C}[t])v = 0$. Then, $(\mathfrak{g} \otimes t^r\mathbb{C}[t])V = 0$.

Proof. Let $x \in \mathfrak{g}, s \geq r$, and $w = (x_1 \otimes t^{r_1}) \cdots (x_2 \otimes t^{r_m})v$ for some $m, r_j \in \mathbb{Z}_{\geq 0}, x_j \in \mathfrak{g}, j = 1, \ldots, m$. We proceed by induction on m. If $m = 0$, we have $(x \otimes t^s)w = 0$ by hypothesis. Assume $m > 0$, let $w' = (x_2 \otimes t^{r_2}) \cdots (x_m \otimes t^{r_m})v$ and assume, by induction hypothesis, that $(y \otimes t^s)w' = 0$ for all $y \in \mathfrak{g}, s \geq r$. Then, given $x \in \mathfrak{g}$ and $s \geq r$, we have

$$(x \otimes t^s)w = (x_1 \otimes t^{r_1})(x \otimes t^s)w' + ([x, x_1] \otimes t^{s+s_1})w'.$$

Both summands are zero by the induction hypothesis on m. $\square$

The next proposition follows immediately from the above lemma and the definition of $M(\lambda)$.

Proposition 3.17. Let $\lambda \in P^+$ and $r > 0$ be such that $R(\lambda, r) = R^+$. Then, $(\mathfrak{g} \otimes t^r\mathbb{C}[t])M(\lambda) = 0.\square$

If V is a $\mathfrak{g}[t]$-module as in Lemma 3.16, then the canonical projection $\mathfrak{g}[t] \to \mathfrak{g}[t:r] := \mathfrak{g}[t]/\mathfrak{g}\otimes t^r\mathbb{C}[t]$ induces a $\mathfrak{g}[t:r]$-module structure on V. Chari and Greenstein in [4, 5] initiated the study of the category $\mathcal{G}_2$ of graded $\mathfrak{g}[t:2]$-modules with finite-dimensional graded pieces (they do not assume $\mathfrak{g}$ is simply laced). Given a subset Γ of $P^+ \times \mathbb{Z}_{\geq}0$, they consider the full subcategories $\mathcal{G}_2(\Gamma)$ of $\mathcal{G}_2$ consisting of modules V such that $V(\mu)$ is an irreducible constituent of $V[r]$ only if $(\mu, r) \in \Gamma$. In particular, they consider subsets Γ of the following form. Given $\Psi \subseteq R^+$ and $\lambda \in P$, set

$$\Gamma(\lambda, \Psi) = \{(\mu, r) \in P \times \mathbb{Z}_{\geq}0 : \lambda - \mu = \sum_{\beta\in\Psi} n_\beta\beta, n_\beta \in \mathbb{Z}_{\geq}0, \sum_{\beta\in\Psi} n_\beta = r\}.$$

Notice that $(\lambda,0)\in\Gamma(\lambda,\Psi)$ for any choice of Ψ and that $\Gamma(\lambda,\emptyset)=\{(\lambda,0)\}$. If we regard $V(\lambda)$ as a module for $\mathfrak{g}[t:2]$ by pulling back the canonical projection $\mathfrak{g}[t:2]\to\mathfrak{g}[t:1]=\mathfrak{g}$, then $V(\lambda)$ is an object of $\mathcal{G}_2(\Gamma(\lambda,\Psi))$. The full strength of the results of [5] is realized when Ψ is either empty or of the form Ψ_ν for some $\nu\in P$ where

$$\Psi_\nu=\{\alpha\in R^+:(\alpha,\nu)=\max\{(\beta,\nu):\beta\in R^+\}\}$$

and $(\cdot,\cdot)$ is the bilinear form on $P\times P$ induced from the Killing form of $\mathfrak{g}$.

For $\lambda\in P^+$ such that $R(\lambda,2)=R^+$, set $\Psi^\lambda=R^+\backslash R(\lambda,1)$. The following theorem is a particular case of [5, Theorem 1].

Theorem 3.18. Let $\lambda\in P^+$ be such that $R(\lambda,2)=R^+$ and suppose that either $\Psi^\lambda=\emptyset$ or $\Psi^\lambda=\Psi_\nu$ for some $\nu\in P$. Then, $M(\lambda)$ is the projective cover of $V(\lambda)$ in the category $\mathcal{G}_2(\Gamma(\lambda,\Psi^\lambda))$. □

For λ as in Theorem 3.18, [5, Theorem 2] gives a formula for computing the graded character of $M(\lambda)$ by induction on the cardinality of the set $\Gamma(\lambda,\Psi^\lambda)$.

Let us return to the case that $\mathfrak{g}$ is of type E_6. It follows from the proof of Theorem 3.14 (see Lemma 4.11 below) that $M(\lambda)$ is a module as in Lemma 3.16 with $r=3$. Moreover, if $\lambda(h_3)=0$, then we can take $r=2$.

Lemma 3.19. Let $\lambda\in P^+$ be such that $\lambda(h_3)=0$ and $\{2,4\}\not\subseteq\mathrm{supp}(\lambda)$. Then, either $\Psi^\lambda=\emptyset$ or there exists $\nu\in P$ such that $\Psi^\lambda=\Psi_\nu$.

Proof. Recalling that $(\alpha_i,\nu)=\frac{1}{2}(\alpha_i,\alpha_i)\nu(h_i)$ and using the characterizations of $R(\lambda,1)$ given by (4.7), one easily checks by inspection of Table 1 below that

(a) $\mathrm{supp}(\lambda)\subseteq\{1,5\}\Rightarrow\Psi^\lambda=\emptyset$.
(b) $6\in\mathrm{supp}(\lambda)\subseteq\{1,5,6\}\Rightarrow\Psi^\lambda=\Psi_{\omega_6}$.
(c) $2\in\mathrm{supp}(\lambda)\subseteq\{1,2,5,6\}\Rightarrow\Psi^\lambda=\Psi_{\omega_2}$.
(d) $4\in\mathrm{supp}(\lambda)\subseteq\{1,4,5,6\}\Rightarrow\Psi^\lambda=\Psi_{\omega_4}$.

Clearly λ satisfies the hypothesis of the lemma iff it satisfies one of the conditions (a)-(d) above. □

This immediately implies the following corollary of Theorem 3.18.

Corollary 3.20. Let λ be as in Lemma 3.19. Then, $M(\lambda)$ is the projective cover of $V(\lambda)$ in the category $\mathcal{G}_2(\Gamma(\lambda,\Psi^\lambda))$. □

Similarly to the proof of Lemma 3.19, one easily checks that if $\{2,4\}\subseteq\mathrm{supp}(\lambda)$, then $\Psi^\lambda\neq\emptyset$ and $\Psi^\lambda\neq\Psi_\nu$ for all $\nu\in P$. Therefore, λ satisfies the hypothesis of Theorem 3.18 iff it satisfies the hypothesis of Lemma 3.19. It follows that every λ as in Theorem 3.18 satisfies the hypothesis of Theorem 3.14. On the other hand, if λ satisfies the hypothesis of Theorem 3.14 but not the one of Theorem 3.18, then $\{2,4\}\subseteq\mathrm{supp}(\lambda)\subseteq\{2,4,6\}$. In this case, we cannot conclude that $M(\lambda)$ is a projective object in some subcategory of $\mathcal{G}_2$ nor can we use [5, Theorem 2] to compute its graded character.

Remark. It is worth remarking that we will perform most of the proof of (3.11) using only the hypothesis $\lambda(h_3)=0$. This provides some evidence that Conjecture 3.13 holds in complete generality. In particular, we conjecture that (3.12) is the graded character of $M(\lambda)$ for all $\lambda\in P^+$ such that $\lambda(h_3)=0$.

4. Proofs

4.1. **On characters for type A_2.** We now record some lemmas about the characters of certain finite-dimensional $\mathfrak{sl}_3$-modules which will be needed in the proof of (3.11). To simplify some formulas, we introduce the notation of divided powers. If A is an associative algebra, $x \in A$, and $r \in \mathbb{Z}_{\geq 0}$, set $x^{(r)} = \frac{1}{r!}x^r$.

We will make use of the following result on representations of the 3-dimensional Heisenberg algebra which will also be used in the proof of (3.10). Thus, consider the three-dimensional Heisenberg Lie algebra $\mathfrak{H}$ spanned by elements x, y, z where z is central and $[x, y] = z$. Part (a) of the following lemma is standard while a proof of part (b) can be found in [10, Lemma 1.5].

Lemma 4.1. Let $r, s \in \mathbb{Z}_{\geq 0}$, V a representation of $\mathfrak{H}$, and suppose $0 \neq v \in V$ is such that $x^r v = 0$.

(a) The following identity holds in $U(\mathfrak{H})$: $x^{(r)}y^{(s)} = \sum_{k=0}^{\min\{r,s\}} z^{(k)}y^{(s-k)}x^{(r-k)}$.
(b) For all $k \in \mathbb{Z}_{\geq 0}$, the element $y^s z^k v$ is in the span of elements of the form $x^a y^b z^c v$ with $0 \leq c < r, a + c = k$, and $b + c = k + s$. Moreover, if $xv = 0$, then $y^s zv = \frac{1}{s+1}\, xy^{s+1}v$. □

Recall that $U(\mathfrak{n}^-)$ is Q^+-graded and denoted by $U(\mathfrak{n}^-)_\eta$ the piece of degree η. For the remainder of this subsection we assume $\mathfrak{g} = \mathfrak{sl}_3$ and $I = \{1, 2\}$. Observe that the map $\mathfrak{n}^- \to \mathfrak{H}$ given by $x_i^- \mapsto x$ and $x_j^- \mapsto y$, where $i, j \in I$ are distinct, is an isomorphism.

Lemma 4.2. Let $i, j \in I, i \neq j$, and $\eta = k_i\alpha_i + k_j\alpha_j \in Q^+$. Then $\{(x_i^-)^{(r)}(x_j^-)^{(k_j)}(x_i^-)^{(k_i-r)} : 0 \leq r \leq \min\{k_i, k_j\}\}$ is a basis of $U(\mathfrak{n}^-)_\eta$.

Proof. Since $\dim(U(\mathfrak{n}^-)_\eta) = p(\eta) = \min\{k_i, k_j\} + 1$, it suffices to show that this set is linearly independent. Let us write $x = x_i^-, y = x_j^-$, and $z = [x, y]$. Then, by part (a) of Lemma 4.1 we have

$$x^{(r)}y^{(k_j)}x^{(k_i-r)} = \sum_{k=0}^{\min\{r,k_j\}} \binom{k_i - k}{r - k} z^{(k)}y^{(k_j-k)}x^{(k_i-k)}.$$

One now easily uses the PBW theorem to prove that these vectors, with $0 \leq r \leq \min\{k_i, k_j\}$, are linearly independent. □

Lemma 4.3. Let $\lambda = m_1\omega_1 + m_2\omega_2 \in P^+, 0 \leq k_1 \leq m_1, 0 \leq k_2 \leq m_2$, and $\mu = \lambda - k_1\alpha_1 - k_2\alpha_2$. Then, $\dim(V(\lambda)_\mu) = \min\{k_1, k_2\} + 1$.

Proof. Straightforward using Kostant's multiplicity formula (cf. [23, Proposition 5.3.10]). □

Lemma 4.4. Let V be a finite-dimensional $\mathfrak{g}$-module, $l \in \mathbb{Z}_{\geq 1}$, and $\mu_1, \dots, \mu_l \in P^+$. Assume $\mu_l < \mu_s$ for all $s < l$, write $\eta_s = \mu_s - \mu_l = k_{s,1}\alpha_1 + k_{s,2}\alpha_2$, and suppose $k_{s,i} \leq \mu_s(h_i), i \in I$. Suppose also that there exists $v_s \in V_{\mu_s}$ such that $V = \sum_{s=1}^{l} U(\mathfrak{n}^-)v_s$. Let $i, j \in I$ be distinct. Then, $V \cong \bigoplus_{s=1}^{l} V(\mu_s)$ iff the vectors $(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}v_s$ for $s = 1, \dots, l$ and $0 \leq r \leq \min\{k_{s,1}, k_{s,2}\}$ are linearly independent.

Proof. By Lemma 4.3 we have $\dim(V(\mu_s))_{\mu_l} = \min\{k_{s,1}, k_{s,2}\} + 1$ and by Lemma 2.4 there exists $m \leq l$ and $s_1, \dots, s_m$ such that $V \cong \bigoplus_{r=1}^{m} V(\mu_{s_r})$. Hence,

$$\dim(V_{\mu_l}) = \sum_{r=1}^{m} \dim(V(\mu_{s_r})_{\mu_l}) = \sum_{r=1}^{m} (\min\{k_{s_r,1}, k_{s_r,2}\} + 1).$$

The if part follows since the cardinality of the set $\{(s,r) : s = 1,\dots,l, 0 \le r \le \min\{k_{s,1},k_{s,2}\}\}$ is $\sum_{s=1}^{l}(\min\{k_{s,1},k_{s,2}\}+1)$.

