HÖLDER CONTINUITY OF HARMONIC QUASICONFORMAL MAPPINGS★

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Abstract. We prove that for harmonic quasiconformal mappings α-Hölder continuity on the boundary implies α-Hölder continuity of the map itself. Our result holds for the class of uniformly perfect bounded domains, in fact we can allow that a portion of the boundary is thin in the sense of capacity. The problem for general bounded domains remains open.

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1. Introduction

The following theorem is the main result in [8].

Theorem 1.1. *Let D be a bounded domain in $\mathbb{R}^n$ and let f be a continuous mapping of $\overline{D}$ into $\mathbb{R}^n$ which is quasiconformal in D. Suppose that, for some $M > 0$ and $0 < \alpha \le 1$,*

$$|f(x) - f(y)| \le M|x - y|^\alpha \tag{1.2}$$

whenever x and y lie on ∂D. Then

$$|f(x) - f(y)| \le M'|x - y|^\beta \tag{1.3}$$

for all x and y on $\overline{D}$, where $\beta = \min(\alpha, K_I^{1/(1-n)})$ and M' depends only on M, α, n, $K(f)$ and $\operatorname{diam}(D)$.

The exponent β is the best possible, as an example of a radial quasiconformal map $f(x) = |x|^{\alpha-1}x$, $0 < \alpha < 1$, of $\overline{\mathbb{B}^n}$ onto itself shows (see [11], p. 49). Also, the assumption of boundedness is essential. Indeed, one can consider $g(x) = |x|^a x$, $|x| \ge 1$ where $a > 0$. Then g is quasiconformal in $D = \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ (see [11], p. 49), it is identity on ∂D and hence Lipschitz continuous on ∂D. However, $|g(te_1) - g(e_1)| \asymp t^{a+1}$, $t \to \infty$, and therefore g is not globally Lipschitz continuous on D.

This paper deals with the following question, suggested by P. Koskela: is it possible to replace β with α if we assume, in addition to quasiconformality, that f is harmonic? In the special case $D = \mathbb{B}^n$ this was proved, for arbitrary moduli of continuity $\omega(\delta)$, in [2]. Our main result is that the answer is positive, if ∂D is a uniformly perfect set (cf. [6]). In fact, we prove a more general result, including domains having a thin, in the sense of capacity, portion of the boundary. However, this generality is in a sense illusory, because any hqc mapping extends harmonically and quasiconformally across such portion of the boundary. Nevertheless, it leads to a natural open question: is the answer positive for arbitrary bounded domain in $\mathbb{R}^n$?

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In the case of smooth boundaries much better regularity up to the boundary can be deduced, see [7]; related results for harmonic functions were obtained by [1].

We denote by $B(x,r)$ and $S(x,r)$ the open ball, respectively sphere, in $\mathbb{R}^n$ with center x and radius $r>0$. We adopt the basic notation, terminology and definitions related to quasiconformal maps from [11]. A condenser is a pair (K,U), where K is a non-empty compact subset of an open set $U\subset\mathbb{R}^n$. The capacity of the condenser (K,U) is defined as

$$\mathrm{cap}(K,U)=\inf\int_{\mathbb{R}^n}|\nabla u|^n dV,$$

where infimum is taken over all continuous real-valued $u\in ACL^n(\mathbb{R}^n)$ such that $u(x)=1$ for $x\in K$ and $u(x)=0$ for $x\in\mathbb{R}^n\setminus U$. In fact, one can replace the ACL^n condition with Lipschitz continuity in this definition. We note that, for a compact $K\subset\mathbb{R}^n$ and open bounded sets U_1 and U_2 containing K we have: $\mathrm{cap}(K,U_1)=0$ iff $\mathrm{cap}(K,U_2)=0$, therefore the notion of a compact set of zero capacity is well defined (see [12], Remarks 7.13) and we can write $\mathrm{cap}(K)=0$ in this situation. For the notion of the modulus $M(\Gamma)$ of a family Γ of curves in $\mathbb{R}^n$ we refer to [11] and [12]. These two notions are related: by results of [5] and [13] we have

$$\mathrm{cap}(K,U)=M(\Delta(K,\partial U;U)),$$

where $\Delta(E,F;G)$ denotes the family of curves connecting E to F within G, see [11] or [12] for details.

