A localization of the Lévy operators arising in mathematical finances.

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1 Introduction

We study the uniform Hölder continuity of the solutions of the following problem.

$$F(x,\nabla v(x),\nabla^2 v(x)) - \int_{\mathbf{R^N}} [v(x+z)-v(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla v(x), z\rangle] c(z)dz - g(x) = 0 \qquad x\in \mathbf{R^N}, \tag{1}$$

where $c(z)dz$ is a positive Radon measure, called Lévy density, defined on $\mathbf{R^N}$ such that

$$\int_{\mathbf{R^N}} \min(|z|^2,1)c(z)dz \leq C_1, \tag{2}$$

$$\frac{C_2}{|z|^{N+\gamma}} \leq |c(z)| \leq \frac{C_3}{|z|^{N+\gamma}} \quad \forall z\in \mathbf{R^N}\cap\{|z|\leq 1\}, \tag{3}$$

where $\gamma\in(0,2)$, $C_i>0$ $(1\leq i\leq 3)$ are constants. We assume that there exists a "uniform" constant $M>1$ such that for a constant $\theta_0\in[0,1]$,

$$|g(x)-g(y)|\leq M|x-y|^{\theta_0} \quad \forall x,y\in\mathbf{R^N}, \tag{4}$$

and

$$\sup_{x\in\mathbf{R^N}} |v| < M. \tag{5}$$

The second-order fully nonlinear partial differential operator F is continuous in $\mathbf{R^N}\times\mathbf{R^N}\times\mathbf{S^N}$, and assumed to satisfy the following two conditions.
(Degenerate ellipticity) :

$$F(x,p,X) \geq F(x,p,Y) \quad \text{if} \quad X\leq Y,$$

$$\forall x\in\mathbf{R^N}, \quad \forall p\in\mathbf{R^N}, \quad \forall X,Y\in\mathbf{S^N}. \tag{6}$$

(Continuity I) : There are modulus of continuity functions w and η from $\mathbf{R}^+\cup\{0\}\to\mathbf{R}^+\cup\{0\}$ such that $\lim_{\sigma\downarrow 0} w(\sigma)=0$, $\lim_{\sigma\downarrow 0}\eta(\sigma)=0$, and

$$|F(x,p,X)-F(y,p,X)|\leq w(|x-y|)|p|^q + \eta(|x-y|)||X|| \tag{7}$$

$$\forall x,y\in\mathbf{R^N}, \quad \forall p\in\mathbf{R^N}, \quad \forall X\in\mathbf{S^N},$$

where $q\geq 1$.

We study this problem in the framework of the viscosity solutions for the integro-differential equations, the definition of which is introduced in Arisawa [5] (see also [6] and [7]). The definition is the following. In order to get rid of the singularity of the Lévy measure, we shall use the following superjet (resp. subjet) and its residue. Let $\hat{x} \in \mathbf{R^N}$, and let $(p, X) \in J^{2,+}_{\mathbf{R^N}} u(\hat{x})$ (resp. $(p, X) \in J^{2,-}_{\mathbf{R^N}} u(\hat{x})$) be a second-order superjet (resp. subjet) of u at $\hat{x}$. Then, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$u(\hat{x} + z) \leq u(\hat{x}) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + \delta|z|^2 \qquad \text{if} \quad |z| \leq \varepsilon \tag{8}$$

(resp.

$$v(\hat{x} + z) \geq v(\hat{x}) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle - \delta|z|^2 \qquad \text{if} \quad |z| \leq \varepsilon \tag{9}$$

) holds. We use this pair of numbers (ε, δ) satisfying (8) (resp. (9)) for any $(p, X) \in J^{2,+}_{\mathbf{R^N}} u(\hat{x})$ (resp. $(p, X) \in J^{2,-}_{\mathbf{R^N}} v(\hat{x})$) in the following definition of viscosity solutions.

Definition 1.1. *Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \mathbf{R^N}$, any $(p, X) \in J^{2,+}_{\mathbf{R^N}} u(\hat{x})$ (resp. $\in J^{2,-}_{\mathbf{R^N}} v(\hat{x})$), and any pair of numbers (ε, δ) satisfying (8) (resp.(9)), the following holds for any $0 < \varepsilon' \leq \varepsilon$*

$$F(\hat{x}, p, X) - \int_{|z|<\varepsilon'} \frac{1}{2}\langle (X + 2\delta I)z, z\rangle c(z)dz$$

$$- \int_{|z|\geq\varepsilon'} [u(\hat{x} + z) - u(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, p\rangle] c(z)dz \leq 0.$$

(resp.

$$F(\hat{x}, p, X) - \int_{|z|<\varepsilon'} \frac{1}{2}\langle (X - 2\delta I)z, z\rangle c(z)dz$$

$$- \int_{|z|\geq\varepsilon'} [v(\hat{x} + z) - v(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, p\rangle] c(z)dz \geq 0.$$

If u is both a viscosity subsolution and a viscosity supersolution , it is called a viscosity solution.

In the framework of the viscosity solutions in Definition 1.1, we have the existence and the comparison results in [5], [6] and [7]. For the convenience of the readres, we shall give typical comparison results and the proof in §2 in below.

Then, we claim the uniform Hölder continuity of u in the following two cases.
(I) $N = 1$.
(II) $N \geq 2$, and F satisfies the following uniform ellipticity.
(Uniform ellipticity) : There exists $\lambda_0 > 0$ such that

$$F(x,p,X) - F(x,p,Y) \geq \lambda_0(Y - X) \quad \text{if} \quad X \leq Y,$$
$$\forall x \in \mathbf{R^N}, \quad \forall p \in \mathbf{R^N}, \quad \forall X, Y \in \mathbf{S^N}. \tag{10}$$

In the case of (I), we claim that for any $\theta \in (0, \min\{1, \theta_0 + \gamma\})$ there exists $C_\theta > 0$ such that

$$|v(x) - v(y)| \leq C_\theta |x - y|^\theta \quad \forall x, y \in \mathbf{R^N}, \tag{11}$$

where $C_\theta > 0$ depends only on M and C_1. (See Theorem 3.1 in below.) In the case of (II), we claim that for any $\theta \in (0, 1)$, there exists $C_\theta > 0$ such that (11) holds. (See Theorem 3.2 in below.) (These results hold for more general problem

$$F(x, \nabla v(x), \nabla^2 v(x)) + \sup_{\alpha \in \mathcal{A}}\{-\int_{\mathbf{R^N}} [v(x+z) - v(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla v(x), z\rangle] c(x,z,\alpha) dz - g(x,\alpha)\} = 0 \qquad x \in \mathbf{R^N},$$

which we do not treat here.)

As for the case other than (I) and (II), that is $N \geq 2$ and F is not necessarily uniformly elliptic (i.e. (10) is not satisfied), we study the following two problems in the torus $\mathbf{T^N}$ instead of (1). The first one is, for $\lambda > 0$,

$$\lambda v(x) + H(\nabla v(x)) - \int_{\mathbf{R^N}} [v(x+z) - v(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla v(x), z\rangle] c(z) dz - g(x) = 0 \qquad x \in \mathbf{T^N}. \tag{12}$$

And the second one is

$$\lambda v(x) + F(x, \nabla v(x), \nabla^2 v(x)) - \int_{\mathbf{R^N}} [v(x+z) - v(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla v(x), z\rangle]c(z)dz - g(x) = 0 \qquad x \in \mathbf{T^N}, \tag{13}$$

where $\lambda > 0$. Here H is a first-order nonlinear operator, and F is a fully nonlinear degenerate elliptic operator, satisfying the following conditions. (Periodicity) :

$$H(\cdot, p), \quad F(\cdot, p, X), \quad \text{and} \quad g(\cdot) \quad \text{are periodic in} \quad x \in \mathbf{T^N},$$
$$\text{for} \quad \forall p \in \mathbf{R^N}, \quad \forall X \in \mathbf{S^N}. \tag{14}$$

(Partial uniform ellipticity) : There exists a constant $\lambda_1 > 0$ such that

$$F(x,p,X) \geq F(x,p,Y) + \lambda_1 Tr(Y' - X') \quad \forall x \in \mathbf{T^N}, \quad \forall p \in \mathbf{R}^N,$$
$$\forall X, Y \in \mathbf{S}^N, \quad X = \begin{pmatrix} X' & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \tag{15}$$

where $\quad X' \leq Y' (X', Y' \in \mathbf{S}^M), \quad 0 < M \leq N.$

(Continuity II) : There are modulus of continuity functions w' and η' from $\mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ such that $\lim_{\sigma \downarrow 0} w'(\sigma) = 0$, $\lim_{\sigma \downarrow 0} \eta'(\sigma) = 0$, and

$$|F(x,p,X) - F(y,p,X)| \leq w'(|x-y|)|p'|^{q'} + \eta'(|x-y|)||X'||$$
$$\forall x, y \in \mathbf{T^N}, \quad \forall p = (p', p'') \in \mathbf{R}^M \times \mathbf{R}^m, \quad \forall X = \begin{pmatrix} X' & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbf{S}^N,$$
$$\text{where} \quad X' \in \mathbf{S}^M, \quad M + m = N, \quad q' \geq 1. \tag{16}$$

Roughly speaking, we claim that for any $\theta \in (0, \theta_0)$ ($\theta_0 > 0$), there exists $C_\theta > 0$ such that

$$|v(x) - v(y)| \leq \frac{C_\theta}{\lambda}|x-y|^\theta \quad \forall x, y \in \mathbf{T^N}, \tag{17}$$

where $C_\theta > 0$ is independent on $\lambda > 0$. (See Theorems 4.1 and 4.2 in below.) The method to derive the above uniform Hölder continuity (11) and the Hölder continuity (17) is based on the argument used in the proof of the comparison result. (See Ishii and Lions [20], for the similar argument in the PDE case.)

