arXiv:1012.3064v1 [math.AG] 14 Dec 2010

# A STRONG ABHYANKAR-MOH THEOREM AND CRITERION OF EMBEDDED LINE

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ABSTRACT. The condition of plane polynomial curve to be a line in well-known Abhyankar-Moh Theorem is replaced by weaker ones. A criterion of embedded line is obtained from this strong theorem.

## INTRODUCTION

Famous Abhyankar-Moh Theorem [1,2] states that for a field $k$ of characteristic zero, if $f(z)$ and $g(z)$ are polynomials and $k[f(z), g(z)] = k[z]$, then either $\deg f(z)$ divides $\deg g(z)$ or $\deg g(z)$ divides $\deg f(z)$. But the considered polynomial curve has to be a line at beginning is too strong and it limits the applications of the Theorem. In this paper, we replace the condition by weaker ones. Therefore we call it Strong Abhyankar-Moh Theorem. Using this strong theorem, we get a criterion for a polynomial plane curve to be an embedded line.

## 1. A STRONG ABHYANKAR-MOH THEOREM AND CRITERION OF EMBEDDED LINE

**Theorem 1.1 (Strong Abhyankar-Moh Theorem).** *Let $k$ be a field of characteristic zero and $F(f, g)$ be a plane curve which is defined by polynomials $f(z)$ and $g(z)$, here $z$ can be an unfaithful parameter. Let $m$ and $n$ be the degrees of $f(z)$ and $g(z)$ respectively. Assume that there is an integer $a > 0$ such that $a \leq \min(m, n)$ and there are polynomials $u(z)$ and $v(z)$ in polynomial ring $k[f(z), g(z)]$ such that $\deg u(z) = m - a$ and $\deg v(z) = n - a$, then we have that either $\deg f(z)$ divides $\deg g(z)$ or $\deg g(z)$ divides $\deg f(z)$.*

*Proof.* First we reduce to faithful parameter case. In fact, if $z$ is not a faithful parameter, from [3, Theorem 3.3.], there exits $h = h(z) \in k[z]$ and $\tilde{f}, \tilde{g} \in k[z]$ such that $f(z) = \tilde{f}(h(z))$ , $g(z) = \tilde{g}(h(z))$ and $h$ is a faithful parameter. We note that $u(z) \in k[f(z), g(z)] = k[\tilde{f}(h(z)), \tilde{g}(h(z))]$ if and only if there exists $\tilde{u}(h) \in k[\tilde{f}(h), \tilde{g}(h)]$ such that $u(z) = \tilde{u}(h(z))$ . We also note that $\deg u(z) = m - a$ if and

1991 *Mathematics Subject Classification.* Primary 13F20, 14H45, 14H50.

*Key words and phrases.* Abhyankar-Moh Theorem, plane polynomial curve, embedded line, criterion of embedded line.

only if $\deg_h \tilde{u}(h) = \frac{m}{\deg h(z)} - \frac{a}{\deg h(z)}$. Therefore we only need to handle faithful parameter case.

We follow terms and notations of [4]. We have approximate roots

$$T_i \in k[f,g], i = 2, \cdots, h$$

For convenient, we denote $T_1 = f$ and $T_0 = g$. Let

$$\deg T_i = -\mu_i,$$

$$d_{i+1} = \gcd(-\mu_0, \cdots, -\mu_i), i = 1, \cdots, h$$

Then we have

$$d_2 > d_3 > \cdots > d_{h+1} = 1$$

By Abhyankar-Moh Semi-group Structure Theorem [2, 4], we have that

$$\deg u(z) = \alpha_0(-\mu_0) + \cdots + \alpha_h(-\mu_h)$$

and

$$\deg v(z) = \beta_0(-\mu_0) + \cdots + \beta_h(-\mu_h)$$

here for $i = 0, \cdots, h$, $\alpha_i$ and $\beta_i$ are nonegative integers, which satisfying

$$0 \leq \alpha_2 < \frac{d_2}{d_3}, \cdots, 0 \leq \alpha_h < \frac{d_h}{d_{h+1}}$$

$$0 \leq \beta_2 < \frac{d_2}{d_3}, \cdots, 0 \leq \beta_h < \frac{d_h}{d_{h+1}}$$

and $\alpha_2, \cdots, \alpha_h, \beta_2, \cdots, \beta_h$ are unique. Hence we have

$$\deg u(z) - \deg v(z) = (\alpha_0 - \beta_0)(-\mu_0) + \cdots + (\alpha_h - \beta_h)(-\mu_h) \tag{1.1.1}$$

As we have that

$$\deg u(z) - \deg v(z) - ((\alpha_0 - \beta_0)(-\mu_0) + \cdots + (\alpha_h - \beta_{h-1})(-\mu_{h-1})) \equiv 0 \mod d_h$$

$$(d_h/d_{h+1}, -\mu_h/d_{h+1}) = 1$$

and

$$|\alpha_h - \beta_h| < d_h/d_{h+1}$$

Therefore we conclude that

$$\alpha_h = \beta_h$$

Similarly we can prove that

$$\alpha_{h-1} = \beta_{h-1}, \cdots, \alpha_2 = \beta_2$$

If $m = n$, we have nothing to prove. Therefore we suppose that $m > n$. It is easy to see that

$$\alpha_1 = 0, \beta_0 = 0, \beta_1 = 0$$

and equation (1.1.1) becomes

$$m - n = \alpha_0 n$$

which proves that $n$ divides $m$. $\square$

**Example 1.2.** Let $f(z) = z^3$ and $g(z) = z^6+z^2$. The plane curve $(f(z), g(z))$ is not an embedded line and we can not apply Abhyankar-Moh Theorem. Let $u(z) = z^2$ and $v(z) = z^5$. As $u(z), v(z) \in k[f(z), g(z)]$, we can apply Theorem 1.1. Therefore Theorem 1.1 is strictly stronger than Abhyankar-Moh Theorem.