Conversely, assume that $V \cong \bigoplus_{s=1}^{l} V(\mu_s)$ and let $V_s, s = 1,\dots,l$, be a submodule of V isomorphic to $V(\mu_s)$ and such that $V = \bigoplus_{s=1}^{l} V_s$. Let also $\pi_s : V \to V_s$ be the associated projection. By Lemma 2.4 we can assume $\pi_s(v_s)$ is a highest weight vector of V_s. Observe that the set $(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}\pi_s(v_s)$ with $0 \le r \le \min\{k_{s,1},k_{s,2}\}$ is a basis of $(V_s)_{\mu_l}$. Indeed, the set $(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}$ is a basis for $U(\mathfrak{n}^-)_{\eta_s}$ by Lemma 4.2. In particular, the vectors $(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}\pi_s(v_s)$ with $0 \le r \le \min\{k_{s,1},k_{s,2}\}$ span $(V_s)_{\mu_l}$. Since we already know that $\dim((V_s)_{\mu_l}) = \min\{k_{s,1},k_{s,2}\}+1$, the claim follows. Let $a_{r,s} \in \mathbb{C}$ be such that

$$\sum_{s=1}^{l}\sum_{r=0}^{\min\{k_{s,1},k_{s,2}\}} a_{r,s}(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}v_s = 0.$$

Given $1 \le t \le l$, we get

$$\pi_t(\sum_{s=1}^{l}\sum_{r=0}^{\min\{k_{s,1},k_{s,2}\}} a_{r,s}(x_i^-)^{(r)}(x_j^-)^{(k_{s,j})}(x_i^-)^{(k_{s,i}-r)}v_s) =$$
$$\sum_{r=0}^{\min\{k_{t,1},k_{t,2}\}} a_{r,t}(x_i^-)^{(r)}(x_j^-)^{(k_{t,j})}(x_i^-)^{(k_{t,i}-r)}\pi_t(v_t) = 0.$$

It follows that $a_{r,t} = 0$ for all $t = 1,\dots,l$ and $0 \le r \le \min\{k_{t,1},k_{t,2}\}$. □

Lemma 4.5. Let $a,b,c,m \in \mathbb{Z}_{\ge 0}, i,j \in I, j \ne i, \lambda = m\omega_i$, and $v \in V(\lambda)_\lambda \backslash \{0\}$. Then,

$$(x_i^-)^a(x_j^-)^b(x_i^-)^c v \ne 0 \quad \Leftrightarrow \qquad b \le c \qquad \text{and} \qquad a + c \le m.$$

Moreover,

$$(x_i^-)^a(x_j^-)^b(x_i^-)^c v = \left(\prod_{s=1}^{a}\frac{c+s-b}{c+s}\right)\ (x_j^-)^b(x_i^-)^{a+c}v.$$

Proof. From the $\mathfrak{sl}_2$ representation theory we have $(x_i^-)^c v \ne 0$ iff $c \le m$. Since $x_j^+(x_i^-)^c v = 0$ and $h_j(x_i^-)^c v = c(x_i^-)^c v$, it follows from the $\mathfrak{sl}_2$ representation theory once more that $(x_j^-)^b(x_i^-)^c v \ne 0$ iff $b \le c$ (and $c \le m$). Notice that this together with the second statement implies the first statement. We prove the second statement by induction on $a \ge 0$. The case $a = 0$ is obvious. The induction step will however depend on the knowledge of the case $a = 1$. For convenience set $x = x_j^-, y = x_i^-$, and $z = [x,y]$. Using the well-known commutation relation in $U(\mathfrak{n}^-)$

$$yx^b = x^b y - bx^{b-1}z$$

we get

$$yx^by^cv = x^by^{c+1}v - bx^{b-1}y^czv = x^by^{c+1}v - \frac{b}{c+1}\ x^by^{c+1}v = \frac{c+1-b}{c+1}\ x^by^{c+1}v$$

where, in the second equality, we used that $xv = 0$ and the last statement of Lemma 4.1. The case $a = 1$ follows. Then, for $a > 1$, using the induction hypothesis we get

$$y^ax^by^cv = y(y^{a-1}x^by^cv) = \left(\prod_{s=1}^{a-1}\frac{c+s-b}{c+s}\right) yx^by^{c+a-1}v.$$

Since, by the case $a = 1$, we have

$$yx^by^{c+a-1}v = \left(\frac{c+a-b}{c+a}\right) x^by^{a+c}v$$

the second statement follows. □

Remark. Notice that if $b \le c$ the number $\prod_{s=1}^{a} \frac{c+s-b}{c+s}$ is a positive rational number.

4.2. **Root data.** Henceforth we assume $\mathfrak{g}$ is of type E_6, set $\lambda = \sum_{i\in I} m_i\omega_i \in P^+$, and assume $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^{\times}$ is such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization of $V_q(\lambda)$. We will need the expression of every positive root in terms of the simple roots and of some of them in terms of the fundamental weights. These expressions are given by Tables 1 and 2 below, respectively.

Table 1

$\beta_1 = \alpha_1 + \alpha_2$	$\beta_{16} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$
$\beta_2 = \alpha_4 + \alpha_5$	$\beta_{17} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$
$\beta_3 = \alpha_2 + \alpha_3$	$\beta_{18} = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
$\beta_4 = \alpha_3 + \alpha_4$	$\beta_{19} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
$\beta_5 = \alpha_3 + \alpha_6$	$\beta_{20} = \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6$
$\beta_6 = \alpha_1 + \alpha_2 + \alpha_3$	$\beta_{21} = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6$
$\beta_7 = \alpha_3 + \alpha_4 + \alpha_5$	$\beta_{22} = \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
$\beta_8 = \alpha_2 + \alpha_3 + \alpha_6$	$\beta_{23} = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
$\beta_9 = \alpha_3 + \alpha_4 + \alpha_6$	$\beta_{24} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6$
$\beta_{10} = \alpha_2 + \alpha_3 + \alpha_4$	$\beta_{25} = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$
$\beta_{11} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6$	$\beta_{26} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
$\beta_{12} = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\beta_{27} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$
$\beta_{13} = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$	$\beta_{28} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$
$\beta_{14} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\beta_{29} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$
$\beta_{15} = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$\beta_{30} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$

Table 2

$\alpha_1 = 2\omega_1 - \omega_2$	$\beta_{24} = \omega_2 - \omega_5$
$\alpha_2 = 2\omega_2 - \omega_1 - \omega_3$	$\beta_{25} = \omega_4 - \omega_1$
$\alpha_3 = 2\omega_3 - \omega_2 - \omega_4 - \omega_6$	$\beta_{26} = \omega_2 - \omega_4 + \omega_5$
$\alpha_4 = 2\omega_4 - \omega_3 - \omega_5$	$\beta_{27} = \omega_1 - \omega_2 + \omega_4$
$\alpha_5 = 2\omega_5 - \omega_4$	$\beta_{28} = \omega_2 - \omega_3 + \omega_4$
$\alpha_6 = 2\omega_6 - \omega_3$	$\beta_{29} = \omega_3 - \omega_6$
$\beta_{23} = \omega_1 - \omega_2 + \omega_3 - \omega_4 + \omega_5$	$\beta_{30} = \omega_6$

4.3. **A smaller set of relations for $M(\lambda)$.** In order to prove Proposition 3.15, we need a version of [21, Proposition 4.6].

Proposition 4.6. Suppose that either $m_3 \neq 0$ or $\overline{\mathrm{supp}}(\lambda)$ is of type A. Then, $M(\lambda)$ is isomorphic to the $\mathfrak{g}[t]$-module $N(\lambda)$ generated by a vector v satisfying

$$h_iv \;=\; m_iv \qquad \text{and} \qquad \mathfrak{n}^+[t]v \;=\; \mathfrak{h}\otimes t\mathbb{C}[t]v \;=\; (x_{\alpha_i}^-)^{m_i+1}v \;=\; x_{\alpha,1}^-v = 0$$

for all $\alpha \in R_1^+ := \{\alpha \in R^+ : \alpha = \sum_{i\in I} n_i\alpha_i \text{ with } n_i \le 1 \text{ for all } i \in I\} = R^+\backslash\{\beta_j : j \ge 20\}$.

Proof. It follows from Lemma 4.11 that $R_1^+ \subseteq R(\lambda, 1)$ and, hence, $M(\lambda)$ is a quotient of $N(\lambda)$. Let us now show that, under the hypothesis assumed on λ, we have an epimorphism in the opposite direction.

Thus, we need to show that $x^-_{\alpha,r}v = 0$ for all $\alpha \in R(\lambda, r)$. In fact, after (3.5), given $\alpha \in R^+$, it suffices to show that

$$x^-_{\alpha,r_\alpha}v = 0 \qquad \text{where} \qquad r_\alpha = \min\{r : \alpha \in R(\lambda, r)\}. \tag{4.1}$$

If $r_\alpha = 0$ this follows immediately from the defining relations of $N(\lambda)$ since they clearly imply that $U(\mathfrak{g})v \cong V(\lambda)$. If $\alpha \in R_1^+$ equation (4.1) is again immediate from the defining relations of $N(\lambda)$. Therefore, we need to prove (4.1) for $\alpha \in R^+\backslash R_1^+$ only. Notice also that Lemma 4.11 implies that $r_\alpha \leq 3$ for all $\alpha \in R^+$.

Assume first that $m_3 \neq 0$. It then follows from (4.7) that $R_1^+ = R(\lambda, 1)$ and (4.1) is immediate for all α such that $r_\alpha = 1$. Equation (4.7) also implies that $R(\lambda, 2) = \{\beta_j : 20 \leq j \leq 28\}$ and $R(\lambda, 3) = \{\beta_{29}, \beta_{30}\}$. Therefore, we are left to show that $x^-_{\beta_j,2}v = 0$ for all $20 \leq j \leq 28$ and $x^-_{\beta_j,3}v = 0$ for all $29 \leq j \leq 30$. This follows from the following commutation relations together with (4.1) for α such that $r_\alpha \leq 1$:

$$\begin{array}{lll}
x^-_{\beta_{20},2} = [x^-_{\alpha_3,1}, x^-_{\beta_{13},1}], & x^-_{\beta_{21},2} = [x^-_{\alpha_3,1}, x^-_{\beta_{17},1}], & x^-_{\beta_{22},2} = [x^-_{\alpha_3,1}, x^-_{\beta_{18},1}], \\
x^-_{\beta_{23},2} = [x^-_{\alpha_3,1}, x^-_{\beta_{19},1}], & x^-_{\beta_{24},2} = [x^-_{\beta_3,1}, x^-_{\beta_{17},1}], & x^-_{\beta_{25},2} = [x^-_{\beta_4,1}, x^-_{\beta_{18},1}], \\
x^-_{\beta_{26},2} = [x^-_{\beta_3,1}, x^-_{\beta_{19},1}], & x^-_{\beta_{27},2} = [x^-_{\beta_4,1}, x^-_{\beta_{19},1}], & x^-_{\beta_{28},2} = [x^-_{\beta_{10},1}, x^-_{\beta_{19},1}], \\
x^-_{\beta_{29},3} = [x^-_{\alpha_3,1}, x^-_{\beta_{28},2}], & x^-_{\beta_{30},3} = [x^-_{\beta_{18},1}, x^-_{\beta_{21},2}]. &
\end{array}$$

Now, assume $m_3 = 0$. In this case, $r_\alpha \leq 2$ for all $\alpha \in R^+$. We consider separately the cases $\mathrm{supp}(\lambda) \subseteq \{1, 2, 4, 5\}$ and $\mathrm{supp}(\lambda) \subseteq \{1, 2, 6\}$ (the case $\mathrm{supp}(\lambda) \subseteq \{4, 5, 6\}$ follows from the latter by the symmetry of the Dynkin diagram). Thus, assume $\mathrm{supp}(\lambda) \subseteq \{1, 2, 6\}$ and consider the following relations

$$\begin{array}{lll}
x^-_{\beta_{20},1} = [x^-_3, x^-_{\beta_{13},1}], & x^-_{\beta_{21},1} = [x^-_3, x^-_{\beta_{17},1}], & x^-_{\beta_{22},1} = [x^-_3, x^-_{\beta_{18},1}], \\
x^-_{\beta_{23},1} = [x^-_3, x^-_{\beta_{19},1}], & x^-_{\beta_{25},1} = [x^-_{\beta_4}, x^-_{\beta_{18},1}], & x^-_{\beta_{27},1} = [x^-_{\beta_4}, x^-_{\beta_{19},1}].
\end{array}$$

Since $\alpha_3, \beta_4 \in R(\lambda, 0)$ in this case, it follows that $x^-_{\beta_j,1}v = 0$ for all $20 \leq j \leq 27, j \neq 24, 26$. If $m_2 = 0$, we need to show that $x^-_{\beta_j,1}v = 0$ for $j \in \{24, 26, 28, 29\}$ and $x^-_{\beta_{30},r}v = 0$ where $r = 1$ if $m_6 = 0$ and $r = 2$ otherwise. Since, in this case, $\beta_3, \beta_{10} \in R(\lambda, 0)$, the former follows from the following relations

$$x^-_{\beta_{24},1} = [x^-_{\beta_3}, x^-_{\beta_{17},1}], \qquad x^-_{\beta_{26},1} = [x^-_{\beta_3}, x^-_{\beta_{19},1}], \qquad x^-_{\beta_{28},1} = [x^-_{\beta_{10}}, x^-_{\beta_{19},1}], \qquad x^-_{\beta_{29},1} = [x^-_3, x^-_{\beta_{28},1}].$$

The latter follows from the relations

$$x^-_{\beta_{30},1} = [x^-_{\beta_{18}}, x^-_{\beta_{21},1}] \qquad \text{and} \qquad x^-_{\beta_{30},2} = [x^-_{\beta_{18},1}, x^-_{\beta_{28},1}]$$

using that $\beta_{18} \in R(\lambda, 0)$ if $m_6 = 0$.