Next, we shall state the strong maximum principle for the Lévy operator. In [18], for the second-order uniformly elliptic integro-differential operator

$$-\sum_{i,j=1}^{N}\overline{a}_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}-\sum_{i=1}^{N}\overline{b}_i\frac{\partial v}{\partial x_i}-\int_{\mathbf{R^N}}[v(x+z)-v(x)-\langle\nabla v(x),z\rangle]c(x,z)dz$$

$$x\in\mathbf{R^N}, \qquad (18)$$

the strong maximum principle was given, where $\lambda_0 I\leq(\overline{a}_{ij})_{1\leq i,j\leq N}\leq\Lambda_0 I$ $(0<\lambda_0\leq\Lambda_0)$. See also, Cancelier [13] for another type of the maximum principle. Here, we shall give the strong maximum principle in $\mathbf{R^N}$ without assuming the uniform ellipticity of the partial differential operator F in (1) (see Theorem 5.1 in below, and M. Arisawa and P.-L. Lions [9]).

Finally, we shall apply these regularity results (11), (17) and the strong maximum principle, to study the so-called ergodic problem. In the case of the Hamilton-Jacobi-Bellman (HJB) operator

$$\sup_{\alpha\in\mathcal{A}}\{-\sum_{i,j=1}^{N}a_{ij}(x,\alpha)\frac{\partial^2 u}{\partial x_i\partial x_j}-\sum_{i=1}^{N}b_i(x,\alpha)\frac{\partial u}{\partial x_i}-f(x,\alpha)\},$$

the ergodicity of the corresponding controlled diffusion process, for example in the torus $\mathbf{T^N}=\mathbf{R^N}\backslash\mathbf{Z^N}$, can be studied by the existence of a unique real number d_f such that the following problem admits a periodic viscosity solution u :

$$d_f+\sup_{\alpha\in\mathcal{A}}\{-\sum_{i,j=1}^{N}a_{ij}(x,\alpha)\frac{\partial^2 u}{\partial x_i\partial x_j}-\sum_{i=1}^{N}b_i(x,\alpha)\frac{\partial u}{\partial x_i}-f(x,\alpha)\}=0 \quad x\in\mathbf{T^N}.$$

We refer the readers to M. Arisawa and P.-L. Lions [8], M. Arisawa [2], [3], for more details. From the analogy of the diffusion case, here we shall formulate the ergodic problem for the integro-differential equations as follows.

(Ergodic problem) Is there a unique number d_f depending only on $f(x)$ such that the following problem has a periodic viscosity solution $u(x)$ defined on $\mathbf{T^N}$?

$$d_f+F(x,\nabla u,\nabla^2 u)-\int_{\mathbf{R^N}}[u(x+z)$$

$$-u(x)-\mathbf{1}_{|z|\leq 1}\langle\nabla u(x),z\rangle]c(z)dz-f(x)=0 \qquad x\in\mathbf{T^N}.$$

The results on the existence of the above number d_f is stated in Theorem 6.1 in below.

2 Comparison results

In this section, we give some typical comparison results for the integro-differential equations in the framework of the solution in Definition 1.1. We consider

$$\lambda u + F(x, \nabla u, \nabla^2 u) - \int_{\mathbf{R^N}} u(x+z) - u(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle z, \nabla u(x)\rangle q(dz) = 0 \qquad \text{in} \quad \Omega, \tag{19}$$

where $\lambda > 0$, and Ω is a bounded domain in $\mathbf{R^N}$, with either the Dirichlet B.C.:

$$u(x) = g(x) \quad \forall x \in \Omega^c, \tag{20}$$

or the Periodic B.C.:

$$\Omega = \mathbf{T^N} = \mathbf{R^N} \backslash \mathbf{Z^N}, \quad u(x) \quad \text{is periodic in} \quad \mathbf{T^N}, \tag{21}$$

where g is a given continuous function in Ω^c. The second-order partial defferential operator F is degenerate elliptic, which satisfies

(Degenerate ellipticity) (cf. [16] (3.14)): There exists a function $w(\cdot)$: $[0,\infty) \to [0,\infty)$, $w(0+) = 0$ such that

$$F(y, r, p, Y) - F(x, r, p, X) \leq w(\alpha|x-y|^2 + |x-y|(|p|+1)) \tag{22}$$
$$\text{for} \quad x, y \in \overline{\Omega}, \quad r \in \mathbf{R}, \quad p \in \mathbf{R^N}$$

for any $\alpha > 0$, and for any $X, Y \in \mathbf{S^N}$ such that

$$-3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{23}$$

For the above, we have the following results.

Theorem 2.1

Assume that Ω *is bounded, and that (2), (22) hold. Let* $u \in USC(\mathbf{R^N})$ *and* $v \in LSC(\mathbf{R^N})$ *be respectively a viscosity subsolution and a supersolution of (19) in* Ω*, which satisfy* $u \leq v$ *on* Ω^c*. Then,*

$$u \leq v \qquad in \quad \Omega.$$

Theorem 2.2

Let $\Omega = \mathbf{T^N}$*. Assume that (2), (22) hold and that* F *is periodic in* $x \in \mathbf{T^N}$*. Let* $u \in USC(\mathbf{T^N})$ *and* $v \in LSC(\mathbf{T^N})$ *be respectively a viscosity subsolution and a supersolution of (19) in* Ω*. Then,*

$$u \leq v \qquad in \quad \Omega.$$

Remark 2.1 The above comparison results hold in more general situations. For example, Ω can be $\mathbf{R^N}$ by assuming that u and v are bounded, or the nonlocal operator can be in the form of

$$-\int_{\{z\in\mathbf{R^N}|x+z\in\overline{\Omega}\}}[u(x+z)-u(x)-\mathbf{1}_{|z|\leq 1}\langle z,\nabla u(x)\rangle]c(z)dz, \quad x\in\mathbf{\Omega},$$

with the Neumann type boundary condition on $\partial\Omega$, etc... We refer the readers to [5], [6] and [7].

In order to prove the above claims, we use the following two Lemmas. (See [7].) The first Lemma is the approximation by the supconvolution and the infconvolution.

Lemma 2.3

Let u *and* v *be respectively a bounded viscosity subsolution and a bounded supersolution of (19). Define for* $r > 0$*, the supconvolution* u^r *and the infconvolution* v_r *of* u *and* v *as follows.*

$$u^r(x) = \sup_{y\in\mathbf{R^N}}\{u(y) - \frac{1}{2r^2}|x-y|^2\} \qquad (supconvolution). \tag{24}$$

$$v_r(x) = \inf_{y\in\mathbf{R^N}}\{v(y) + \frac{1}{2r^2}|x-y|^2\} \qquad (\textit{infconvolution}). \tag{25}$$

Then, for any $\nu > 0$ there exists $r > 0$ such that u^r and v_r are respectively a subsolution and a supersolution of the following problems.

$$\lambda u_r + F(x, \nabla u^r, \nabla^2 u^r) - \int_{\mathbf{R^N}} u^r(x+z) - u^r(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle z, \nabla u^r(x)\rangle q(dz)\}\leq\nu \qquad in \quad \Omega_r. \tag{26}$$
$$\lambda v_r + F(x, \nabla v_r, \nabla^2 v_r) - \int_{\mathbf{R^N}} v_r(x+z) - v_r(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle z, \nabla v_r(x)\rangle q(dz)\} \geq -\nu \qquad in \quad \Omega_r, \tag{27}$$

where $\Omega_r = \{x \in \Omega| \quad dist(x, \partial\Omega) > \sqrt{2Mr}\}$ *for* $M = \max\{\sup_{\overline{\Omega}}|u|, \sup_{\overline{\Omega}}|v|\}$.

Remark that u^r is semiconvex, v_r is semiconcave, and both are Lipschitz continuous in $\mathbf{R^N}$. The second lemma comes from the Jensen's maximum principle and the Alexandrov's theorem (see [16] and [17]). The last claim of this lemma is quite important in the limit procedure in the nonlocal term.