In addition to this notion of capacity, related to quasiconformal mappings, we need Wiener capacity, related to harmonic functions. For a compact $K\subset\mathbb{R}^n$ it is defined by

$$\mathrm{cap}_W(K)=\inf\int_{\mathbb{R}^n}|\nabla u|^2 dV,$$

where infimum is taken over all Lipschitz continuous compactly supported functions u on $\mathbb{R}^n$ such that $u=1$ on K. Let us note that every compact $K\subset\mathbb{R}^n$ which has capacity zero has Wiener capacity zero. Indeed, choose an open ball $B_R=B(0,R)\supset K$. Since $n\geq 2$ we have, by Hölder inequality,

$$\int_{\mathbb{R}^n}|\nabla u|^2 dV\leq |B_R|^{1-2/n}\left(\int_{\mathbb{R}^n}|\nabla u|^n dV\right)^{2/n}$$

for any Lipschitz continuous u vanishing outside U, our claim follows immediately from definitions.

A compact set $K\subset\mathbb{R}^n$, consisting of at least two points, is α-uniformly perfect ($\alpha>0$) if there is no ring R separating K (i.e. such that both components of $\mathbb{R}^n\setminus R$ intersect K) such that $\mathrm{mod}(R)>\alpha$. We say that a compact $K\subset\mathbb{R}^n$ is uniformly perfect if it is α-uniformly perfect for some $\alpha>0$.

We denote the α-dimensional Hausdorff measure of a set $F\subset\mathbb{R}^n$ by $\Lambda_\alpha(F)$.

2. The main result

In this section D denotes a bounded domain in $\mathbb{R}^n$. Let

$$\Gamma_0=\{x\in\partial D:\mathrm{cap}\,\overline{B}(x,\epsilon)\cap\partial D=0 \text{ for some } \epsilon>0\},$$

and $\Gamma_1=\partial D\setminus\Gamma_0$. Using this notation we can state our main result.

Theorem 2.1. *Assume $f:\overline{D}\to\mathbb{R}^n$ is continuous on $\overline{D}$, harmonic and quasiconformal in D. Assume f is Hölder continuous with exponent α, $0<\alpha\leq 1$, on ∂D and Γ_1 is uniformly perfect. Then f is Hölder continuous with exponent α on $\overline{D}$.*

If Γ_0 is empty we obtain the following

Corollary 2.2. *If $f : \overline{D} \to \mathbb{R}^n$ is continuous on $\overline{D}$, Hölder continuous with exponent α, $0 < \alpha \leq 1$, on ∂D, harmonic and quasiconformal in D and if ∂D is uniformly perfect, then f is Hölder continuous with exponent α on $\overline{D}$.*

The first step in proving Theorem 2.1 is reduction to the case $\Gamma_0 = \emptyset$. In fact, we show that existence of a hqc extension of f across Γ_0 follows from well known results. Let $D' = D \cup \Gamma_0$. Then D' is an open set in $\mathbb{R}^n$, Γ_0 is a closed subset of D' and $\partial D' = \Gamma_1$.

Clearly $\operatorname{cap}(K \cap \Gamma_0) = 0$ for each compact $K \subset D'$, and therefore, by Lemma 7.14 in [12], $\Lambda_\alpha(K \cap \Gamma_0) = 0$ for each $\alpha > 0$. In particular, Γ_0 has σ-finite $(n-1)$-dimensional Hausdorff measure. Since it is closed in D', we can apply Theorem 35.1 in [11] to conclude that f has a quasiconformal extension F across Γ_0 which has the same quasiconformality constant as f.