Assume $\mathrm{supp}(\lambda) \subseteq \{1, 2, 4, 5\}$. As in the previous case, one sees that $x^-_{\beta_j,1}v = 0$ for all $20 \leq j \leq 23$. If both m_2 and m_4 are nonzero, we are left to show that $x^-_{\beta_j,2}v = 0$ for all $24 \leq j \leq 30$. For $24 \leq j \leq 28$, this is done as in the case $m_3 \neq 0$ while for $j = 29, 30$ this then follows from the relations

$$x^-_{\beta_{29},2} = [x^-_3, x^-_{\beta_{28},2}] \qquad \text{and} \qquad x^-_{\beta_{30},2} = [x^-_{\beta_{18},1}, x^-_{\beta_{21},1}].$$

If $m_2 = 0$ and $m_4 \neq 0$, we need to show that $x^-_{\beta_{24},1}v = x^-_{\beta_{26},1}v = 0$. This is done as in the case $\mathrm{supp}(\lambda) \subseteq \{1, 2, 6\}$. The case $m_2 \neq 0$ and $m_4 = 0$ is treated similarly. In particular, if $m_4 = 0$ we have $x^-_{\beta_{25},1}v = x^-_{\beta_{27},1}v = 0$. Finally, if $m_2 = m_4 = 0$, we need to prove in addition that $x^-_{\beta_j,1}v = 0$ for $j = 28, 29, 30$. This is done as in the case $\mathrm{supp}(\lambda) \subseteq \{1, 6\}$. $\square$

4.4. **Quantized relations.** The goal of this subsection is to prove Proposition 3.15. We proceed as in the proof of [21, Proposition 3.22] where a similar statement for orthogonal Lie algebras was proved. First we record several previously proved results which will be used in the proof.

Lemma 4.7 ([21, Lemma 4.18])**.** Suppose w is a highest-ℓ-weight vector of $V_q(\boldsymbol{\omega}_{i,a,m})$ for some $i \in I, a \in \mathbb{C}(q)^\times$, and $m \in \mathbb{Z}_{\geq 0}$. Then, $x_{i,1}^- w = aq^m x_i^- w$. □

The following proposition follows from the results of [3, Section 6].

Proposition 4.8. Let $l \in \mathbb{Z}_{\geq 1}, i_j \in I, m_j \in \mathbb{Z}_{\geq 1}, a_j \in \mathbb{C}(q)^\times$ for $j = 1, \dots, l$. If $\frac{a_j}{a_k} \notin q^{\mathbb{Z}_{>0}}$ for $j > k$, then $V_q(\boldsymbol{\omega}_{i_1,a_1,m_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i_l,a_l,m_l})$ is a highest-ℓ-weight module. □

Corollary 4.9 ([21, Corollary 4.4])**.** Let $\lambda \in P^+, a_i \in \mathbb{C}(q)^\times, i \in I$, and $\boldsymbol{\lambda} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$. Then, there exists an ordering $i_1, \dots, i_n$ of I such that $V_q(\boldsymbol{\lambda})$ is isomorphic to the $U_q(\tilde{\mathfrak{g}})$-submodule of $V_q(\boldsymbol{\omega}_{i_1,a_{i_1},\lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i_n,a_{i_n},\lambda(h_{i_n})})$ generated by the top weight space. □

Proposition 4.10 ([21, Proposition 3.13])**.** Suppose $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^\times$ is such that $V_q(\boldsymbol{\lambda})$ is a minimal affinization and that $J \subseteq I$ is an admissible subdiagram. Let v be a highest-ℓ-weight vector of $V = \overline{V_q(\boldsymbol{\lambda})}, \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $a \in \mathbb{C}^\times$ be such that $\overline{\boldsymbol{\lambda}} = \boldsymbol{\omega}_{\lambda,a}$. Then $x_{\alpha,r}^- v = a^r x_\alpha^- v$ for every $\alpha \in R_J^+$. □

If $\alpha \in R_J^+$ for some admissible diagram J, we shall refer to α as an admissible root.

Proof of Proposition 3.15. Let $a \in \mathbb{C}$ be such that $\bar{\boldsymbol{\lambda}} = \boldsymbol{\omega}_{\lambda,a}$. We fix a highest-$\ell$-weight vector v of $V = V_q(\boldsymbol{\lambda})$ and $a_i \in \mathbb{A}^\times, i \in I$, such that $\boldsymbol{\lambda} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,m_i}$. Let also $\bar{v}$ be the image of v in $\overline{V}$ and v' be the image of $\bar{v}$ in $L(\boldsymbol{\lambda})$. By Proposition 4.6, we need to show that $x_{\alpha,1}^- v' = 0$ for all $\alpha \in R_1^+$. This is equivalent to showing that

$$x_{\alpha,1}^- \bar{v} = a\bar{v} \qquad \text{for all} \qquad \alpha \in R_1^+. \tag{4.2}$$

By Proposition 4.10, (4.2) holds if α is an admissible root. Therefore, it remains to show that

$$x_{\beta_j,1}^- \bar{v} = a\bar{v} \qquad \text{for all} \qquad 7 < j < 20. \tag{4.3}$$

Assume first that $\mathrm{supp}(\lambda) \subseteq \{1, 2, 3, 4, 5\}$. In this case $\alpha_6 \in R(\lambda, 0)$ and (4.3) with $j \in \{8, 9, 11, 12\}$ follows from the following relations

$$x_{\beta_8,1}^- = [x_6^-, x_{\beta_3,1}^-], \quad x_{\beta_9,1}^- = [x_6^-, x_{\beta_4,1}^-], \quad x_{\beta_{11},1}^- = [x_6^-, x_{\beta_6,1}^-], \quad x_{\beta_{12},1}^- = [x_6^-, x_{\beta_7,1}^-]$$

together with the fact that $\beta_3, \beta_4, \beta_6$, and β_7 are admissible roots. Next, assume that we have proved (4.3) for $j \in \{10, 14, 15, 16\}$. Then, (4.3) for the remaining values of j follows from the following relations

$$x_{\beta_{13},1}^- = [x_6^-, x_{\beta_{10},1}^-], \quad x_{\beta_{17},1}^- = [x_6^-, x_{\beta_{14},1}^-], \quad x_{\beta_{18},1}^- = [x_6^-, x_{\beta_{15},1}^-], \quad x_{\beta_{19},1}^- = [x_6^-, x_{\beta_{16},1}^-].$$

In order to prove (4.3) for $j \in \{10, 14, 15, 16\}$, it suffices to find elements $X_j, X_{j,1} \in U_{\mathbb{A}}(\tilde{\mathfrak{n}}^-)$ such that

$$\overline{X_j} = x_{\beta_j}^-, \quad \overline{X_{j,1}} = x_{\beta_j,1}^-, \quad \text{and} \quad X_{j,1}v = a_j(q)X_j v + x_j v \tag{4.4}$$

for some $a_j(q) \in \mathbb{A}$ and $x_j \in U_{\mathbb{A}}(\mathfrak{g})$ satisfying $a_j(1) = a$ and $\overline{x}_j = 0$. We prove the existence of such elements assuming

$$a_{i+1} = a_i q^{m_i+m_{i+1}+1} \qquad \text{for all} \qquad i < 5. \tag{4.5}$$

The case $a_{i+1} = a_i q^{-(m_i+m_{i+1}+1)}, i < 5$, is proved similarly using part (b) of Proposition 2.10 instead of part (a). Let $i_0 = \max\{i \in I : m_i \neq 0\}$ (in the case $a_{i+1} = a_i q^{-(m_i+m_{i+1}+1)}, i < 5$, we would use $i_0 = \min\{i \in I : m_i \neq 0\}$). The relations $X_{j,1}v = a_j(q)X_j v + x_j v$ of (4.4) are the quantized relations alluded to in the title of this subsection.

Let $\boldsymbol{\lambda}'$ be such that $\boldsymbol{\lambda} = \boldsymbol{\lambda}'\boldsymbol{\omega}_{i_0,a_{i_0},m_{i_0}}$. Let also v_1, v_2 be highest-ℓ-weight vectors of $V_q(\boldsymbol{\lambda}')$ and $V_q(\boldsymbol{\omega}_{i_0,a_{i_0},m_{i_0}})$, respectively. By (4.5), Proposition 4.8, and Corollary 4.9, the assignment $v \mapsto v_1 \otimes v_2$ extends to an isomorphism $V \cong U_q(\tilde{\mathfrak{g}})(v_1 \otimes v_2) \subseteq V_q(\boldsymbol{\lambda}') \otimes V_q(\boldsymbol{\omega}_{i_0,a_{i_0},m_{i_0}})$. Henceforth, we identify v with $v_1 \otimes v_2$. We write down the proof of the existence of elements as in (4.4) for $j = 16$ assuming $i_0 = 5$ (the other cases are proved similarly and the computations are simpler). Set

$$X_{14} = [x_4^-, [x_3^-, [x_1^-, x_2^-]]], \qquad X_{16} = [x_5^-, X_{14}], \quad \text{and} \quad X_{16,1} = [x_{5,1}^-, X_{14}].$$

Quite clearly, $X_{16}, X_{16,1} \in U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ satisfy the first two identities in (4.4). By Lemmas 1.3 and 1.4, modulo an element of the form xv with $x \in U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \otimes U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ such that $\bar{x} = 0$, we have

$$\begin{aligned}
X_{16}v &= x_5^- X_{14}(v_1 \otimes v_2) - X_{14}x_5^-(v_1 \otimes v_2)\\
&= x_5^-((X_{14}v_1) \otimes v_2) - X_{14}(v_1 \otimes (x_5^- v_2))\\
&= (x_5^- X_{14}v_1) \otimes (k_5^{-1}v_2) + (X_{14}v_1) \otimes (x_5^- v_2)\\
&- (X_{14}v_1) \otimes ((k_1k_2k_3k_4)^{-1}x_5^- v_2) - v_1 \otimes (X_{14}x_5^- v_2)\\
&= q^{-m_5}(x_5^- X_{14}v_1) \otimes v_2 + (1-q^{-m_5})(X_{14}v_1) \otimes (x_5^- v_2) - v_1 \otimes (X_{14}x_5^- v_2)
\end{aligned}$$

while

$$\begin{aligned}
X_{16,1}v &= x_{5,1}^- X_{14}(v_1 \otimes v_2) - X_{14}x_{5,1}^-(v_1 \otimes v_2)\\
&= x_{5,1}^-((X_{14}v_1) \otimes v_2) - X_{14}(v_1 \otimes (x_{5,1}^- v_2))\\
&= (x_{5,1}^- X_{14}v_1) \otimes (k_5v_2) + (X_{14}v_1) \otimes (x_{5,1}^- v_2)\\
&- (X_{14}v_1) \otimes ((k_1k_2k_3k_4)^{-1}x_{5,1}^- v_2) - v_1 \otimes (X_{14}x_{5,1}^- v_2)\\
&= q^{m_5}(x_{5,1}^- X_{14}v_1) \otimes v_2 + (1-q^{-m_5})(X_{14}v_1) \otimes (x_{5,1}^- v_2) - v_1 \otimes (X_{14}x_{5,1}^- v_2).
\end{aligned}$$

Using Lemma 4.7 we get

$$\begin{aligned}
X_{16,1}v &= q^{m_5}(x_{5,1}^- X_{14}v_1) \otimes v_2 + (1-q^{-m_5})(X_{14}v_1) \otimes (a_5q^{m_5}x_5^- v_2) - v_1 \otimes (X_{14}(a_5q^{m_5}v_2))\\
&= a_5q^{m_5}X_{16}v + q^{m_5}(x_{5,1}^- X_{14}v_1) \otimes v_2 - a_5(x_5^- X_{14}v_1) \otimes v_2.
\end{aligned}$$

Since $a_{16}(q) := a_5q^{m_5}$ satisfies $a_{16}(1) = a$, in order to prove that X_{16} and $X_{16,1}$ satisfy the last identity of (4.4), it suffices to show that

$$q^{m_5}(x_{5,1}^- X_{14}v_1) \otimes v_2 = a_5(x_5^- X_{14}v_1) \otimes v_2. \tag{4.6}$$

Notice that $x_{5,r}^+ X_{14}v_1 = 0$ for all $r \in \mathbb{Z}$ and let W be the $U_q(\tilde{\mathfrak{g}}_5)$-submodule of $V_q(\boldsymbol{\lambda}')$ generated by $X_{14}v_1$. Then, by Proposition 2.10(a), W is a highest-ℓ-weight module with highest ℓ-weight $\boldsymbol{\omega}_{5,a_4q^{m_4}}$. It then follows from Lemma 4.7 that

$$x_{5,1}^- X_{14}v_1 = a_4q^{m_4+1}x_5^- X_{14}v_1.$$

This and (4.5) imply (4.6).