Lemma 2.4 *Let U be semiconvex and V be semiconcave in Ω. For $\phi(x,y) = \alpha|x-y|^2$ ($\alpha > 0$) consider $\Phi(x,y) = U(x) - V(y) - \phi(x,y)$, and assume that $(\overline{x}, \overline{y})$ is an interior maximum of Φ in $\overline{\Omega}\times\overline{\Omega}$. Assume also that there is an open precompact subset O of $\Omega\times\Omega$ containing $(\overline{x}, \overline{y})$, and that $\mu = \sup_O \Phi(x,y) - \sup_{\partial O}\Phi(x,y) > 0$. Then, the following holds.*
(i) There exists a sequence of points $(x_m, y_m) \in O$ ($m \in \mathbf{N}$) such that $\lim_{m\to\infty}(x_m, y_m) = (\overline{x}, \overline{y})$, and $(p_m, X_m) \in J^{2,+}_{\Omega}U(x_m)$, $(p'_m, Y_m) \in J^{2,-}_{\Omega}V(y_m)$ such that $\lim_{m\to\infty} p_m = \lim_{m\to\infty} p'_m = 2\alpha(x_m - y_m) = p$, and $X_m \leq Y_m \quad \forall m$.
(ii) For $P_m = (p_m - p, -(p'_m - p))$, $\Phi_m(x,y) = \Phi(x,y) - \langle P_m, (x,y)\rangle$ takes a maximum at (x_m, y_m) in O.
(iii) The following holds for any $z \in \mathbf{R^N}$ such that $(x_m + z, y_m + z) \in O$.

$$U(x_m+z) - U(_m) - \langle p_m, z\rangle\leq V(y_m+z) - V(y_m) - \langle p'_m, z\rangle. \tag{28}$$

We admit the above claims here. (In fact, the proofs of Lemma 2.3 and 2.4 are not so difficult, see for example [16] and [17].)

Proof of Theorem 2.1. We use the argument by contradiction, and assume that $\max_{\overline{\Omega}}(u-v)=(u-v)(x_0)=M_0>0$ for $x_0\in\Omega$. Then, we approximate u by u^r (supconvolution) and v by v_r (infconvolution), which are a subsolution and a supersolution of (26) and (27), respectively. Clearly, $\max_{\overline{\Omega}}(u^r-v_r)\geq M_0>0$. Let $\overline{x}\in\Omega$ be the maximizer of u^r-v_r. In the following, we abbreviate the index and write $u=u^r$, $v=v_r$ without any confusion. As in the PDE theory, consider $\Phi(x,y)=u(x)-v(y)-\alpha|x-y|^2$, and let $(\hat{x},\hat{y})$ be the maximizer of Φ. Then, from Lemma 2.3 there exists $(x_m,y_m)\in\Omega$ ($m\in\mathbf{N}$) such that $\lim_{m\to\infty}(x_m,y_m)=(\hat{x},\hat{y})$, and we can take (ε_m,δ_m) a pair of positive numbers such that $u(x_m+z)\leq u(x_m)+\langle p_m,z\rangle+\frac{1}{2}\langle X_mz,z\rangle+\delta_m|z|^2$, $v(y_m+z)\geq v(y_m)+\langle p'_m,z\rangle+\frac{1}{2}\langle Y_mz,z\rangle-\delta_m|z|^2$, for $\forall|z|\leq\varepsilon_m$. From the definition of the viscosity solutions, we have

$$F(x_m,u(x_m),p_m,X_m)-\int_{|z|\leq\varepsilon_m}\frac{1}{2}\langle(X_m+2\delta_mI)z,z\rangle dq(z)$$

$$-\int_{|z|\geq\varepsilon_m}u(x_m+z)-u(x_m)-\mathbf{1}_{|z|\leq 1}\langle z,p_m\rangle q(dz)\leq\nu,$$

$$F(y_m,v(y_m),p'_m,Y_m)-\int_{|z|\leq\varepsilon_m}\frac{1}{2}\langle(Y_m-2\delta_mI)z,z\rangle dq(z)$$

$$-\int_{|z|\geq\varepsilon_m}v(y_m+z)-v(y_m)-\mathbf{1}_{|z|\leq 1}\langle z,p'_m\rangle q(dz)\geq-\nu.$$

By taking the difference of the above two inequalities, by using (28), and by passing $m\to\infty$ (thanking to (28), it is available), we can obtain the desired contradiction. The claim $u\leq v$ is proved.

Remark 2.2 As for the usage of (28) in the limit procedure $m\to\infty$ in the proof of Theorem 2.1, we refer the interested readers to the similar argument in the proof of Theorem 3.2 in below.

3 Uniform Hölder continuities of viscosity solutions

In this section, we study the uniform Hölder continuities of viscosity solutions of (1) in the cases of (I) and (II).

Theorem 3.1.

Let $N = 1$, and let v be a viscosity solution of (1) satisfying (5). Assume that (2), (3) and (4) hold, where $\gamma \in (0, 2)$. Assume also that F satisfies (6) and (7), where there exist constants $L > 0$, $\rho_i > 0$ $(i = 1, 2)$ such that

$$\lim_{s \downarrow 0} w(s)s^{-\rho_1} \le L, \quad \lim_{s \downarrow 0} \eta(s)s^{-\rho_2} \le L, \tag{29}$$

and $\rho_1 + \gamma > q$, $\rho_2 + \gamma > 2$. Then for any $\theta \in (0, \min\{1, \theta_0 + \gamma\})$, there exists a constant $C_\theta > 0$ such that (11) holds. The constant C_θ depends only on $M > 0$ and C_i $(1 \le i \le 3)$.

Theorem 3.2.

Let $N \geq 2$, and let v be a viscosity solution of (1) satisfying (5). Assume that (2), (3) and (4) hold, where $\gamma \in (0, 2)$.

If F satisfies (7) and (10), where there exist constants $L > 0$, $\rho_1 > 0$ such that

$$\lim_{s \downarrow 0} w(s)s^{-\rho_1} \le L, \tag{30}$$

and $\rho_1 + 2 > q$, then for any $\theta \in (0, 1)$, there exists a constant $C_\theta > 0$ such that (11) holds. The constant C_θ depends only on $M > 0$ and C_i $(1 \le i \le 3)$.

The following lemma gives the relationship between δ and ε in Definition 1.1, and is used in the proofs in below.

Lemma 3.3.

Let $\phi(z) = C_\theta |z|^\theta$ $(z \in \mathbf{R^N})$, $\theta \in (0, 1)$, $r > 0$, and let $\hat{z} \in \{z \in \mathbf{R^N} | \quad |z| < r, \quad z \neq 0\}$ be fixed. Then, there exists $\overline{C} > 0$ such that for any $\delta > 0$, and for any $z \in \mathbf{R^N}$ such that $|z| \le \frac{|\hat{z}|}{2}$, if z satisfies

$$|z| \le \delta \overline{C} |\hat{z}|^{3-\theta}, \tag{31}$$

we have

$$|\phi(\hat{z} + z) - \phi(\hat{z}) - \langle \nabla\phi(\hat{z}), z\rangle - \frac{1}{2}\langle \nabla^2\phi(\hat{z})z, z\rangle| \le \delta |z|^2. \tag{32}$$

The constant $\overline{C}$ is independent on r, θ, and $\hat{z}$.

Proof of Lemma 3.3. From the Taylor expansion of ϕ at $\hat{z}$

$$\phi(\hat{z}+z)-\phi(\hat{z})-\langle\nabla\phi(\hat{z}),z\rangle-\frac{1}{2}\langle\nabla^2\phi(\hat{z})z,z\rangle=\frac{1}{3!}\sum_{i,j,k=1}^{N}\frac{\partial^3\phi(\hat{z}+\rho(z)z)}{\partial z_i\partial z_j\partial z_k}z_iz_jz_k$$

$$\text{for}\quad z\in\{z\in\mathbf{R^N}|\quad |z|<\frac{|\hat{z}|}{2}\},$$

where $\rho=\rho(z)\in(0,1)$. By calculating $\frac{\partial^3\phi}{\partial x_i\partial x_j\partial x_k}$, we see that there exists a constant $C>0$ independent on r, θ, δ such that

$$|\phi(\hat{z}+z)-\phi(\hat{z})-\langle\nabla\phi(\hat{z}),z\rangle-\frac{1}{2}\langle\nabla^2\phi(\hat{z})z,z\rangle|\leq C|\hat{z}+\rho z|^{\theta-3}|z|^3$$

$$\text{for}\quad z\in\{z\in\mathbf{R^N}|\quad |z|<\frac{|\hat{z}|}{2}\}.$$

Then, if $|z|\leq\frac{\delta}{C}|\hat{z}+\rho z|^{3-\theta}$

$$C|z|^3|\hat{z}+\rho z|^{\theta-3}\leq\delta|z|^2. \tag{33}$$

Since for $|z|\leq\frac{|\hat{z}|}{2}$,

$$\frac{1}{2}|\hat{z}|\leq|\hat{z}+\rho z|\leq 2|\hat{z}|,$$

there exists $\overline{C}>0$ independent on r, θ, δ, and ρ such that, if

$$|z|\leq\delta\overline{C}|\hat{z}|^{3-\theta}\leq\frac{\delta}{C}|\hat{z}+\rho z|^{3-\theta}, \tag{31$'$}$$

then (33) holds. Therefore, if z satisfies (31) with the above $\overline{C}>0$, and if $|z|<\frac{|\hat{z}|}{2}$, the inequality (32) holds.