Since Γ_0 is a countable union of compact subsets K_j of capacity zero and $\operatorname{cap}_W(K_j) = \operatorname{cap}(K_j)$ we conclude that Γ_0 has Wiener capacity zero. Hence, by a classical result (see [4]), there is a (unique) extension $G : \overline{D'} \to \mathbb{R}^n$ of f which is harmonic in D'. Obviously, $F = G$ is a harmonic quasiconformal extension of f to $\overline{D'}$ which has the same quasiconformality constant as f.

In effect, we reduced the proof of Theorem 2.1 to the proof of Corollary 2.2. We begin the proof of Corollary 2.2 with the following

Lemma 2.3. *Let $D \subset \mathbb{R}^n$ be a bounded domain with uniformly perfect boundary. There exists a constant $m > 0$ such that for every $y \in D$ we have*

$$\operatorname{cap}(\overline{B}(y, \frac{d}{2}), D) \geq m \ , \qquad d = \operatorname{dist}(y, \partial D). \tag{2.4}$$

Proof. Fix $y \in D$ as above and $z \in \partial D$ such that $|y - z| = d \equiv r$. Clearly $diam(\partial D) = diam(D) > 2r$. Set $F_1 = \overline{B}(z, r) \cap (\partial D)$ and $F_2 = \overline{B}(z, r) \cap \overline{B}(y, \frac{d}{2})$, $F_3 = S(z, 2r)$. Let $\Gamma_{i,j} = \Delta(F_i, F_j; \mathbb{R}^n)$ for $i, j = 1, 2, 3$. By [6, Thm 4.1(3)] there exists a constant $a = a(E, n) > 0$ such that

$$M(\Gamma_{1,3}) \geq a$$

while by standard estimates [11, 7.5] there exists $b = b(n) > 0$ such that

$$M(\Gamma_{2,3}) \geq b\,.$$

Next, by [12, Cor 5.41] there exists $m = m(E, n) > 0$ such that

$$M(\Gamma_{1,2}) \geq m\,.$$

Finally, with $B = \overline{B}(y, d/2)$ we have

$$\operatorname{cap}(B, D) = M(\Delta(B, \partial D; \mathbb{R}^n)) \geq M(\Gamma_{1,2}) \geq m\,.$$

□

In conclusion, from the above lemma, our assumption

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\alpha \ , \qquad x_1, x_2 \in \partial D,$$

and Lemma 8 in [8] we conclude that there is a constant M, depending on m, n, $K(f)$, C and α only such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha \ , \quad y \in D,\ x \in \partial D,\ \operatorname{dist}(y, \partial D) = |x - y|.$$

However, an argument presented in [8] shows that the above estimate holds for $y \in D$, $x \in \partial D$ without any further conditions, but with possibly different constant:

$$|f(x) - f(y)| \le M'|x-y|^\alpha\ , \quad y \in D,\ x \in \partial D. \tag{2.5}$$

The following lemma was proved in [3] for real valued functions, but the proof relies on the maximum principle which holds also for vector valued harmonic functions, hence lemma holds for harmonic mappings as well.

Lemma 2.6. *Assume $h : \overline{D} \to \mathbb{R}^n$ is continuous on $\overline{D}$ and harmonic in D. Assume for each $x_0 \in \partial D$ we have*

$$\sup_{B_r(x_0)\cap D'} |h(x) - h(x_0)| \le \omega(r) \qquad \text{for} \quad 0 < r \le r_0.$$

Then $|h(x) - h(y)| \le \omega(|x-y|)$ whenever $x, y \in D$ and $|x-y| \le r_0$.

Now we combine (2.5) and the above lemma, with $r_0 = \mathrm{diam}(D)$, to complete the proof of Corollary 2.2 and therefore of Theorem 2.1 as well.

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