The case $\mathrm{supp}(\lambda) \subseteq \{1,2,3,6\}$ is dealt with similarly and the case $\mathrm{supp}(\lambda) \subseteq \{3,4,5,6\}$ then follows using the symmetry of the Dynkin diagram. We omit the details. $\square$

4.5. **Upper bounds.** In this subsection we prove (3.10). Let $v \in M(\lambda)_\lambda$ be nonzero.

Lemma 4.11. For every $i \in I, m \in \mathbb{Z}_{\geq 0}$, and $\alpha = \sum_{j \in I} a_j\alpha_j \in R^+$ we have $\alpha \in R(i,m,a_i)$. In particular:

(a) $R(1,m,1) = R(5,m,1) = R^+$.
(b) $R(6,m,1) \supseteq R^+ \setminus \{\beta_{30}\}$ and $R(6,m,2) = R^+$.
(c) $R(2,m,1) \supseteq R^+ \setminus \{\beta_{24}, \beta_{26}, \beta_{28}, \beta_{29}, \beta_{30}\}$ and $R(2,m,2) = R^+$.
(d) $R(4,m,1) \supseteq R^+ \setminus \{\beta_{25}, \beta_{27}, \beta_{28}, \beta_{29}, \beta_{30}\}$ and $R(4,m,2) = R^+$.

(e) $R(3,m,1) \supseteq R^+ \setminus \{\beta_j : j \geq 20\}, R(3,m,2) \supseteq R^+ \setminus \{\beta_{29}, \beta_{30}\}$, and $R(3,m,3) = R^+$.

Proof. Statements (a)-(e) follow from the first statement by inspection of Table 1. Conversely, clearly items (a)-(e) together imply the first statement. The proof is analogous to that of [2, Proposition 1.2] (see also [23, Lemma 5.2.8]). We omit the details. $\square$

Observe that the above Lemma together with (3.5) imply

$$x^-_{\alpha_i,r}v = x^-_{\beta_j,r}v = x^-_{\beta_k,s}v = 0$$

for all $i \in I, j < 20, k < 29, r \geq 1, s \geq 2$ and $R(\lambda, 3) = R^+$. Let $R'(i,m,r)$ be the set on the right-hand-side of the inclusion symbol of the appropriate item of Lemma 4.11. It will follow from Section 4.6 below that, if $m > 0$, then

$$R(i,m,r) = R'(i,m,r). \tag{4.7}$$

Set $R'(\lambda, r) = \cap_{i\in I} R'(i, m_i, r)$ and let $\mathfrak{r}(\lambda)$ be the subspace of $\mathfrak{g}[t]$ spanned by $\{x^-_{\alpha,1}, x^-_{\beta,2} : \alpha \in R^+\backslash R'(\lambda,1), \beta \in R^+\backslash R'(\lambda,2)\}$ which is clearly an abelian ideal of $\mathfrak{n}^-[t]$. Since we are assuming $m_3 = 0$, we have $R'(\lambda,2) = R^+$ and, therefore, $\mathfrak{r}(\lambda)$ is the subspace of $\mathfrak{g}[t]$ spanned by $\{x^-_{\alpha,1} : \alpha \in R^+\backslash R'(\lambda,1)\}$. Since $R(\lambda,r) = R^+$ for all $r \geq 2$ by (3.5), a straightforward application of the PBW Theorem implies

$$M(\lambda) = U(\mathfrak{n}^-[t])v = U(\mathfrak{n}^-)U(\mathfrak{r}(\lambda))v. \tag{4.8}$$

Moreover,

$$R(\lambda,1) \supseteq R^+ \setminus \{\beta_{24}, \beta_{25}, \beta_{26}, \beta_{27}, \beta_{28}, \beta_{29}, \beta_{30}\}. \tag{4.9}$$

by Lemma 4.11 and, therefore,

$$M(\lambda) = U(\mathfrak{n}^-)U(x^-_{\beta_{30},1})U(x^-_{\beta_{29},1})U(x^-_{\beta_{28},1})U(x^-_{\beta_{27},1})U(x^-_{\beta_{26},1})U(x^-_{\beta_{25},1})U(x^-_{\beta_{24},1})v. \tag{4.10}$$

We now apply Lemma 4.1 to prove that

$$M(\lambda) = U(\mathfrak{n}^-)U(x^-_{\beta_{30},1})U(x^-_{\beta_{28},1})U(x^-_{\beta_{27},1})U(x^-_{\beta_{26},1})U(x^-_{\beta_{25},1})U(x^-_{\beta_{24},1})v. \tag{4.11}$$

Indeed, let $x = x^-_3, y = x^-_{\beta_{28},1}, z = x^-_{\beta_{29},1}$ which generates a three-dimensional Heisenberg subalgebra of $\mathfrak{g}[t]$. Since $xv = 0$, it follows from Lemma 4.1 that $(x^-_{\beta_{28},1})^r(x^-_{\beta_{29},1})^s v$ is a multiple of $(x^-_3)^s(x^-_{\beta_{28},1})^{r+s}v$ for every $r,s \in \mathbb{Z}_{\geq}0$. Since $[x^-_3, x^-_{\beta_j,1}] = 0$ for all $24 \leq j \leq 30, j \neq 28$, (4.11) follows.

Given $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathbb{Z}^6_{\geq 0}$, set

$$\mathbf{x_r} = (x^-_{\beta_{30},1})^{r_6}(x^-_{\beta_{28},1})^{r_5}(x^-_{\beta_{27},1})^{r_4}(x^-_{\beta_{26},1})^{r_3}(x^-_{\beta_{25},1})^{r_2}(x^-_{\beta_{24},1})^{r_1}$$

so that (4.11) is equivalent to

$$M(\lambda) = \sum_{\mathbf{r}\in\mathbb{Z}^6_{\geq 0}} U(\mathfrak{n}^-)\mathbf{x_r}v. \tag{4.12}$$

Recall the definition of $\mathrm{wt}(\mathbf{r})$ in Subsection 3.2 and use Table 2 to observe that $\mathbf{x_r}v \in M(\lambda)[\mathrm{gr}(\mathbf{r})]_{\mathrm{wt}(\mathbf{r})}$.

Consider the Heisenberg subalgebra of $\mathfrak{g}[t]$ generated by $\{x^-_1, x^-_{\beta_{25},1}, x^-_{\beta_{27},1}\}$. Since $(x^-_1)^{m_1+1}v = 0$ and $[x^-_1, x^-_{\beta_j,1}] = 0$ for all $24 \leq j \leq 30, j \neq 25$, it follows from Lemma 4.1 that we can restrict the sum of (4.12) to $\mathbf{r} \in \mathbb{Z}^6_{\geq 0}$ such that $r_4 \leq m_1$. Similarly, by working with the Heisenberg subalgebra generated by $\{x^-_5, x^-_{\beta_{24},1}, x^-_{\beta_{26},1}\}$ we can assume $r_3 \leq m_5$.

Next, we show that we can restrict the sum of (4.12) to $\mathbf{r} \in \mathbb{Z}^6_{\geq 0}$ such that $r_1 + r_3 + r_5 \leq m_2$ and $r_2 + r_4 + r_5 \leq m_4$. By contradiction, assume this is not the case. It then follows from Lemma 2.4 that there exists $\mathbf{r} \in \mathbb{Z}^6_{\geq 0}$ satisfying either $r_1 + r_3 + r_5 > m_2$ or $r_2 + r_4 + r_5 > m_4$ and such that $V(\mathrm{wt}(\mathbf{r}))$ is an irreducible summand of $M(\lambda)$. Moreover, the injectivity of $\mathrm{wt} : \mathbb{Z}^6 \to P$ implies that the projection of $\mathbf{x_r}v$ on this summand is non zero. Fix such $\mathbf{r}$ and suppose $r_1+r_3+r_5>m_2$ (the other case follows from the symmetry of the Dynkin diagram). Let $\mathbf{s}=(r_1,r_2,r_3,0,r_4+r_5,r_6)$ and notice that

$$(x_2^+)^{r_4}\mathbf{x_s}v = c\mathbf{x_r}v \qquad \text{for some} \qquad c\in\mathbb{C}^\times. \tag{4.13}$$

This easily follows from the relations

$$[x_2^+, x_{\beta_j,1}^-]=0 \qquad \text{for all} \qquad 24\le j\le 30, \quad j\ne 26,28,$$
$$[x_2^+, x_{\beta_{26},1}^-]=x_{\beta_{23},1}^-, \qquad [x_2^+, x_{\beta_{28},1}^-]=x_{\beta_{27},1}^-,$$
$$[x_{\beta_{23},1}^-, x_{\beta_j,1}^-]=0 \qquad \text{for all} \qquad 24\le j\le 30, \qquad \text{and} \qquad x_{\beta_{23},1}^-v=0.$$

It follows from (4.13) that the projection of $\mathbf{x_s}v$ on $V(\mathrm{wt}(\mathbf{r}))$ is non zero and, hence, $V(\mathrm{wt}(\mathbf{r}))_{\mathrm{wt}(\mathbf{s})}\ne 0$. We claim that this is a contradiction. Indeed, notice that $\mathrm{wt}(\mathbf{s})(h_2)=(m_2-r_1-r_3-r_4-r_5)$. Hence, $\sigma_2\mathrm{wt}(\mathbf{s})=\mathrm{wt}(\mathbf{s})-(m_2-r_1-r_3-r_4-r_5)\alpha_2$ is a weight of $V(\mathrm{wt}(\mathbf{r}))$. Here, $\sigma_2\in\mathcal{W}$ is the simple reflection associated with α_2. Since $\mathrm{wt}(\mathbf{s})=\mathrm{wt}(\mathbf{r})-r_4\alpha_2$, it follows that

$$\sigma_2\mathrm{wt}(\mathbf{s})=\mathrm{wt}(\mathbf{r})+(r_1+r_3+r_5-m_2)\alpha_2>\mathrm{wt}(\mathbf{r}),$$

contradicting $V(\mathrm{wt}(\mathbf{r}))_{\sigma_2\mathrm{wt}(\mathbf{s})}\ne 0$.

So far we proved that the sum in (4.12) can be restricted to $\mathbf{r}\in\mathbb{Z}_{\ge 0}^6$ such that $r_4\le m_1, r_3\le m_5, r_1+r_3+r_5\le m_2$, and $r_2+r_4+r_5\le m_4$. Now, observe that, for such $\mathbf{r}$, $\mathrm{wt}(\mathbf{r})\in P^+$ iff $\mathbf{r}\in\mathcal{A}$. Therefore, by Lemma 2.4, we must have a surjective homomorphism of $\mathfrak{g}$-modules

$$\bigoplus_{\mathbf{r}\in\mathcal{A}_r} V(\mathrm{wt}(\mathbf{r}))\to M(\lambda)[r]$$

for every $r\in\mathbb{Z}_{\ge 0}$ and (3.10) follows.

Remark. Let $w\in T(\lambda)_\lambda$ be nonzero and notice that, since $T(\lambda)$ is a quotient of $M(\lambda)$, equations (4.12) remain valid after replacing $M(\lambda)$ by $T(\lambda)$ on the left-hand-side and v by w on the right-hand-side.

4.6. **The Kirillov-Reshetikhin case.** In this subsection we assume $\lambda=m_i\omega_i$ for some $i\in I, i\ne 3$, and prove (3.11) in this case. As mentioned earlier, for such λ, (3.12) (and hence (3.11)) follows from [17, 16] (see also [2]). However, in order to prove (3.11) for more general λ later, we will need further details about this case than just (3.11). Hence, we consider it separately. We split the proof in cases according to the value of i. We keep denoting by v a nonzero vector in $M(\lambda)_\lambda$.

4.6.1. Assume $i=1$ or $i=5$ and notice that Lemma 4.11 implies $\mathfrak{r}(\lambda)=0$ in this case. Hence, $M(\lambda)=U(\mathfrak{n}^-)v$ and it follows that $M(\lambda)$ is isomorphic to the pullback of $V(\lambda)$ by the map $\mathfrak{g}[t]\to\mathfrak{g}, x\otimes f(t)\mapsto f(0)x$. Since $\mathcal{A}=\{\lambda\}$ in this case, (3.11) follows.