Proof of Theorem 3.1. Fix an arbitrary number $\theta\in(0,1)$. Let $r_0>0$ be a small enough number which will be determined in the end of the proof. For $C_\theta>0$ such that

$$C_\theta r_0^\theta=2M, \tag{34}$$

we shall prove (11), by the contradiction's argument. For $x,y\in\mathbf{R^N}$ such that $|x-y|\geq r_0$, from (5) we have

$$|v(x)-v(y)|\leq 2M\leq C_\theta|x-y|^\theta.$$

Assume that there exist $x', y' \in \mathbf{R^N}$ ($|x'-y'| < r_0$) such that

$$|v(x') - v(y')| > C_\theta |x'-y'|^\theta,$$

and we shall look for a contradiction. Consider for $\tau \in (0,1)$

$$\Phi(x,y) = v(x) - v(y) - C_\theta |x-y|^\theta - \frac{\tau}{2}|x|^2,$$

and let $(\hat{x}, \hat{y})$ be a maximum point of Φ. Let us write $\phi(x,y) = C_\theta |x-y|^\theta$, and calculate

$$\nabla_x \phi(x,y) = C_\theta \theta |x-y|^{\theta-2}(x-y) = -\nabla_y \phi(x,y)$$

$$\nabla^2_{xx}\phi(x,y) = C_\theta\theta|x-y|^{\theta-2}I + C_\theta\theta(\theta-2)|x-y|^{\theta-4}(x-y)\otimes(x-y) = \nabla^2_{yy}\phi(x,y).$$

Put $p = \nabla_x \phi(\hat{x},\hat{y}) = -\nabla_y \phi(\hat{x},\hat{y})$, and $Q = \nabla^2_{xx}\phi(\hat{x},\hat{y}) = \nabla^2_{yy}\phi(\hat{x},\hat{y})$. Since

$$\Phi(\hat{x}+z,\hat{y}) = v(\hat{x}+z) - v(\hat{y}) - C_\theta|\hat{x}+z-\hat{y}|^\theta - \frac{\tau}{2}(\hat{x}+z)^2$$

$$\le \Phi(\hat{x},\hat{y}) = v(\hat{x}) - v(\hat{y}) - C_\theta|\hat{x}-\hat{y}|^\theta - \frac{\tau}{2}\hat{x}^2,$$

for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$v(\hat{x}+z) - v(\hat{x}) \le C_\theta|\hat{x}+z-\hat{y}|^\theta - C_\theta|\hat{x}-\hat{y}|^\theta + \frac{\tau}{2}(\hat{x}+z)^2 - \frac{\tau}{2}\hat{x}^2 \tag{35}$$

$$\le (p+\tau\hat{x})z + \frac{1}{2}(Q+\tau)z^2 + \delta z^2 \quad \text{for} \quad |z| < \varepsilon.$$

Samely, since

$$\Phi(\hat{x},\hat{y}+z) = v(\hat{x}) - v(\hat{y}+z) - C_\theta|\hat{x}-(\hat{y}+z)|^\theta - \frac{\tau}{2}\hat{x}^2$$

$$\le \Phi(\hat{x},\hat{y}) = v(\hat{x}) - v(\hat{y}) - C_\theta|\hat{x}-\hat{y}|^\theta - \frac{\tau}{2}\hat{x}^2,$$

for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$v(\hat{y}+z) - v(\hat{y}) \ge -(C_\theta|\hat{x}-(\hat{y}+z)|^\theta - C_\theta|\hat{x}-\hat{y}|^\theta) \tag{36}$$

$$\ge -(-pz + \frac{1}{2}Qz^2) - \delta z^2 = pz + \frac{1}{2}(-Q)z^2 - \delta z^2 \quad \text{for} \quad |z| < \varepsilon.$$

From the definition of viscosity solutions, by using the pair of numbers (ε, δ) in (35) and (36), we have

$$F(\hat{x}, p+\tau\hat{x}, Q+\tau) - \int_{|z|\leq\varepsilon} \frac{1}{2}(Q+\tau+2\delta)z^2 c(z)dz$$

$$- \int_{|z|\geq\varepsilon} [v(\hat{x}+z) - v(\hat{x}) - \mathbf{1_{|z|\leq 1}}(p+\tau\hat{x})z]c(z)dz - g(\hat{x}) \leq 0,$$

and

$$F(\hat{y}, p, -Q) - \int_{|z|\leq\varepsilon} \frac{1}{2}(-Q-2\delta)z^2 c(z)dz$$

$$- \int_{|z|\geq\varepsilon} [v(\hat{y}+z) - v(\hat{y}) - \mathbf{1_{|z|\leq 1}}pz]c(z)dz - g(\hat{y}) \geq 0.$$

By taking the difference of the above two inequalities, we have the following.

$$\begin{aligned} &F(\hat{x}, p+\tau\hat{x}, Q+\tau) - F(\hat{y}, p, -Q) \\ &\quad -\frac{1}{2}\int_{|z|\leq\varepsilon} (Q+\tau+2\delta)z^2 c(z)dz - \frac{1}{2}\int_{|z|\leq\varepsilon} (Q+2\delta)z^2 c(z)dz \\ &- \int_{|z|\geq\varepsilon} [v(\hat{x}+z) - v(\hat{x}) - \mathbf{1_{|z|\leq 1}}(p+\tau\hat{x})z]c(z)dz \\ &\quad + \int_{|z|\geq\varepsilon} [v(\hat{y}+z) - v(\hat{y}) - \mathbf{1_{|z|\leq 1}}pz]c(z)dz \leq g(\hat{x}) - g(\hat{y}) + \nu. \end{aligned} \tag{37}$$

We need the estimates.

Lemma 3.4.

The inequalities (35) and (36) hold with

$$(\varepsilon, \delta) = (\frac{|\hat{x}-\hat{y}|}{4}, \frac{\overline{C}^{-1}}{4}|\hat{x}-\hat{y}|^{\theta-2}). \tag{38}$$

With this pair of numbers, by taking $\tau > 0$ small enough, there exists a constant $C > M$ such that the following inequalities hold.
(a)

$$F(\hat{x}, p+\tau\hat{x}, Q+\tau) - F(\hat{y}, p, -Q) \geq -C(|\hat{x}-\hat{y}|^{\rho_1}|p|^q + |\hat{x}-\hat{y}|^{\rho_2}||Q||). \tag{39}$$

(b)

$$\int_{|z|\geq\varepsilon}[v(\hat{x}+z)-v(\hat{x})-\mathbf{1}_{|\mathbf{z}|\leq\mathbf{1}}(p+\tau\hat{x})z]c(z)dz-\int_{|z|\geq\varepsilon}[v(\hat{y}+z)-v(\hat{y})$$

$$-\mathbf{1}_{|\mathbf{z}|\leq\mathbf{1}}pz]c(z)dz\leq C\tau^{\frac{1}{2}}|\hat{x}-\hat{y}|^{-\gamma}. \quad (40)$$

Proof of Lemma 3.4. By putting $\hat{z}=\hat{x}-\hat{y}$ in Lemma 3.3, for $\delta=\frac{\overline{C}^{-1}}{4}|\hat{x}-\hat{y}|^{\theta-2}$, we can take

$$\varepsilon=\min\{\delta\overline{C}|\hat{x}-\hat{y}|^{3-\theta},\frac{1}{2}|\hat{x}-\hat{y}|\}=\frac{1}{4}|\hat{x}-\hat{y}|,$$

so that (35), (36) hold.
(a) From the continuity of F, (6), and (7), since $Q\leq O$, for $r_0>0$ ($|\hat{x}-\hat{y}|<r_0$) small enough,

$$F(\hat{x},p+\tau\hat{x},Q+\tau)-F(\hat{y},p,-Q)$$

$$=F(\hat{x},p,Q)-F(\hat{x},p,-Q)+F(\hat{x},p,-Q)-F(\hat{y},p,-Q)+o(\tau)$$

$$\geq -w(\hat{x}-\hat{y})|p|^q-\eta(\hat{x}-\hat{y})||Q||+o(\tau)\geq -C(|\hat{x}-\hat{y}|^{\rho_1}|p|^q+|\hat{x}-\hat{y}|^{\rho_2}||Q||), \quad (41)$$

where $C>M$ is a constant.
(b) Since $\Phi(\hat{x},\hat{y})=v(\hat{x})-v(\hat{y})-C_\theta|\hat{x}-\hat{y}|^\theta-\frac{\tau}{2}\hat{x}^2\geq\Phi(0,0)=0$, from (5),

$$\frac{\tau}{2}\hat{x}^2\leq 2M. \quad (42)$$

Thus, for $\tau\in(0,1)$

$$v(\hat{x}+z)-v(\hat{y}+z)-(v(\hat{x})-v(\hat{y}))\leq\frac{\tau}{2}(\hat{x}+z)^2-\frac{\tau}{2}\hat{x}^2\leq\tau^{\frac{1}{2}}(2M|z|+|z|^2),$$

and from this

$$\int_{|z|\geq\varepsilon}[v(\hat{x}+z)-v(\hat{x})-\mathbf{1}_{|z|\leq 1}(p+\tau\hat{x})z]c(z)dz-\int_{|z|\geq\varepsilon}[v(\hat{y}+z)-v(\hat{y})$$

$$-\mathbf{1}_{|z|\leq 1}pz]c(z)dz\leq\int_{|z|\geq\varepsilon}\tau^{\frac{1}{2}}(3M|z|+z^2)c(z)dz \quad (43)$$