**Theorem 1.3 (Criterion of Embedded Line).** *Let $k$ be a field of characteristic zero and $f(z)$ and $g(z)$ be polynomials. Then $k[f(z), g(z)] = k[z]$ if and only if $k[f(z), g(z)] \neq k$ and $f'(z)$ and $g'(z)$ are in the polynomial ring $k[f(z), g(z)]$.*

*Proof.* $\Longrightarrow$ Trivial.

$\Longleftarrow$ There are two cases.

Case 1: $f(z)$ or $g(z)$ is in the field $k$, say $f(z) \in k$ and $g(z) \notin k$. As $g'(z) \in k[f(z), g(z)] = k[g(z)]$, $g(z)$ is linear in $z$, therefore $k[f(z), g(z)] = k[z]$.

Case 2: Both $f(z)$ and $g(z)$ are not in $k$. As $\deg f'(z) = \deg f(z) - 1$ and $\deg g'(z) = \deg g(z) - 1$ , we can apply Theorem 1.1 to get the conclusion that either $\deg f(z)$ divides $\deg g(z)$ or $\deg g(z)$ divides $\deg f(z)$. We induct on the sum of $\deg f(z)$ and $\deg g(z)$. Let us say $\deg f(z) \geq \deg g(z)$. From Theorem 1.1, we can write $\deg f(z) = l \deg g(z)$, here $l$ is a positive integer. Let $a$ and $b$ be the leading coefficients of $f(z)$ and $g(z)$ respectively. We define $f_1(z) = f(z) - a(b^{-1}g(z))^l$ and $g_1(z) = g(z)$. It is obvious that either $f_1(z) = 0$ or $\deg f_1(z) < \deg f(z)$. It is easy to verify that we still have that $f_1(z)$ is in the polynomial ring $k[f(z), g(z)] = k[f_1(z), g_1(z)]$. We can continue to apply Theorem 1.1. This process must finish in finite steps, say $n$, and, say $f_n(z) \in k$ and $g_n(z) \in k[f(z), g(z)] = k[f_n(z), g_n(z)] = k[g_n(z)]$. We reduce to case 1, which is proved. $\square$

## 2. Applications

Using the Criterion of Embedded Line, we can give a new equivalence of plane Jacobian Conjecture.

**Proposition 2.1.** *Plane Jacobian Conjecture is equivalent to the following:*

*Let $k$ be a field of characteristic zero and $f(x, y)$ and $g(x, y)$ be polynomials over $k$. Assume that $f_x g_y - f_y g_x \in k^*$, then $f_y(x, y)$ and $g_y(x, y)$ are in polynomial ring $k(x)[f(x, y), g(x, y)]$.*

*Proof.* It is well known that Jacobian Conjecture is equivalent to $\deg f | \deg g$ or $\deg g | \deg f$ if $f_x g_y - f_y g_x \in k^*$ [4]. The proposition follows the Criteria of Embedded Line immediately. $\square$

With Criteria of Embedded Line, we can completely characterize one kind of plane curves.

**Proposition 2.2.** *Let $k$ be a field of characteristic zero and $f(z)$ and $g(z)$ be monic polynomials such that $\deg f = m$ , $\deg g = n$ and $\gcd(m, n) = d$. Suppose that*

$$nf'g - mfg' = a \in k^* \tag{2.2.1}$$

*and*

$$f^{n/d} - g^{m/d} = b \in k \tag{2.2.2}$$

*Then plane curve* $(f(z), g(z))$ *is an embedded line. Moreover, if* $m \leq n$ *then we have*

$$f(z) = z + c, g(z) = (z + c)^n - b, \tag{2.2.3}$$

*here* $c \in k$ *and* $b \in k^*$.

*Proof.* From (2.2.1), we have that $f$ and $g$ are coprime. Hence we have $b \neq 0$. From (2.2.2) we have

$$\frac{n}{d} f^{n/d-1} f' - \frac{m}{d} g^{m/d-1} g' = 0 \tag{2.2.4}$$

Multiplying (2.2.4) by $g$ and substituting (2.2.1), we have

$$a f^{n/d-1} + m g' (f^{n/d} - g^{m/d}) = 0 \tag{2.2.5}$$

Therefore

$$g' = -m^{-1} b^{-1} a f^{n/d-1} \tag{2.2.6}$$

Similarly we have

$$f' = -n^{-1} b^{-1} a g^{m/d-1} \tag{2.2.7}$$

By Criterion of Embedded Line, the plane curve $(f(z), g(z))$ is an embedded line. Hence $d = \min(m, n) = m$ and therefore $f' \in k^*$. As we suppose that $f$ is monic, therefore $f(z) = z + c$ for some $c \in k$ . From (2.2.2), we have $g(z) = (z+c)^n - b$. □

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