4.6.2. Now suppose $i=6$. Notice that $\mathcal{A}_r=\{(0,0,0,0,0,r)\}$ for all $0\le r\le m_6$ and $\mathcal{A}_r=\emptyset$ otherwise. Since $\mathrm{wt}((0,0,0,0,0,r))=(m_6-r)\omega_6$, (3.11) becomes

$$t_{(m_6-r)\omega_6,r}\ne 0 \quad \text{for all} \quad 0\le r\le m_6. \tag{4.14}$$

We begin proving this in the case $m_6=1$ in which case we have $T(\lambda)=M(\lambda)$ by definition. Observe that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\omega_6),V(\omega_6))\ne 0$ which is true since $V(\omega_6)$ is isomorphic to the adjoint representation. Hence, we can apply the construction given by (3.15) with $V_0=V(\omega_6)$ and $V_1=V(0)$. One easily checks that the highest weight vector w_0 of V_0 satisfies the relations satisfied by v and, hence, the module V constructed in this way is a quotient of $M(\omega_6)$. Since $V[0]\cong V(\omega_6)$ and $V[1]\cong V(0)$, (4.14) follows. Moreover, we clearly have $x_{\beta_{30},1}^-w_0\ne 0$ (otherwise the map p_0 would be zero) and, hence, $x_{\beta_{30},1}^-v\ne 0$. In particular, (4.7) holds for $i=6$.

For $m_6 > 1$, let $w \in M(\omega_6)_{\omega_6}$ be nonzero. Since $T(\lambda)$ is generated by $w^{\otimes m_6} \in M(\omega_6)^{\otimes m_6}$ one easily checks that $(x^-_{\beta_{30},1})^r w^{\otimes m_6} \neq 0$ for all $r \leq m_6$. In particular,

$$(x^-_{\beta_{30},1})^r v \neq 0 \qquad \text{iff} \qquad r \leq m_6. \tag{4.15}$$

By the remark closing Subsection 4.5, $T(\lambda) = \sum_{r=0}^{m_6} U(\mathfrak{n}^-)(x^-_{\beta_{30},1})^r w^{\otimes m_6}$. Hence, $(x^-_{\beta_{30},1})^r w^{\otimes m_6}$ must be a highest-weight vector in $T(\lambda)[r]$ which implies (4.14).

4.6.3. Next, let $i = 2$. The proof is parallel to the previous case. Namely, (3.11) becomes equivalent to

$$t_{(m_2-r)\omega_2+r\omega_5,r} \neq 0 \quad \text{for all} \quad 0 \leq r \leq m_2. \tag{4.16}$$

Notice that $x^-_{\beta_{24},1}$ plays the role that $x^-_{\beta_{30},1}$ did in the case $i = 6$. If $m_2 = 1$, we again use the construction given by (3.15) this time with $V_0 = V(\omega_2)$ and $V_1 = V(\omega_5)$. In particular, it follows that $x^-_{\beta_{24},1} v \neq 0$. For $m_2 > 1$, let $w \in M(\omega_2)_{\omega_2}$ be nonzero. As before, we conclude that $(x^-_{\beta_{24},1})^r w^{\otimes m_2} \neq 0$ for all $0 \leq r \leq m_2$. Equation (4.16) follows as in the previous case by using that $T(\lambda) = \sum_{r=0}^{m_2} U(\mathfrak{n}^-)(x^-_{\beta_{24},1})^r w^{\otimes m_2}$.

We now record the following lemma which, in particular, proves (4.7) for $i = 2$.

Lemma 4.12. Let $r_j \in \mathbb{Z}_{\geq 0}, j = 1, \ldots, 5$, and $w = (x^-_{\beta_{24},1})^{r_1}(x^-_{\beta_{26},1})^{r_2}(x^-_{\beta_{28},1})^{r_3}(x_{\beta_{29},1})^{r_4}(x^-_{\beta_{30},1})^{r_5} v$. Then w is a nonzero scalar multiple of

$$(x_6^-)^{r_5}(x_3^-)^{r_5+r_4}(x_4^-)^{r_5+r_4+r_3}(x_5^-)^{r_5+r_4+r_3+r_2}(x^-_{\beta_{24},1})^{r_1+r_2+r_3+r_4+r_5} v.$$

Moreover, w is nonzero iff $r_1 + \cdots + r_5 \leq m_2$. In particular, $R(2, m_2, 1) = R'(2, m_2, 1)$.

Proof. The last statement follows immediately from the second. The first statement follows from straightforward successive applications of Lemma 4.1. Namely, we first consider the Heisenberg subalgebra generated by $x = x_6^-, y = x^-_{\beta_{29},1}$, and $z = x^-_{\beta_{30},1}$ together with the relation $xv = 0$ to get

$$(x^-_{\beta_{29},1})^{r_4}(x^-_{\beta_{30},1})^{r_5} v = \eta (x_6^-)^{r_5}(x^-_{\beta_{29},1})^{r_4+r_5} v$$

for some nonzero scalar η. Since $[x_6^-, x^-_{\beta_j,1}] = 0$ for $j = 24, 26, 28$, it follows that

$$(x^-_{\beta_{24},1})^{r_1}(x^-_{\beta_{26},1})^{r_2}(x^-_{\beta_{28},1})^{r_3}(x_{\beta_{29},1})^{r_4}(x^-_{\beta_{30},1})^{r_5} v = \eta (x_6^-)^{r_5}(x^-_{\beta_{24},1})^{r_1}(x^-_{\beta_{26},1})^{r_2}(x^-_{\beta_{28},1})^{r_3}(x_{\beta_{29},1})^{r_4+r_5} v.$$

By similarly considering the subalgebras generated by $\{x_3^-, x^-_{\beta_{28},1}, x^-_{\beta_{29},1}\}$, $\{x_4^-, x^-_{\beta_{26},1}, x^-_{\beta_{28},1}\}$, and $\{x_5^-, x^-_{\beta_{24},1}, x^-_{\beta_{26},1}\}$ in this order, the first statement follows.

We have seen above that $(x^-_{\beta_{24},1})^r w^{\otimes m_2} \neq 0$ iff $r \leq m_2$. This implies $(x^-_{\beta_{24},1})^{r_1+r_2+r_3+r_4+r_5} v \neq 0$ iff $r_1 + \cdots + r_5 \leq m_2$. Since $x_5^+(x^-_{\beta_{24},1})^r v = (x^-_{\beta_{24},1})^r x_5^+ v = 0$ and $h_5(x^-_{\beta_{24},1})^r v = rv$, it follows that $(x_5^-)^s(x^-_{\beta_{24},1})^r v \neq 0$ for all $0 \leq s \leq r$. In particular, $(x_5^-)^{r_5+r_4+r_3+r_2}(x^-_{\beta_{24},1})^{r_1+r_2+r_3+r_4+r_5} v \neq 0$. The proof is completed proceeding similarly. $\square$

4.6.4. The case $i = 4$ is obtained from the previous case by using the nontrivial Dynkin diagram automorphism of $\mathfrak{g}$. In particular we have:

Lemma 4.13. Let $r_j \in \mathbb{Z}_{\geq 0}, j = 1, \ldots, 5$, and $w = (x^-_{\beta_{25},1})^{r_1}(x^-_{\beta_{27},1})^{r_2}(x^-_{\beta_{28},1})^{r_3}(x_{\beta_{29},1})^{r_4}(x^-_{\beta_{30},1})^{r_5} v$. Then w is a nonzero scalar multiple of

$$(x_6^-)^{r_5}(x_3^-)^{r_5+r_4}(x_2^-)^{r_5+r_4+r_3}(x_1^-)^{r_5+r_4+r_3+r_2}(x^-_{\beta_{25},1})^{r_1+r_2+r_3+r_4+r_5} v.$$

Moreover, w is nonzero iff $r_1 + \cdots + r_5 \leq m_4$. In particular, $R(4, m_4, 1) = R'(4, m_4, 1)$. $\square$

4.7. **Lower bounds.** We now complete the proof of (3.11) for λ as in Theorem 3.14. In fact, we will carry out most of the proof assuming only that $\lambda(h_3)=0$. Recall the notation $\mathbf{x_r}, \mathbf{r}\in\mathcal{A}$, developed in Section 4.5. In addition, we shall use the following notation. Denote by v_{i,m_i} a nonzero vector in $M(m_i\omega_i)_{m_i\omega_i}$ and by v_{i,m_i}^s the image of v_{i,m_i} in $M(m_i\omega_i)(s)$. By definition of the truncated module $M(m_i\omega_i)(s)$ we have

$$M(m_i\omega_i)(s)[r]=0 \qquad \text{if} \qquad r>s. \tag{4.17}$$

Given $\boldsymbol{s}=(s_i)_{i\in I}\in\mathbb{Z}_{\geq 0}^I$, let $T_{\boldsymbol{s}}(\lambda)$ be the submodule of $\bigotimes_{i\in I} M(m_i\omega_i)(s_i)$ generated by $v_{\boldsymbol{s}} := \bigotimes_{i\in I} v_{i,m_i}^{s_i}$. Since $T(\lambda)$ is the submodule of $\bigotimes_{i\in I} M(m_i\omega_i)$ generated by $v := \bigotimes_{i\in I} v_{i,m_i}$, there exists a unique epimorphism from $T(\lambda)$ onto $T_{\boldsymbol{s}}(\lambda)$ such that $v\mapsto v_{\boldsymbol{s}}$. Let $t_{\mu,r}^{\boldsymbol{s}}$ denote the multiplicity of $V(\mu)$ as an irreducible constituent of $T_{\boldsymbol{s}}(\lambda)[r]$. Observe that, since $|\mathcal{A}_{\mathrm{wt}(\mathbf{r}),\mathrm{gr}(\mathbf{r})}|=1$ for all $\mathbf{r}\in\mathcal{A}$, in order to prove (3.11), it suffices to prove that

$$\text{for each} \quad \mathbf{r}\in\mathcal{A} \quad \text{there exists} \quad \boldsymbol{s}\in\mathbb{Z}_{\geq 0}^I \quad \text{such that} \quad t^{\boldsymbol{s}}_{\mathrm{wt}(\mathbf{r}),\mathrm{gr}(\mathbf{r})}\geq 1. \tag{4.18}$$

It will be convenient to write the tensor product $\bigotimes_{i\in I} M(m_i\omega_i)$ in the following order: $M(m_2\omega_2)\otimes M(m_4\omega_4)\otimes M(m_6\omega_6)\otimes M(m_1\omega_1)\otimes M(m_5\omega_5)$, where we already used that $m_3=0$ and, hence, $M(m_3\omega_3)\cong V(0)\cong\mathbb{C}$. In particular, $v=v_{2,m_2}\otimes v_{4,m_4}\otimes v_{6,m_6}\otimes v_{1,m_1}\otimes v_{5,m_5}$ and similarly for $v_{\boldsymbol{s}}, \boldsymbol{s}\in\mathbb{Z}_{\geq 0}^I$. To shorten notation we write $w=v_{1,m_1}\otimes v_{5,m_5}$ when convenient so that

$$v=v_{2,m_2}\otimes v_{4,m_4}\otimes v_{6,m_6}\otimes w.$$

Let $\{\mathbf{e}_j : j=1,\dots,6\}$ be the canonical basis of $\mathbb{Z}^6_{\geq 0}$. Given $r\in\mathbb{Z}$, set $\mathbb{Z}^6[r]=\{\mathbf{r}\in\mathbb{Z}^6:\mathrm{gr}(\mathbf{r})=r\}$, and observe that $\mathbb{Z}^6[0]$ is a free $\mathbb{Z}$-module having $\mathbf{b}:=\{(\boldsymbol{e}_1-\boldsymbol{e}_5),(\boldsymbol{e}_2-\boldsymbol{e}_5),(\boldsymbol{e}_5-\boldsymbol{e}_3),(\boldsymbol{e}_5-\boldsymbol{e}_4),(\boldsymbol{e}_5-\boldsymbol{e}_6)\}$ as an ordered $\mathbb{Z}$-basis. Define $\mathbf{b}_j\in\mathbf{b}, j=1,\dots,5$, by requiring that $\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_5\}$ as an ordered set. Clearly, $\mathbf{r},\mathbf{r}'\in\mathbb{Z}^6[r]$ iff $\mathbf{r}-\mathbf{r}'\in\mathbb{Z}^6[0]$. Given $\boldsymbol{j}=(j_1,j_2,j_3,j_4,j_5)\in\mathbb{Z}^5$ and $\boldsymbol{s}\in\mathbb{Z}^I_{\geq 0}$ such that $s_2\leq m_2, s_4\leq m_4, s_6\leq m_6$, observe that $\mathbf{r_0}=(s_2,s_4,0,0,0,s_6)\in\mathcal{A}_{s_2+s_4+s_6}$ and set

$$\mathbf{r}_{\boldsymbol{j}}=\mathbf{r_0}-\sum_{l=1}^5 j_l\mathbf{b}_l=(s_2-j_1,s_4-j_2,j_3,j_4,j_1+j_2-j_3-j_4-j_5,s_6+j_5).$$