From (3) and (38)

$$\int_{|z|\geq\varepsilon}\tau^{\frac{1}{2}}(3M|z|+z^2)c(z)dz\leq\tau^{\frac{1}{2}}C\max\{1,|\hat{x}-\hat{y}|^{1-\gamma}\}\leq C\tau^{\frac{1}{2}}|\hat{x}-\hat{y}|^{-\gamma}, \quad (44)$$

where $C>M$ is the constant. By plugging (44) into (43), we get (40).

We put the estimates (39)-(40) in (37), and since $\nu>0$ can be taken arbitrarily small,

$$C^{-1}C_\theta|\hat{x}-\hat{y}|^{\theta-\gamma}\leq C(|\hat{x}-\hat{y}|^{\rho_1}|p|^q+|\hat{x}-\hat{y}|^{\rho_2}||Q|| \\ +2\tau^{\frac{1}{2}}|\hat{x}-\hat{y}|^{-\gamma}+M|\hat{x}-\hat{y}|^{\theta_0}). \quad (45)$$

From (34), $\theta\in(0,\min\{1,\theta_0+\gamma\})$, $\rho_1+\gamma>q$, $\rho_2+\gamma>2$, and since we can take $\tau\in(0,1)$ arbitrarily small, for $r_0>0$ ($|\hat{x}-\hat{y}|<r_0$) small enough, we get a contradiction. Thus, the claim in Theorem 3.1 is proved.

Proof of Theorem 3.2. We use the similar contradiction argument as in the proof of Theorem 3.1. For an arbitrary fixed number $\theta\in(0,1)$, and for $r_0>0$ small enough, let $C_\theta>0$ be such that (34):

$$C_\theta r^\theta=2M,$$

and we shall prove (11) by the contradiction's argument. As before, assume that there exist $x',y'\in\mathbf{R^N}$ ($|x'-y'|<r_0$) such that

$$v(x')-v(y')>C_\theta|x'-y'|^\theta,$$

and we shall look for a contradiction. However, we must modify the preceding argument, because for $N\geq 2$ the matrix $Q=\nabla^2_{xx}\phi(\hat{x},\hat{y})=\nabla^2_{yy}\phi(\hat{x},\hat{y})$ (ϕ is the function in the proof of Theorem 3.1) is no longer negatively definite, and we have to use Lemma 2.4. For this reason, let us consider the supconvolution v^r and the infconvolution v_r of v defined by (24) and (25), respectively. From Lemma 2.3, for any $\nu>0$ there exists $r_1>0$ such that v^r ($\forall r\in(0,r_1)$) is a subsolution of

$$F(x,\nabla v^r(x),\nabla^2v^r(x))-\int_{\mathbf{R^N}}[v^r(x+z)-v^r(x) \\ -\mathbf{1}_{|z|\leq 1}\langle\nabla v^r(x),z\rangle]c(z)dz-g(x)\leq\nu \qquad x\in\mathbf{R^N}, \quad (46)$$

and v_r $(\forall r \in (0, r_1))$ is a supersolution of

$$F(x, \nabla v_r(x), \nabla^2 v_r(x)) - \int_{\mathbf{R^N}} [v_r(x+z) - v_r(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla v_r(x), z\rangle] c(z)dz - g(x) \geq -\nu \qquad x \in \mathbf{R^N}. \tag{47}$$

Of course from the preceding assumption, for $\forall r \in (0, r_0)$

$$v^r(x') - v_r(y') > C_\theta |x' - y'|^\theta.$$

Now, consider for $\tau \in (0, 1)$

$$\Phi(x, y) = v^r(x) - v_r(y) - C_\theta |x - y|^\theta - \frac{\tau}{2}|x|^2,$$

and let $(\hat{x}, \hat{y})$ be a maximum point of Φ. Put $p = \nabla_x \phi(\hat{x}, \hat{y}) = -\nabla_y \phi(\hat{x}, \hat{y})$, and $Q = \nabla^2_{xx}\phi(\hat{x}, \hat{y}) = \nabla^2_{yy}\phi(\hat{x}, \hat{y})$. We use Lemma 2.4 for $U = v^r - \frac{\tau}{2}|x|^2$, and $V = v_r$, and for $O = \{(x, y) \in \mathbf{R^{2N}} | |x - y| < r_0\}$, and we know that there exists $(x_m, y_m) \in \mathbf{R^{2N}}$ such that $\lim_{m\to\infty}(x_m, y_m) = (\hat{x}, \hat{y})$. There also exist $(p_m + \tau x_m, X_m + \tau I) \in J^{2,+}_{\mathbf{R^N}} v^r(x_m)$, $(p'_m, Y_m) \in J^{2,-}_{\mathbf{R^N}} v_r(y_m)$ such that $\lim_{m\to\infty} p_m = \lim_{m\to\infty} p'_m = 2\alpha(x_m - y_m) = p$, and $X_m \leq Y_m \quad \forall m$. Moreover, the claim in Lemma 2.4 (iii) leads the following for any $z \in \mathbf{R^N}$ such that $(x_m + z, y_m + z) \in O$

$$v^r(x_m + z) - v^r(x_m) - \langle p_m, z\rangle - \{v_r(y_m + z) - v_r(y_m) - \langle p'_m, z\rangle\} \tag{48}$$

$$\leq \frac{\tau}{2}|x_m + z|^2 - \frac{\tau}{2}|x_m|^2 = \frac{\tau}{2}\{2\langle x_m, z\rangle + |z|^2\}.$$

Let $(\varepsilon_m, \delta_m)$ be a pair of positive numbers such that

$$v^r(x_m+z) \leq v^r(x_m) + \langle (p_m + \tau x_m), z\rangle + \frac{1}{2}\langle (X_m + \tau I)z, z\rangle + \delta_m |z|^2 \qquad \text{if} \quad |z| \leq \varepsilon_m, \tag{49}$$

and

$$v_r(y_m+z) \geq v_r(y_m) + \langle p'_m, z\rangle + \frac{1}{2}\langle Y_m z, z\rangle - \delta_m |z|^2 \qquad \text{if} \quad |z| \leq \varepsilon_m. \tag{50}$$

Then, from the definition of viscosity solutions, we have

$$F(x_m, p_m + \tau x_m, X_m + \tau I) - \int_{|z|<\varepsilon_m} \frac{1}{2}\langle (X_m + (\tau + 2\delta_m)I)z, z\rangle c(z)dz$$

$$-\int_{|z|\geq\varepsilon_m}[v^r(x_m+z)-v^r(x_m)-\mathbf{1}_{|z|\leq 1}\langle(p_m+\tau x_m),z\rangle]c(z)dz-g(x_m)\leq\nu,$$

and

$$F(y_m,p_m,Y_m)-\int_{|z|<\varepsilon_m}\frac{1}{2}\langle(Y_m-2\delta_m I)z,z\rangle c(z)dz$$

$$-\int_{|z|\geq\varepsilon_m}[v_r(y_m+z)-v_r(y_m)-\mathbf{1}_{|z|\leq 1}\langle p'_m,z\rangle]c(z)dz-g(y_m)\geq-\nu.$$