Thus, $\mathbf{r}\in\mathbb{Z}^6[s_2+s_4+s_6]$ iff $\mathbf{r}=\mathbf{r}_{\boldsymbol{j}}$ for some $\boldsymbol{j}\in\mathbb{Z}^5$. For shortening some expressions, given $\boldsymbol{j}\in\mathbb{Z}^5$, we may use the notation $j_0=j_1+j_2-j_3-j_4-j_5$. Notice that $\mathbf{r}_{\boldsymbol{j}}\in\mathcal{A}$ iff

$$\begin{gathered}0\leq j_3\leq m_5, \qquad 0\leq j_4\leq m_1, \qquad j_1\leq s_2, \qquad j_2\leq s_4, \qquad j_0\geq 0,\\ j_5\leq m_6-s_6, \qquad j_1-j_3-j_5\leq m_4-s_4, \qquad j_2-j_4-j_5\leq m_2-s_2.\end{gathered}\tag{4.19}$$

Set

$$\mathcal{A}(\boldsymbol{s})=\{\mathbf{r}\in\mathcal{A}:\mathbf{x_r}v_{\boldsymbol{s}}\neq 0\}\cap\mathbb{Z}^6[s_2+s_4+s_6]$$

and let $\mathcal{B}(\boldsymbol{s})$ be the set of tuples $\boldsymbol{j}\in\mathbb{Z}^5_{\geq 0}$ satisfying

$$\begin{gathered}j_3\leq j_1\leq s_2, \qquad j_4\leq j_2\leq s_4, \qquad j_3\leq m_5, \qquad j_4\leq m_1, \qquad j_0\geq 0,\\ j_5\leq m_6-s_6, \qquad j_1-j_3-j_5\leq m_4-s_4, \qquad j_2-j_4-j_5\leq m_2-s_2.\end{gathered}\tag{4.20}$$

In Subsection 4.8 we will show that

$$\mathbf{r}_{\boldsymbol{j}}\in\mathcal{A}(\boldsymbol{s}) \qquad \Leftrightarrow \qquad \boldsymbol{j}\in\mathcal{B}(\boldsymbol{s}). \tag{4.21}$$

It follows from (4.21) that

$$T_{\boldsymbol{s}}(\lambda)[s_2+s_4+s_6] = \sum_{\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s})} U(\mathfrak{n}^-)\mathbf{x}_{\boldsymbol{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}}. \tag{4.22}$$

For $\boldsymbol{j},\boldsymbol{k}\in\mathcal{B}(\boldsymbol{s})$ we have

$$\begin{aligned}\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}}) - \mathrm{wt}(\mathbf{r}_{\boldsymbol{k}}) = \;& (k_2-j_2)\alpha_1 + (k_1-j_1)\alpha_5 + (k_5-j_5)(\alpha_3+\alpha_6)+\\ &+ (k_2-j_2+j_4-k_4)\alpha_2 + (k_1-j_1+j_3-k_3)\alpha_4.\end{aligned} \tag{4.23}$$

In particular, $\mathrm{wt}(\mathbf{r}_{\mathbf{O}})$ is the unique maximal weight of $T_{\boldsymbol{s}}(\lambda)[s_2+s_4+s_6]$ and, hence,

$$t^{\boldsymbol{s}}_{\mathrm{wt}(\mathbf{r}_{\mathbf{O}}),s_2+s_4+s_6} \geq 1. \tag{4.24}$$

Lemma 4.14. Let $\mathbf{r}\in\mathcal{A}$. Then, there exists $\boldsymbol{s}\in\mathbb{Z}^I_{\geq 0}$ and $\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s})$ such that $j_5=0$ and $\mathbf{r}=\mathbf{r}_{\boldsymbol{j}}$. In particular, $\mathbf{r}\in\mathcal{A}(\boldsymbol{s})$.

Proof. Let $s_1=s_3=s_5=0, s_2=r_1+r_3+r_5, s_4=r_2+r_4$, and $s_6=r_6$. As before, set $\mathbf{r}_{\mathbf{O}} = (s_2,s_4,0,0,0,s_6)$ and notice that $\mathbf{r}_{\mathbf{O}}\in\mathcal{A}$. One easily checks that $\mathbf{r}=\mathbf{r}_{\boldsymbol{j}}$ where $\boldsymbol{j}=(r_3+r_5,r_4,r_3,r_4,0)$. By (4.21), $\boldsymbol{r}\in\mathcal{A}(\boldsymbol{s})$ iff $\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s})$. The checking of the latter is straightforward. ☐

The above lemma shows that it suffices to show (4.18) in the case that $\mathbf{r}=\mathbf{r}_{\boldsymbol{j}}$ for some $\boldsymbol{s}\in\mathbb{Z}^I_{\geq 0}$ and $\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s})$ such that $j_5=0$. In this case, it follows from the proof of (4.21) (see the last line of Subsection 4.8) that $\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}}$ is a nonzero scalar multiple of

$$v_{\boldsymbol{j}} := (x^-_{\beta_{28},1})^{j_1-j_3}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2}\otimes(x^-_{\beta_{28},1})^{j_2-j_4}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4}\otimes w'$$

where $w'=(x^-_{\beta_{30},1})^{s_6}v^{s_6}_{6,m_6}\otimes w$. Notice $v_{\boldsymbol{j}}\neq 0$ since $\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s})$. From now on we fix $\boldsymbol{s}\in\mathbb{Z}^I_{\geq 0}$, write $\mathcal{B}=\mathcal{B}(\boldsymbol{s})$, and set

$$\mathcal{B}_0=\{\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s}) : j_5=0\}.$$

Given $\boldsymbol{k}\in\mathcal{B}$, let

$$\mathcal{B}^+_{\boldsymbol{k}} = \{\boldsymbol{j}\in\mathcal{B}(\boldsymbol{s}) : \mathrm{wt}(\mathbf{r}_{\boldsymbol{k}}) < \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})\} \qquad\text{and}\qquad \mathcal{B}_{\boldsymbol{k}} = \mathcal{B}^+_{\boldsymbol{k}}\cup\{\boldsymbol{k}\}.$$

It easily follows from (4.23) that

$$\boldsymbol{k}\in\mathcal{B}_0 \quad\Rightarrow\quad \mathcal{B}_{\boldsymbol{k}}\subseteq\mathcal{B}_0. \tag{4.25}$$

By Lemma 2.4, (4.22), and the injectivity of $\mathrm{wt}:\mathcal{A}\to P$, (4.18) holds for $\mathbf{r}=\mathbf{r}_{\boldsymbol{k}}$ iff

$$v_{\boldsymbol{k}}\notin V^+_{\boldsymbol{k}} := \sum_{\boldsymbol{j}\in\mathcal{B}^+_{\boldsymbol{k}}} U(\mathfrak{n}^-)v_{\boldsymbol{j}}. \tag{4.26}$$

Equivalently, (4.18) holds for $\mathbf{r}=\mathbf{r}_{\boldsymbol{k}}$ iff we have an isomorphism of $\mathfrak{g}$-modules

$$V_{\boldsymbol{k}} := \sum_{\boldsymbol{j}\in\mathcal{B}_{\boldsymbol{k}}} U(\mathfrak{n}^-)v_{\boldsymbol{j}} \cong \bigoplus_{\boldsymbol{j}\in\mathcal{B}_{\boldsymbol{k}}} V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})). \tag{4.27}$$

Given $\boldsymbol{j}\in\mathcal{B}_0$, define the height of $\boldsymbol{j}$ to be

$$\mathrm{ht}(\boldsymbol{j}) = \mathrm{ht}(\mathrm{wt}(\mathbf{r}_{\mathbf{O}}) - \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})) = 2(j_1+j_2)-(j_3+j_4) = j_1+j_2+j_0.$$

We prove (4.27) by induction on $k = \mathrm{ht}(\boldsymbol{k})$. Equation (4.24) implies that (4.27) holds for $k = 0$. Thus, assume $k > 0$ and, by induction hypothesis, that (4.27) holds for $\boldsymbol{j} \in \mathcal{B}_0$ such that $\mathrm{ht}(\boldsymbol{j}) < k$. It follows from the induction hypothesis and (4.25) that

$$\dim((V^+_{\boldsymbol{k}})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}) = \sum_{\boldsymbol{j}\in\mathcal{B}^+_{\boldsymbol{k}}} \dim(V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}}))_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}). \tag{4.28}$$

We are left to show that

$$\dim((V_{\boldsymbol{k}})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}) = \dim((V^+_{\boldsymbol{k}})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}) + 1. \tag{4.29}$$

Let $J_- = \{1,2\}, J_+ = \{4,5\}, J = J_- \cup J_+ \subseteq I$ so that $\mathfrak{g}_{J_\pm} \cong \mathfrak{sl}_3$ and $\mathfrak{g}_J \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_3$. By Proposition 2.2, $U(\mathfrak{g}_{J_\pm})V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}}))_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})} \cong V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_{J_\pm})$ and similarly for J in place of $J_\pm$. Moreover, we have isomorphisms of vector spaces

$$V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}}))_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})} \cong V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_J)_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})_J} \cong V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_{J_-})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})_{J_-}} \otimes V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_{J_+})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})_{J_+}}. \tag{4.30}$$

The first isomorphism above is clear and the second follows from Proposition 2.3. If $\boldsymbol{j} \in \mathcal{B}_{\boldsymbol{k}}$, it easily follows from (4.23) that

$$\begin{aligned} &k_2 - j_2 \le \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})(h_1) = m_1 + s_4 - j_2 - j_4, && k_2 - j_2 + j_4 - k_4 \le \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})(h_2) = m_2 - s_2 - j_2 + 2j_4,\\ &k_1 - j_1 \le \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})(h_5) = m_5 + s_2 - j_1 - j_3, && k_1 - j_1 + j_3 - k_3 \le \mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})(h_4) = m_4 - s_4 - j_1 + 2j_3, \end{aligned}$$

Hence, we can use Lemma 4.3 to compute

$$\begin{gathered} \dim(V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_{J_-})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})_{J_-}}) = \min\{k_2 - j_2, k_2 - j_2 + j_4 - k_4\} + 1\\ \text{and}\\ \dim(V(\mathrm{wt}(\mathbf{r}_{\boldsymbol{j}})_{J_+})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})_{J_+}}) = \min\{k_1 - j_1, k_1 - j_1 + j_3 - k_3\} + 1. \end{gathered} \tag{4.31}$$

Plugging this in (4.28) we get

$$\dim((V^+_{\boldsymbol{k}})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}) = \sum_{\boldsymbol{j}\in\mathcal{B}^+_{\boldsymbol{k}}} (\min\{k_2-j_2, k_2-j_2+j_4-k_4\}+1)(\min\{k_1-j_1, k_1-j_1+j_3-k_3\}+1). \tag{4.32}$$

We will need the following notation. Given, $i_1, i_2, \dots, i_l \in I$, and $a_1, \dots, a_l \in \mathbb{Z}_{\ge 0}$, set

$$\mathbf{x}^{a_1,\dots,a_l}_{i_1,\dots,i_l} = (x^-_{i_1})^{(a_1)} \cdots (x^-_{i_l})^{(a_l)}.$$

Also, given $\boldsymbol{j} \in \mathcal{B}_{\boldsymbol{k}}$, set

$$l_-(\boldsymbol{j}) = \min\{k_2 - j_2, k_2 - j_2 + j_4 - k_4\}, \quad l_+(\boldsymbol{j}) = \min\{k_1 - j_1, k_1 - j_1 + j_3 - k_3\}$$

so that (4.29) can be rewritten as

$$\dim((V_{\boldsymbol{k}})_{\mathrm{wt}(\mathbf{r}_{\boldsymbol{k}})}) = \sum_{\boldsymbol{j}\in\mathcal{B}_{\boldsymbol{k}}} (l_-(\boldsymbol{j}) + 1)(l_+(\boldsymbol{j}) + 1). \tag{4.33}$$

It now follows from Lemma 4.4 and (4.22) that (4.33) holds iff the vectors

$$\mathbf{x}^{p_5, p_4(\boldsymbol{j}), k_1 - j_1 - p_5}_{5,4,5}\mathbf{x}^{p_1, p_2(\boldsymbol{j}), k_2 - j_2 - p_1}_{1,2,1} v_{\boldsymbol{j}} \qquad \text{are linearly independent} \tag{4.34}$$

for $\boldsymbol{j} \in \mathcal{B}_{\boldsymbol{k}}, 0 \le p_1 \le l_-(\boldsymbol{j}), 0 \le p_5 \le l_+(\boldsymbol{j})$. Here $p_2(\boldsymbol{j}) = k_2 - j_2 + j_4 - k_4$ and $p_4(\boldsymbol{j}) = k_1 - j_1 + j_3 - k_3$. We will prove (4.34) only for λ as in Theorem 3.14. However, let us develop for a little longer the general case. In particular, we will show that all the vectors in (4.34) are nonzero.