By taking the difference of the two inequalities,

$$F(x_m,p_m+\tau x_m,X_m+\tau I)-F(y_m,p_m,Y_m)$$

$$-\frac{1}{2}\int_{|z|\leq\varepsilon_m}\langle(X_m-Y_m+(\tau+4\delta_m)I)z,z\rangle c(z)dz$$

$$\leq 2\nu+\int_{|z|\geq\varepsilon_m}[v^r(x_m+z)-v^r(x_m)-\mathbf{1}_{|z|\leq 1}\langle p_m+\tau x_m,z\rangle]c(z)dz$$

$$-\int_{|z|\geq\varepsilon_m}[v_r(y_m+z)-v_r(y_m)-\mathbf{1}_{|z|\leq 1}\langle p'_m,z\rangle]c(z)dz+g(x_m)-g(y_m)$$

$$\leq 2\nu+\int_{|z|\geq\varepsilon_m\cap O_m(z)}\frac{\tau}{2}|z|^2c(z)dz\int_{|z|\geq\varepsilon_m\cap O_m(z)^c}[\{v^r(x_m+z)-v^r(x_m)$$

$$-\mathbf{1}_{|z|\leq 1}\langle p_m+\tau x_m,z\rangle\}-\{v_r(y_m+z)-v_r(y_m)-\mathbf{1}_{|z|\leq 1}\langle p'_m,z\rangle\}],$$

where

$$O_m(z)=\{z\in\mathbf{R}^\mathbf{N}|\quad(x_m+z,y_m+z)\in\Omega\times\Omega\}.$$

Since $\lim_{m\to\infty}(x_m,y_m)=(\hat{x},\hat{y})\in\Omega\times\Omega$, there exists a ball $B(0)\subset\mathbf{R}^\mathbf{N}$, centered at the origin, independent on $m\in\mathbf{N}$, such that $B(0)\subset O(z)=\lim_{m\to\infty}O_m(z)$, i.e.

$$(x_m+z,y_m+z)\in\Omega\times\Omega\quad\forall z\in B(0),\quad\forall m\in\mathbf{N}. \tag{51}$$

Then, by pussing $m\to\infty$ in the above inequality, we get

$$F(\hat{x},p+\tau\hat{x},X+\tau I)-F(\hat{y},p,Y)$$

$$\leq 2\nu+\int_{O(z)}\frac{\tau}{2}|z|^2c(z)dz+\int_{O(z)^c}[\{v^r(\hat{x}+z)-v^r(\hat{x})$$

$$-\mathbf{1}_{|z|\leq 1}\langle p+\tau\hat{x},z\rangle\}-\{v_r(\hat{y}+z)-v_r(\hat{y})-\mathbf{1}_{|z|\leq 1}\langle p,z\rangle\}]c(z)dz.$$

$$\leq C(\nu+M)+\int_{\mathbf{R}^\mathbf{N}}\frac{\tau}{2}|z|^2c(z)dz+\int_{\{|z|<1\}\cap O(z)^c}\tau|\hat{x}||z|c(z)dz,$$

where $C > 0$ is a constant, and we have used the fact that $(\hat{x}, \hat{y})$ is the maximizer of Φ. From (2) and (34), and since $O(z)^c \subset B(0)^c$, for $0 < \tau < 1$,

$$F(\hat{x}, p + \tau\hat{x}, X + \tau I) - F(\hat{y}, p, Y) \leq C(\nu + M + \tau^{\frac{1}{2}}), \tag{52}$$

where $C > 0$ is a constant. We shall give the estimate of the left-hand side of the above.

Lemma 3.5.

There exists a constant $C > M$ such that the following holds.

$$F(\hat{x}, p + \tau\hat{x}, X + \tau I) - F(\hat{y}, p, -Y) \geq \frac{C_\theta}{C}|\hat{x} - \hat{y}|^{\theta-2} + o(\tau). \tag{53}$$

Proof of Lemma 3.5. From the continuity of F, (7), (10), and (42) (which is also true for $N \geq 2$),

$$F(\hat{x}, p+\tau\hat{x}, X+\tau I) - F(\hat{y}, p, -Y) = F(\hat{x}, p, \overline{X}) - F(\hat{y}, p, -\overline{Y}) + o(\tau) + o(\nu')$$

$$= F(\hat{x}, p, \overline{X}) - F(\hat{x}, p, -\overline{Y}) + F(\hat{x}, p, -\overline{Y}) - F(\hat{y}, p, -\overline{Y}) + o(\tau) + o(\nu')$$

$$\geq -\lambda_0 \mathrm{Tr}(\overline{X} + \overline{Y}) - w(|\hat{x} - \hat{y}|)|p|^q - \eta(|\hat{x} - \hat{y}|)||\overline{Y}|| + o(\tau) + o(\nu'). \tag{54}$$

We need the following lemma, the proof of which is delayed in the end.

Lemma 3.6.

If A, B, and $Q \in \mathbf{S^N}$ satisfy

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \leq \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix}, \tag{55}$$

then there exists a constant $L > 0$ such that

$$||A||, \quad ||B|| \quad \leq L||Q||^{\frac{1}{2}}|Tr(A + B)|^{\frac{1}{2}}.$$

The constant L depends only on N.

Remark that

$$-Tr(X - Y) \geq C_\theta\theta(1 - \theta)|\hat{x} - \hat{y}|^{\theta-2} > 0, \tag{56}$$

because , $X-Y\leq 2Q$, $X-Y\leq O$, and for $O\leq P=\frac{(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})}{|\hat{x}-\hat{y}|^2}\leq I$,

$$\mathrm{Tr}(X-Y)\leq\mathrm{Tr}(P(X-Y))\leq 2\mathrm{Tr}(PQ)=2C_\theta\theta(\theta-1)|\hat{x}-\hat{y}|^{\theta-2}<0.$$

Therefore, by putting $A=X$ and $B=Y$ in Lemma 3.6, and by taking $r_0>0$ ($|\hat{x}-\hat{y}|<r_0$) small enough, from (56)

$$\eta(|\hat{x}-\hat{y}|)||Y||\leq K'C_\theta\eta(|\hat{x}-\hat{y}|)|\hat{x}-\hat{y}|^{\theta-2}\leq KC_\theta|\hat{x}-\hat{y}|^{\theta-2},$$

where $K,K'>0$ are constants. For $r_0>0$ ($|\hat{x}-\hat{y}|<r_0$) small enough, from (29) and (34)

$$w(|\hat{x}-\hat{y}|)|p|^q=w(|\hat{x}-\hat{y}|)C_\theta^q|\hat{x}-\hat{y}|^{q(\theta-1)}$$

$$\leq L(C_\theta|\hat{x}-\hat{y}|^\theta)^q|\hat{x}-\hat{y}|^{\rho_1-q}\leq\frac{\lambda_0}{4}C_\theta|\hat{x}-\hat{y}|^{\theta-2}.$$

Therefore, from (54) and (56),

$$F(\hat{x},p+\tau\hat{x},X+\tau I)-F(\hat{y},p,Y)\geq\frac{C_\theta}{C}|\hat{x}-\hat{y}|^{\theta-2}+o(\tau),$$

where $C>M$ is a constant. We showed (53).

By plugging (53) into (52), since $\nu>0$ can be taken arbitrarily small, for any $0<\theta<1$, we get a contradiction for $r_0>0$ ($|\hat{x}-\hat{y}|<r_0$) small enough. We have proved (11).

Finally, we are to prove Lemma 3.6.

Proof of Lemma 3.6 By multiplying the matrix

$$\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

to the both hand sides of (55) first from right and then from left, we get

$$\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix}\leq\begin{pmatrix} O & O \\ O & 4Q \end{pmatrix}.$$

Thus, for any $t\in\mathbf{R}$ and $\xi\in\mathbf{R^N}$

$$\begin{pmatrix} t\xi & \xi \end{pmatrix}\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix}\begin{pmatrix} t\xi \\ \xi \end{pmatrix}\leq\begin{pmatrix} t\xi & \xi \end{pmatrix}\begin{pmatrix} O & O \\ O & 4Q \end{pmatrix}\begin{pmatrix} t\xi \\ \xi \end{pmatrix},$$

and

$$t^2\langle\xi,(A+B)\xi\rangle + 2t\langle\xi,(A-B)\xi\rangle + \langle\xi,(A+B)\xi\rangle - 4\langle\xi,Q\xi\rangle\leq 0.$$

Hence, for any $|\xi| = 1$,

$$\langle\xi,(A-B)\xi\rangle^2\leq\langle\xi,(A+B)\xi\rangle(4\langle\xi,Q\xi\rangle - \langle\xi,(A+B)\xi\rangle).$$

This yields $\langle\xi,(A+B)\xi\rangle^2\leq 4||A+B||\cdot||Q||$, and since $||A+B||\leq C|\mathrm{Tr}(A+B)|\cdot||Q||$ where $C > 0$ is a constant depending only on $N > 0$, we proved the claim.

4 Other Hölder continuities of viscosity solutions

In this section, we shall study (12):

$$\lambda v(x) + H(\nabla v(x)) - \int_{\mathbf{R^N}}[v(x+z) - v(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle\nabla v(x), z\rangle]c(z)dz - g(x) = 0 \qquad x\in\mathbf{T^N},$$

and (13):

$$\lambda v(x) + F(x,\nabla v(x),\nabla^2 v(x)) - \int_{\mathbf{R^N}}[v(x+z) - v(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle\nabla v(x), z\rangle]c(z)dz - g(x) = 0 \qquad x\in\mathbf{T^N},$$

where $\lambda > 0$. We consider the case other than (I) $N = 1$, and (II) F is uniformly elliptic. So, we are interested in the case of $N\geq 2$, and F (or H) is degenerate elliptic. We assume the conditions (14)-(16).