Set $\boldsymbol{p}=(p_1,p_5), \mathbf{x}_{\boldsymbol{j},\boldsymbol{p}}=\mathbf{x}_{5,4,5}^{p_5,p_4(\boldsymbol{j}),k_1-j_1-p_5}\mathbf{x}_{1,2,1}^{p_1,p_2(\boldsymbol{j}),k_2-j_2-p_1}$, and $v_{\boldsymbol{j},\boldsymbol{p}}=\mathbf{x}_{\boldsymbol{j},\boldsymbol{p}}v_{\boldsymbol{j}}$. Thus, we want to show that the vectors $v_{\boldsymbol{j},\boldsymbol{p}}$ are linearly independent for $\boldsymbol{j}$ and $\boldsymbol{p}$ as above. From now on, when now confusion arises, we simplify notation and write l_- in place of $l_-(\boldsymbol{j})$, etc. Recall that

$$v_{\boldsymbol{j}}=(x_{\beta_{28},1}^-)^{j_1-j_3}(x_{\beta_{26},1}^-)^{j_3}(x_{\beta_{24},1}^-)^{s_2-j_1}v_{2,m_2}^{s_2}\otimes(x_{\beta_{28},1}^-)^{j_2-j_4}(x_{\beta_{27},1}^-)^{j_4}(x_{\beta_{25},1}^-)^{s_4-j_2}v_{4,m_4}^{s_4}\otimes w'.$$

To simplify the expression above, set $v_6=(x_{\beta_{30},1}^-)^{s_6}v_{6,m_6}^{s_6}$, $\mathbf{x}_{\boldsymbol{j}}^2=(x_{\beta_{28},1}^-)^{j_1-j_3}(x_{\beta_{26},1}^-)^{j_3}(x_{\beta_{24},1}^-)^{s_2-j_1}$, and $\mathbf{x}_{\boldsymbol{j}}^4=(x_{\beta_{28},1}^-)^{j_2-j_4}(x_{\beta_{27},1}^-)^{j_4}(x_{\beta_{25},1}^-)^{s_4-j_2}$ so that

$$v_{\boldsymbol{j}}=\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}\otimes\mathbf{x}_{\boldsymbol{j}}^4v_{4,m_4}^{s_4}\otimes v_6\otimes w. \tag{4.35}$$

Also, using Lemma 4.12 we get

$$\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=(x_4^-)^{j_1-j_3}(x_5^-)^{j_1}(x_{\beta_{24},1}^-)^{s_2}v_{2,m_2}^{s_2}\quad\text{and}\quad\mathbf{x}_{\boldsymbol{j}}^4v_{4,m_4}^{s_4}=(x_2^-)^{j_2-j_4}(x_1^-)^{j_2}(x_{\beta_{25},1}^-)^{s_4}v_{4,m_4}^{s_4} \tag{4.36}$$

up to nonzero scalar multiples. By applying the comultiplication one sees that $v_{\boldsymbol{j},\boldsymbol{p}}$ is equal to

$$\sum_{\chi}\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}\otimes\mathbf{x}_{5,4,5}^{d_4,e_4,f_4}\mathbf{x}_{1,2,1}^{a_4,b_4,c_4}\mathbf{x}_{\boldsymbol{j}}^4v_{4,m_4}^{s_4}\otimes v_6\otimes\mathbf{x}_{1,2,1}^{a_1,b_1,c_1}v_{1,m_1}\otimes\mathbf{x}_{5,4,5}^{d_5,e_5,f_5}v_{5,m_5} \tag{4.37}$$

where χ runs over the set of collections of nonnegative integers a_l,b_l,c_l,d_l,e_l,f_l satisfying

$$\begin{gathered}a_2+a_4+a_1=p_1,\qquad b_2+b_4+b_1=p_2,\qquad c_2+c_4+c_1=k_2-j_2-p_1,\\ d_2+d_4+d_5=p_5,\qquad e_2+e_4+e_5=p_4,\qquad f_2+f_4+f_5=k_1-j_1-p_5.\end{gathered} \tag{4.38}$$

Above we also used that $\mathbf{x}_{1,2,1}^{a,b,c}v_{5,m_5}=\mathbf{x}_{5,4,5}^{a,b,c}v_{1,m_1}=\mathbf{x}_{1,2,1}^{a,b,c}v_6=\mathbf{x}_{5,4,5}^{a,b,c}v_6=0$ whenever $a+b+c>0$. We will need to study the summands on the right-hand-side of (4.37).

Using Lemma 4.5 we see that $\mathbf{x}_{1,2,1}^{a_1,b_1,c_1}v_{1,m_1}\neq0$ iff $a_1+c_1\leq m_1$ and $b_1\leq c_1$ and, in that case, $\mathbf{x}_{1,2,1}^{a_1,b_1,c_1}v_{1,m_1}=\eta\mathbf{x}_{2,1}^{b_1,a_1+c_1}v_{1,m_1}$ for some positive rational number η (depending on a_1,b_1,c_1). Similarly, $\mathbf{x}_{5,4,5}^{d_5,e_5,f_5}v_{5,m_5}\neq0$ iff $d_5+f_5\leq m_5$ and $e_5\leq f_5$ and, in that case, $\mathbf{x}_{5,4,5}^{d_5,e_5,f_5}v_{5,m_5}$ is a positive multiple of $\mathbf{x}_{4,5}^{e_5,d_5+f_5}v_{5,m_5}$. Next, we study the factor $\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}$. Notice that $x_5^+(x_{\beta_{24},1}^-)^{s_2}v_{2,m_2}^{s_2}=h_4(x_{\beta_{24},1}^-)^{s_2}v_{2,m_2}^{s_2}=0$, and $h_5(x_{\beta_{24},1}^-)^{s_2}v_{2,m_2}^{s_2}=s_2$. Therefore, we can use Lemma 4.5 together with (4.36) to see that $\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}$ is a nonnegative rational multiple of

$$\mathbf{x}_{4,5}^{e_2+j_1-j_3,j_1+f_2+d_2}(x_{\beta_{24},1}^-)^{s_2}v_{2,m_2}^{s_2}$$

and it is nonzero provided $e_2\leq j_3+f_2$ and $j_1+d_2+f_2\leq s_2$. Since $d_2\leq p_5, f_2\leq k_1-j_1-p_5$ by (4.38), and $k_1\leq s_2$, the latter is always satisfied. One easily checks that

$$x_1^+\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=h_1\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=0$$

which implies $\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=0$ if $c_2\neq0$. Next, using the relations

$$[x_2^+,x_{\beta_{24},1}^-]=x_{\beta_{21},1}^-,\quad[x_{\beta_{21},1}^-,x_{\beta_{24},1}^-]=0,\quad x_{\beta_{21},1}^-v_{2,m_2}^{s_2}=0,$$

one sees that $x_2^+\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=0$. Since $h_2\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}=(m_2-s_2)\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}$, it follows from Lemma 4.5 that

$$\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}_{\boldsymbol{j}}^2v_{2,m_2}^{s_2}\neq0\quad\text{iff}\quad c_2=0\quad\text{and}\quad a_2\leq b_2\leq m_2-s_2.$$

Since we anyway have $b_2 \le p_2 = (k_2 - k_4) - (j_2 - j_4) \le m_2 - s_2$, the relevant conditions are $c_2 = 0$ and $a_2 \le b_2$. Therefore, we find that $\mathbf{x}_{5,4,5}^{d_2,e_2,f_2}\mathbf{x}_{1,2,1}^{a_2,b_2,c_2}\mathbf{x}^2_{\boldsymbol{j}} v_{2,m_2}^{s_2}$ is a nonnegative rational multiple of

$$\mathbf{x}_{4,5}^{e_2+j_1-j_3,j_1+f_2+d_2}\mathbf{x}_{1,2}^{a_2,b_2}(x^-_{\beta_{24},1})^{s_2}v_{2,m_2}^{s_2}$$

which is nonzero iff

$$a_2 \le b_2 \qquad \text{and} \qquad e_2 \le j_3 + f_2.$$

Similarly, we get that $\mathbf{x}_{5,4,5}^{d_4,e_4,f_4}\mathbf{x}_{1,2,1}^{a_4,b_4,c_4}\mathbf{x}^4_{\boldsymbol{j}} v_{4,m_4}^{s_4}$ is a nonnegative rational multiple of

$$\mathbf{x}_{2,1}^{b_4+j_2-j_4,j_2+c_4+a_4}\mathbf{x}_{5,4}^{d_4,e_4}(x^-_{\beta_{25},1})^{s_4}v_{4,m_4}^{s_4}$$

which is nonzero iff

$$d_4 \le e_4 \qquad \text{and} \qquad b_4 \le j_4 + c_4.$$

Therefore, the sum in (4.37) is a linear combination of the vectors

$$\mathbf{x}_{4,5}^{e_2',f_2'}\mathbf{x}_{1,2}^{a_2,b_2}(x^-_{\beta_{24},1})^{s_2}v_{2,m_2}^{s_2} \otimes \mathbf{x}_{5,4}^{d_4,e_4}\mathbf{x}_{2,1}^{b_4',c_4'}(x^-_{\beta_{25},1})^{s_4}v_{4,m_4}^{s_4} \otimes v_6 \otimes \mathbf{x}_{2,1}^{b_1,c_1'}v_{1,m_1} \otimes \mathbf{x}_{4,5}^{e_5,f_5'}v_{5,m_5} \tag{4.39}$$

where

$$\begin{aligned} c_1' &= c_1 + a_1, & b_4' &= b_4 + j_2 - j_4, & c_4' &= c_4 + a_4 + j_2,\\ f_5' &= f_5 + d_5, & e_2' &= e_2 + j_1 - j_3, & f_2' &= f_2 + d_2 + j_1, \end{aligned}$$

with the numbers $a_l, b_l, \ldots, f_l$ satisfying (4.38) as well as

$$\begin{aligned} &a_1 + c_1 \le m_1, \quad && b_1 \le c_1, \quad && a_2 \le b_2, \quad && b_4 \le c_4 + j_4, \quad && c_2 = 0,\\ &d_5 + f_5 \le m_5, && e_5 \le f_5, && d_4 \le e_4, && e_2 \le f_2 + j_3, && f_4 = 0. \end{aligned} \tag{4.40}$$

Notice that

$$\begin{aligned} &a_2 = a_1 = b_1 = b_4 = c_1 = 0, \quad && a_4 = p_1, \quad && b_2 = p_2, \quad && c_4 = k_2 - j_2 - p_1,\\ &d_4 = d_5 = e_5 = e_2 = f_5 = 0, && d_2 = p_5, && e_4 = p_4, && f_2 = k_1 - j_1 - p_5, \end{aligned}$$

satisfy (4.38) and (4.40), which implies that the set of nonzero summands in (4.37) is nonempty. One easily sees that the vectors in (4.39), for distinct values of $(a_2, b_1, b_2, b_4', c_1', c_4', d_4, e_2', e_4, e_5, f_2', f_5')$, are linearly independent by looking at the weights of their tensor factors. Since $v_{\boldsymbol{j},\boldsymbol{p}}$ is a linear combination of these vectors with positive rational coefficients, it follows that $v_{\boldsymbol{j},\boldsymbol{p}} \ne 0$ for all choices of $\boldsymbol{j}$ and $\boldsymbol{p}$.

We now restrict ourselves to λ as in Theorem 3.14. To simplify notation, we rewrite the vectors in (4.39) as

$$v_2^{a_2,b_2,e_2',f_2'} \otimes v_4^{b_4',c_4',d_4,e_4} \otimes v_6 \otimes v_1^{b_1,c_1'} \otimes v_5^{e_5,f_5'}. \tag{4.41}$$

If $\{2,4\} \not\subseteq \operatorname{supp}(\lambda)$, the argument reduces to one identical to the one used in the proof of [21, Proposition 5.7] (all the details can be found in [23, Lemma 5.3.9]). From now on we assume $\operatorname{supp}(\lambda) \subseteq \{2,4,6\}$ which is the remaining case to consider. In this case, we must have $j_3 = j_4 = k_3 = k_4 = 0, p_2 = l_-, p_4 = l_+$. In particular, (4.38) and (4.40) reduce to

$$\begin{aligned} &a_1 = b_1 = c_1 = c_2 = 0, \quad a_2 + a_4 = p_1, \quad b_2 + b_4 = k_2 - j_2, \quad c_4 = k_2 - j_2 - p_1, \quad a_2 \le b_2, \quad b_4 \le c_4,\\ &d_5 = e_5 = f_5 = f_4 = 0, \quad d_4 + d_2 = p_5, \quad e_4 + e_2 = k_1 - j_1, \quad f_2 = k_1 - j_1 - p_5, \quad d_4 \le e_4, \quad e_2 \le f_2. \end{aligned}$$

Therefore, $v_{\boldsymbol{j},\boldsymbol{p}}$ is a linear combination of vectors of the form

$$v_2^{a_2,b_2,k_1-e_4,k_1-d_4} \otimes v_4^{k_2-b_2,k_2-a_2,d_4,e_4} \otimes v_6 \quad \text{with} \quad 0 \le a_2 \le p_1 \le b_2 \le l_-, 0 \le d_4 \le p_5 \le e_4 \le l_+. \tag{4.42}$$

Set

$$v_{a,b,d,e} = v_2^{a,b,k_1-e,k_1-d} \otimes v_4^{k_2-b,k_2-a,d,e} \otimes v_6 \tag{4.43}$$

and observe that the coefficient of $v_{a,b,d,e}$ in $v_{\boldsymbol{j},\boldsymbol{p}}$ is nonzero iff $j_1 \leq k_1 - e, j_2 \leq k_2 - b, a \leq p_1, d \leq p_5$.