Example 4.1. The following is an example of F satisfying the conditions (14)-(16).

$$-\sum_{i=1}^{N-1} a_i(x)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{i=1}^{N-1} b_i(x)\frac{\partial u}{\partial x_i}(x) + |\frac{\partial u}{\partial x_N}(x)| \qquad x\in\mathbf{T^N},$$

where $a_i(x) > \exists\lambda_1 > 0$ and $b_i(x)$ $(1\underline{<}i\underline{<}N-1)$ are periodic in $\mathbf{T^N}$. Or, more generally the following Hamilton-Jacobi-Bellman operator satisfies (14)-(16).

$$F(x,u,\nabla u,\nabla^2 u) = \sup_{\alpha\in\mathcal{A}}\{-\sum_{ij=1}^{N} a_{ij}(x,\alpha)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{N} b_i(x,\alpha)\frac{\partial u}{\partial x_i}$$

$$+c(x,\alpha)u - f(x,\alpha)\} \qquad x\in\mathbf{T^N}, \quad (57)$$

where $\mathcal{A}$ a given set (controls), $(a_{ij}(x,\alpha)\in\mathbf{S}^N$ $(\alpha\in\mathcal{A})$ non-negative matrices periodic in $\mathbf{T^N}$ such that there exist matrices σ^α $(\alpha\in A)$ of the size $N\times k$,

$$A_\alpha = (a_{ij}(x,\alpha)) = \sigma(x,\alpha)^T\sigma(x,\alpha),$$

$$A'_\alpha \geq \lambda_1 I_M, \quad A_\alpha = \begin{pmatrix} A'_\alpha & A_{\alpha 12} \\ A_{\alpha 21} & A_{\alpha 22} \end{pmatrix}, \quad A'_\alpha\in\mathbf{S}^M \quad (M<N),$$

where $\lambda_1 > 0$, and $b(x,\alpha) = (b_i(x,\alpha))\in\mathbf{R}^N$, $c(x,\alpha)\in\mathbf{R}$ are bounded, periodic in $\mathbf{T^N}$, and regular enough.

We shall give the results.

Theorem 4.1.

Let v be a periodic viscosity solution of (12) satisfying (5). Assume that (2), (3), (4), and (14) hold, where $\gamma\in(0,2)$, and $\lambda>0$. Let $H(p)=|p|^q$, where $q\geq 1$. Then, for any $\theta\in(0,\theta_0)$, there exists a constant $C_\theta>0$ such that (17) holds. The constant C_θ does not depend on $\lambda\in(0,1)$.

Theorem 4.2.

Let v be a periodic viscosity solution of (13) satisfying (5). Assume that (2), (3), (4), (14), (15) and (16) hold, where $\gamma\in(0,2)$, $\lambda>0$, and that there exist constants $L>0$, $\rho_i>0$ $(i=1,2)$ such that

$$\lim_{s\downarrow 0} w(s)s^{-\rho_1}\underline{<}L, \quad \lim_{s\downarrow 0}\eta(s)s^{-\rho_2}\underline{<}L, \qquad (58)$$

where $\rho_1+\gamma>q$, $\rho_2+\gamma>2$. Then, for any $\theta\in(0,\theta_0)$ there exists a constant $C_\theta>0$ such that (17) holds. The constant C_θ does not depend on $\lambda\in(0,1)$.

Proof of Theorem 4.1. We use the contradiction argument similar to that of Theorem 2.1. Fix $\theta \in (0, \theta_0)$, and let $r_0 > 0$ be small enough. Let us take $\overline{C}_\theta > 0$ such that

$$\overline{C}_\theta r^\theta = 2M, \tag{59}$$

and we shall prove (17) (for $\frac{C_\theta}{\lambda} = \overline{C_\theta}$) by contradiction. For $x, y \in \mathbf{T^N}$ such that $|x - y| \geq r_0$, from (5) we have

$$|v(x) - v(y)| \leq 2M \leq \overline{C}_\theta |x - y|^\theta.$$

Thus, assume that there exist $x', y' \in \mathbf{T^N}$ ($|x' - y'| < r_0$) such that

$$v(x') - v(y') > \overline{C}_\theta |x' - y'|^\theta,$$

and we shall look for a contradiction. As in the proof of Theorem 3.2, take the supconvolution v^r and the infconvolution v_r of v, which are respectively the subsolution and the supersolution of the following problems.

$$\begin{aligned}
\lambda v^r(x) + H(\nabla v^r(x)) - \int_{\mathbf{R^N}} [v^r(x+z) - v^r(x) \\
-\mathbf{1}_{|z|\leq 1} \langle \nabla v^r(x), z \rangle] c(z) dz - g(x) \leq \nu \qquad x \in \mathbf{T^N}, \\
\lambda v_r(x) + H(\nabla v_r(x)) - \int_{\mathbf{R^N}} [v_r(x+z) - v_r(x) \\
-\mathbf{1}_{|z|\leq 1} \langle \nabla v_r(x), z \rangle] c(z) dz - g(x) \geq -\nu \qquad x \in \mathbf{T^N},
\end{aligned}$$

where $\nu > 0$ is an arbitrary small constant. Remark that

$$v^r(x') - v_r(y') > \overline{C}_\theta |x' - y'|^\theta$$

holds for $|x' - y'| < r_0$. Consider

$$\Phi(x, y) = v^r(x) - v_r(y) - \overline{C}_\theta |x - y|^\theta,$$

and let $(\hat{x}, \hat{y})$ be a maximum point of Φ. Let us write $\phi(x, y) = \overline{C}_\theta |x - y|^\theta$. For

$$\nabla_x \phi(x, y) = \overline{C}_\theta \theta |x - y|^{\theta - 2}(x - y) = -\nabla_y \phi(x, y),$$

$$\nabla^2_{xx} \phi(x, y) = \overline{C}_\theta \theta |x-y|^{\theta-2} I + \overline{C}_\theta \theta(\theta - 2)|x-y|^{\theta-2}(x-y) \otimes (x-y) = \nabla^2_{yy} \phi(x, y),$$

put $p = \nabla_x \phi(\hat{x}, \hat{y}) = -\nabla_y \phi(\hat{x}, \hat{y})$, and $Q = \nabla^2_{xx} \phi(\hat{x}, \hat{y}) = \nabla^2_{yy} \phi(\hat{x}, \hat{y})$. As in the proof of Theorem 3.2, by using Lemma 2.4 for $U = v^r$, $V = v_r$, and

$O=\{(x,y)\in \mathbf{R^{2N}} | |x-y|<r_0\}$, we know that there exists $(x_m,y_m)\in \mathbf{T^{2N}}$ such that $\lim_{m\to\infty}(x_m,y_m)=(\hat{x},\hat{y})$. There also exist $(p_m,X_m)\in J^{2,+}_{\mathbf{T^N}}v^r(x_m)$, $(p'_m,Y_m)\in J^{2,-}_{\mathbf{T^N}}v_r(y_m)$ such that $\lim_{m\to\infty}p_m=\lim_{m\to\infty}p'_m=2\alpha(x_m-y_m)=p$, and $X_m\leq Y_m \quad \forall m$. The claim in Lemma 2.4 (iii) leads for any $z\in\mathbf{R^N}$ such that $(x_m+z,y_m+z)\in O$,

$$v^r(x_m+z)-v^r(x_m)-\langle p_m,z\rangle-\{v_r(y_m+z)-v_r(y_m)-\langle p'_m,z\rangle\}\leq 0. \quad (60)$$

Let (ε_m,δ_m) be a pair of positive numbers such that

$$v^r(x_m+z)\leq v^r(x_m)+\langle p_m,z\rangle+\frac{1}{2}\langle X_mz,z\rangle+\delta_m|z|^2 \qquad \text{if} \quad |z|\leq\varepsilon_m, \quad (61)$$

and

$$v_r(y_m+z)\geq v_r(y_m)+\langle p'_m,z\rangle+\frac{1}{2}\langle Y_mz,z\rangle-\delta_m|z|^2 \qquad \text{if} \quad |z|\leq\varepsilon_m. \quad (62)$$

By using the similar argument as in Theorem 3.2, from the definition of viscosity solutions, we have the following.