To complete the proof, we now show by induction on $n_1 \in \mathbb{Z}_{\geq 0}$ that the set $\{v_{\boldsymbol{j},\boldsymbol{p}} : (k_1 - j_1) \leq n_1\}$ is linearly independent. We prove this performing a further induction on $n_2 \in \mathbb{Z}_{\geq 0}$ to show that the set $\{v_{\boldsymbol{j},\boldsymbol{p}} : (k_1 - j_1) \leq n_1, (k_2 - j_2) \leq n_2\}$ is linearly independent. Set

$$S(n_1, n_2) = \{(\boldsymbol{j}, \boldsymbol{p}) : k_1 - j_1 \leq n_1, k_2 - j_2 \leq n_2\}, \quad S[n_1, n_2) = \{(\boldsymbol{j}, \boldsymbol{p}) : k_1 - j_1 = n_1, k_2 - j_2 \leq n_2\},$$
$$S(n_1, n_2] = \{(\boldsymbol{j}, \boldsymbol{p}) : k_1 - j_1 \leq n_1, k_2 - j_2 = n_2\}, \quad S[n_1, n_2] = \{(\boldsymbol{j}, \boldsymbol{p}) : k_1 - j_1 = n_1, k_2 - j_2 = n_2\}.$$

The inductions clearly start when $n_1 = n_2 = 0$ since $\{v_{\boldsymbol{j},\boldsymbol{p}} : (\boldsymbol{j}, \boldsymbol{p}) \in S(0,0)\} = \{v_{\boldsymbol{k}}\}$. Assume now that $n_2 > 0$ and, by induction hypothesis, that the set $\{v_{\boldsymbol{j},\boldsymbol{p}} : (\boldsymbol{j}, \boldsymbol{p}) \in S(n_1, n_2 - 1)\}$ is linearly independent. Let $c_{\boldsymbol{j},\boldsymbol{p}} \in \mathbb{C}$ be such that

$$\sum_{(\boldsymbol{j},\boldsymbol{p}) \in S(n_1,n_2)} c_{\boldsymbol{j},\boldsymbol{p}} v_{\boldsymbol{j},\boldsymbol{p}} = 0. \tag{4.44}$$

By the induction hypothesis, it remains to show that

$$c_{\boldsymbol{j},\boldsymbol{p}} = 0 \quad \text{for all} \quad (\boldsymbol{j}, \boldsymbol{p}) \in S(n_1, n_2]. \tag{4.45}$$

Set

$$S[n_1, n_2](m) = \{(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1, n_2] : (p_1, p_5) = (n_2 - r, n_1 - s), r + s \leq m\}.$$

Observe that if $(\boldsymbol{j}, \boldsymbol{p}) \in S(n_1, n_2)$ is such that the coefficient of v_{n_2-r,n_2,n_1-s,n_1} in $v_{\boldsymbol{j},\boldsymbol{p}}$ is nonzero, then $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1, n_2]$ and $(p_1, p_5) = (n_2 - r', n_1 - s'), 0 \leq r' \leq r, 0 \leq s' \leq s$. An easy induction on $r + s \geq 0$ shows that $c_{\boldsymbol{j},\boldsymbol{p}} = 0$ for all $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1, n_2](r + s)$. This implies $c_{\boldsymbol{j},\boldsymbol{p}} = 0$ for all $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1, n_2]$. Similarly, if $(\boldsymbol{j}, \boldsymbol{p}) \in S(n_1, n_2) \backslash S[n_1, n_2]$ is such that the coefficient of $v_{n_2-r,n_2,n_1-1-s,n_1-1}$ in $v_{\boldsymbol{j},\boldsymbol{p}}$ is nonzero, then $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1 - 1, n_2]$ and $(p_1, p_5) = (n_2 - r', n_1 - 1 - s'), 0 \leq r' \leq r, 0 \leq s' \leq s$. Again, an easy induction on $r + s \geq 0$ shows that $c_{\boldsymbol{j},\boldsymbol{p}} = 0$ for all $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1 - 1, n_2](r + s)$. Proceeding recursively in this way one proves $c_{\boldsymbol{j},\boldsymbol{p}} = 0$ for all $(\boldsymbol{j}, \boldsymbol{p}) \in S[n_1 - j, n_2], 0 \leq j \leq n_1$. Since $S(n_1, n_2] = \cup_j S[n_1 - j, n_2]$, (4.45) follows.

The above paragraph proves the induction step on n_2. It remains to show that the induction on n_2 starts when $n_1 > 0$. Thus, assume $n_1 > 0, n_2 = 0$ and, by induction hypothesis on n_1, that $\{v_{\boldsymbol{j},\boldsymbol{p}} : (\boldsymbol{j}, \boldsymbol{p}) \in S(n_1 - 1, 0)\}$ is linearly independent. Let $c_{\boldsymbol{j},\boldsymbol{p}} \in \mathbb{C}$ be such that

$$\sum_{(\boldsymbol{j},\boldsymbol{p}) \in S(n_1,0)} c_{\boldsymbol{j},\boldsymbol{p}} v_{\boldsymbol{j},\boldsymbol{p}} = 0. \tag{4.46}$$

By the induction hypothesis, it remains to show that

$$c_{\boldsymbol{j},\boldsymbol{p}} = 0 \quad \text{for all} \quad (\boldsymbol{j}, \boldsymbol{p}) \in S[n_1, 0]. \tag{4.47}$$

The proof of (4.47) is similar to that of (4.45) and we omit the details.

Remark. Observe that the above proof of (4.45) is based on finding values of a, b, d, e such that $v_{a,b,d,e}$ appears with nonzero coefficient in $v_{\boldsymbol{j},\boldsymbol{p}}$ for exactly one value of of the pair $(\boldsymbol{j}, \boldsymbol{p}) \in S(n_1, n_2)$ and so on. The difficult in adapting the above proof for proving (4.34) for all λ not supported in the trivalent node resides in the fact that, if $\{2, 4\} \subseteq \operatorname{supp}(\lambda)$ and either $m_1 \neq 0$ or $m_5 \neq 0$, one can give examples of $(\boldsymbol{j}, \boldsymbol{p}) \neq (\boldsymbol{j}', \boldsymbol{p}')$ such that the summands of the form $v_{a,b,d,e}$ with nonzero coefficients appearing in $v_{\boldsymbol{j},\boldsymbol{p}}$ are exactly the same as those appearing in $v_{\boldsymbol{j}',\boldsymbol{p}'}$. Hence, one would need to keep a very efficient control of the coefficients.

4.8. **Proof of** (4.21)**.** By (4.19), in order to prove that $\mathbf{r}_{\boldsymbol{j}} \in \mathcal{A}(\boldsymbol{s}) \Rightarrow \boldsymbol{j} \in \mathcal{B}(s)$, it remains to show that $\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}} \neq 0$ only if $j_3 \leq j_1, j_4 \leq j_2$, and $j_5 \geq 0$. It follows from Lemma 4.11 that

$$\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}} = (x^-_{\beta_{30},1})^{s_6+j_5}(x^-_{\beta_{28},1})^{j_0}\left((x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} \otimes (x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4} \otimes v^{s_6}_{6,m_6} \otimes w\right)$$

Notice that if $s_2 - j_1 + j_3 > s_2$ we have $(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} = 0$ by (4.17). In other words, $\mathbf{r}_{\boldsymbol{j}} \in \mathcal{A}(\boldsymbol{s})$ only if $j_3 \leq j_1$. Similarly, we must have $j_4 \leq j_2$. Continuing the above computation we get that $\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}} = (x^-_{\beta_{30},1})^{s_6+j_5}v'$ where v' is the vector

$$\sum_{k=0}^{j_0}\tbinom{j_0}{k}(x^-_{\beta_{28},1})^{j_0-k}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} \otimes (x^-_{\beta_{28},1})^{k}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4} \otimes v^{s_6}_{6,m_6} \otimes w.$$

By (4.17), $(x^-_{\beta_{28},1})^{j_0-k}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} = 0$ if $(j_0-k)+j_3+(s_2-j_1) > s_2$. Hence, the summand corresponding to k in the above summation is nonzero only if $j_2 - j_4 - j_5 \leq k$. Similarly, $(x^-_{\beta_{28},1})^{k}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4} = 0$ if $k+j_4+(s_4-j_2) > s_4$, i.e., if $k > j_2 - j_4$. Thus, the summand corresponding to k in the above summation is nonzero only if $j_2 - j_4 - j_5 \leq k \leq j_2 - j_4$. In particular, we must have $j_5 \geq 0$.

To complete the proof of (4.21), we need to show that $\boldsymbol{j} \in \mathcal{B}(\boldsymbol{s}) \Rightarrow \mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}} \neq 0$. Set $j_- = \max\{0, j_2 - j_4 - j_5\}$ and $j_+ = \min\{j_0, j_2 - j_4\}$ and observe that $\boldsymbol{j} \in \mathcal{B}(\boldsymbol{s}) \Rightarrow j_- \leq j_+$. Given $j_- \leq k \leq j_+$, set

$$v_k = \tbinom{j_0}{k}(x^-_{\beta_{28},1})^{j_0-k}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} \otimes (x^-_{\beta_{28},1})^{k}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4}.$$

Notice that Lemmas 4.12 and 4.13 imply that $v_k \neq 0$. Continuing the above computation we see that

$$\begin{aligned}\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}} &= \sum_{l=0}^{s_6+j_5}\sum_{k=j_-}^{j_+}\tbinom{s_6+j_5}{l}(x^-_{\beta_{30},1})^{s_6+j_5-l}v_k \otimes (x^-_{\beta_{30},1})^{l}v^{s_6}_{6,m_6} \otimes w = \\ &= \tbinom{s_6+j_5}{s_6}\sum_{k=j_-}^{j_+}(x^-_{\beta_{30},1})^{j_5}v_k \otimes (x^-_{\beta_{30},1})^{s_6}v^{s_6}_{6,m_6} \otimes w.\end{aligned}$$

The second equality above is proved as follows. By (4.17), $(x^-_{\beta_{30},1})^{s_6+j_5-l}v_k = 0$ if $(s_6+j_5-l)+(j_0-k)+j_3+(s_2-j_1)+k+j_4+(s_4-j_2) > s_2+s_4$, i.e., if $l < s_6$. Similarly, $(x^-_{\beta_{30},1})^{l}v^{s_6}_{6,m_6} = 0$ if $l > s_6$. By (4.15), $(x^-_{\beta_{30},1})^{s_6}v^{s_6}_{6,m_6} \neq 0$ and, therefore, it remains to show that

$$(x^-_{\beta_{30},1})^{j_5}\sum_{k=j_-}^{j_+}v_k \neq 0. \tag{4.48}$$

Indeed, $\tbinom{j_0}{k}^{-1}(x^-_{\beta_{30},1})^{j_5}v_k$ is equal to

$$\sum_{l=0}^{j_5}\tbinom{j_5}{l}(x^-_{\beta_{30},1})^{l}(x^-_{\beta_{28},1})^{j_0-k}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} \otimes (x^-_{\beta_{30},1})^{j_5-l}(x^-_{\beta_{28},1})^{k}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4}.$$

Making use of (4.17) once more we see that

$$(x^-_{\beta_{30},1})^{j_5}v_k = \tbinom{j_0}{k}\tbinom{j_5}{k-j_2+j_4+j_5}v_2^k \otimes v_4^k$$

where

$$v_2^k = (x^-_{\beta_{30},1})^{k-j_2+j_4+j_5}(x^-_{\beta_{28},1})^{j_0-k}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2}$$

and
$$v_4^k = (x^-_{\beta_{30},1})^{j_2-j_4-k}(x^-_{\beta_{28},1})^k(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4}.$$
Lemma 4.12 implies that $v_2^k \neq 0$ while Lemma 4.13 implies that $v_4^k \neq 0$. Observing that v_2^k are weight vectors of distinct weight and similarly for v_4^k, (4.48) follows. This completes the proof of (4.21). Notice also that, if $j_5 = 0$, it follows from the computations above that $\mathbf{x}_{\mathbf{r}_{\boldsymbol{j}}} v_{\boldsymbol{s}}$ is a nonzero scalar multiple of
$$(x^-_{\beta_{28},1})^{j_1-j_3}(x^-_{\beta_{26},1})^{j_3}(x^-_{\beta_{24},1})^{s_2-j_1}v^{s_2}_{2,m_2} \otimes (x^-_{\beta_{28},1})^{j_2-j_4}(x^-_{\beta_{27},1})^{j_4}(x^-_{\beta_{25},1})^{s_4-j_2}v^{s_4}_{4,m_4} \otimes (x^-_{\beta_{30},1})^{s_6}v^{s_6}_{6,m_6} \otimes w.$$

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