$$\begin{aligned}
&\lambda(v^r(x_m)-v_r(y_m))+H(p_m)-H(p'_m)\\
&-\frac{1}{2}\int_{|z|\leq\varepsilon_m}\langle (X_m+2\delta_mI)z,z\rangle c(z)dz-\frac{1}{2}\int_{|z|\leq\varepsilon_m}\langle (Y_m-2\delta_mI)z,z\rangle c(z)dz\\
&-\int_{|z|\geq\varepsilon_m}[v^r(x_m+z)-v^r(x_m)-\mathbf{1}_{|z|\leq 1}\langle p_m,z\rangle]c(z)dz\\
&+\int_{|z|\geq\varepsilon_m}[v_r(y_m+z)-v_r(y_m)-\mathbf{1}_{|z|\leq 1}\langle p'_m,z\rangle]c(z)dz\leq g(x_m)-g(y_m)+2\nu.
\end{aligned}$$

Remarking that $X_m\leq Y_m$ and (60) hold, and by using the similar argument as in Theorem 3.2, we can pass $m\to\infty$ in the above inequality to have

$$\lambda(v^r(\hat{x})-v_r(\hat{y}))\leq g(\hat{x})-g(\hat{y})+2\nu,$$

and since $\nu>0$ is arbitrary, we have

$$\lambda\overline{C}_\theta|\hat{x}-\hat{y}|^\theta\leq M|\hat{x}-\hat{y}|^{\theta_0}.$$

Since $\overline{C}_\theta=\frac{2M}{r_0^\theta}$, the above leads

$$2\lambda\leq r_0^{\theta_0}.$$

However, if we take for an arbitrarily fixed $c > \frac{1}{\theta_0}$,

$$r_0 = \lambda^c, \quad \overline{C}_\theta = \frac{2M}{\lambda^{c\theta}}, \tag{63}$$

we get a contradiction for any $\lambda \in (0,1)$. Therefore, for $0 < \theta < \theta_0$, by taking $c = \frac{1}{\theta}$ and thus $\overline{C}_\theta = \frac{2M}{\lambda}$, we proved our claim for $C_\theta = 2M$

$$v(x) - v(y) \leq \overline{C}_\theta |x-y|^\theta = \frac{2M}{\lambda} |\hat{x} - \hat{y}|^\theta \quad \forall x, y \in \mathbf{T}^\mathbf{N}.$$

Proof of Theorem 4.2. The argument is similar to that of Theorem 4.1, and we omit the proof.

5 Strong maximum principle

In this section, we consider

$$F(x, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^\mathbf{N}} [u(x+z) - u(x) - \langle \nabla u(x), z \rangle] c(z) = 0 \qquad \forall x \in \mathbf{R}^\mathbf{N}, \tag{64}$$

where F satisfies (6) and

$$F(x, 0, O) \quad \geq \quad 0 \quad \forall x \in \mathbf{R}^\mathbf{N}. \tag{65}$$

We assume the following condition.
(Almost everywhere positivity) : For any open set $D \in \mathbf{R}^\mathbf{N}$,

$$\int_{z \in D} 1 c(z) dz > 0. \tag{66}$$

Our strong maximum principle is the following.
Theorem 5.1 ([9]).

Consider the integro-differential equation (64), and assume that (2), (3), (65), and (66) hold. Let u be a viscosity subsolution of (64), and assume that it takes a maximum at a point $x_0 \in \mathbf{R}^\mathbf{N}$, i.e.

$$u(x) \leq u(x_0) \qquad \forall x \in \mathbf{R}^\mathbf{N}. \tag{67}$$

Then, u is constant in $\mathbf{R}^\mathbf{N}$ *almost everywhere.*

Proof. From (67), for $p = 0$ and $X = O$,

$$u(x_0 + z) \leq u(x_0) + \langle 0, z\rangle + \frac{1}{2}\langle Oz, z\rangle + \delta|z|^2 \quad \text{if} \quad |z| \leq \varepsilon$$

holds for any $\delta > 0$ and $\varepsilon > 0$. Hence, from the definition of viscosity subsolution

$$F(x_0, 0, O) - \int_{|z|\leq\varepsilon} \frac{1}{2}\langle (O + 2\delta I)z, z\rangle c(z)dz$$

$$- \int_{|z|\geq\varepsilon} [u(x_0 + z) - u(x_0) - \langle 0, z\rangle]c(z)\} \leq 0,$$

holds for any $\delta > 0$ and $\varepsilon > 0$. So, from (65) we have

$$\int_{|z|\geq\varepsilon} [u(x_0) - u(x_0 + z)]c(z) \leq 0$$

holds for any $\varepsilon > 0$. Therefore, from (3), (66), and (67),

$$u(x) \leq u(x_0) \leq u(x) \qquad \text{almost everywhere in} \quad x \in \mathbf{R}^\mathbf{N},$$

and the claim is proved.

Remark 5.1. We shall use the above strong maximum principle to solve the ergodic problem in the next section.

6 Ergodic problem for integro-differential equations

In this section, we apply the results in preceding sections to solve the ergodic problem in $\mathbf{T}^\mathbf{N}$. We shall study the existence of a unique number d_f such that the following problem has a periodic viscosity solution.

$$d_f + F(x, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^\mathbf{N}} [u(x + z) - u(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u(x), z\rangle]c(z)dz - f(x) = 0 \qquad x \in \mathbf{T}^\mathbf{N}. \tag{68}$$

For this purpose, we consider the approximated problem:

$$\lambda u_\lambda + F(x, \nabla u_\lambda, \nabla^2 u_\lambda) - \int_{\mathbf{R^N}} [u_\lambda(x+z) - u_\lambda(x) \tag{69}$$

$$-\mathbf{1}_{|z|\le 1}\langle \nabla u_\lambda(x), z\rangle] c(z) dz - f(x) = 0 \qquad x \in \mathbf{T^N},$$

where $\lambda \in (0,1)$, and we shall see whether there exists the following unique limit number

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d_f \qquad \text{uniformly in} \quad \mathbf{T^N}.$$

We assume that F satisfies (22), and that the following hold.

(Periodicity) :

$$F(\cdot, p, X), \quad f(\cdot) \quad \text{are periodic in} \quad x \in \mathbf{T^N}, \quad \forall (p, X) \in (\mathbf{R^N} \times \mathbf{S^N}). \tag{70}$$

(Homogeneity) : The partial differential operator F is positively homogenious in degree one

$$F(x, \xi p, \xi X) = \xi F(x, p, X)$$

$$\forall \xi > 0, \quad \forall x \in \mathbf{T^N}, \quad \forall p \in \mathbf{R^N}, \quad \forall X \in \mathbf{S^N}. \tag{71}$$

As we have seen in Theorem 2.2, under (22) and (70), the comparison result holds. From the Perron's method (see [5] and [6]), it is known that there exists a unique periodic viscosity solution u_λ of (69) for any $\lambda \in (0,1)$. Now, we state our main result.

Theorem 6.1.

Let u_λ ($\lambda \in (0,1)$) be the periodic viscosity solution of (69). Assume that the conditions in Theorem 5.1, (22), (70), and (71) hold. Fix an arbitrary point $x_0 \in \mathbf{T^N}$. Then, the following hold.
(i) Assume that the conditions in Theorem 3.1, or those in Theorem 3.2 hold. Then, there exist a unique number d_f and a periodic function u such that

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d_f, \quad \lim_{\lambda \downarrow 0} (u_\lambda(x) - u_\lambda(x_0)) = u(x) \quad \textit{uniformly in} \quad \mathbf{T^N}, \tag{72}$$

such that (68) holds in the sense of viscosity solutions.
(ii) Assume that the conditions in Theorem 4.1, or those in Theorem 4.2 hold. Then, there exists a unique number d_f such that

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d_f \quad \textit{uniformly in} \quad \mathbf{T^N},$$

which is characterized by the following. For any $\nu > 0$ there exist a periodic viscosity subsolution $\underline{u}$ and a periodic viscosity supersolution $\overline{u}$ of

$$d_f + F(x, \nabla \underline{u}, \nabla^2 \underline{u}) - \int_{\mathbf{R^N}} [\underline{u}(x+z)$$

$$-\underline{u}(x) - \mathbf{1}_{|z|\leq 1}\langle \nabla \underline{u}(x), z\rangle] c(z)dz - f(x) \leq \nu \qquad x \in \mathbf{T^N},$$

$$d_f + F(x, \nabla \overline{u}, \nabla^2 \overline{u}) - \int_{\mathbf{R^N}} [\overline{u}(x+z)$$

$$-\overline{u}(x) - \mathbf{1}_{|z|\leq 1}\langle \nabla \overline{u}(x), z\rangle] c(z)dz - f(x) \geq -\nu \qquad x \in \mathbf{T^N}.$$

Proof of Theorem 6.1. (i) We shall prove the claim in the following three steps.
(Step 1.) We prove the uniform boundedness of $v_\lambda(x) = u_\lambda(x) - u_\lambda(x_0)$:

$$|v_\lambda(x)| = |u_\lambda(x) - u_\lambda(x_0)| \leq \exists M' \quad \forall x \in \mathbf{T^N}, \quad \forall \lambda \in (0,1), \qquad (73)$$

by a contradiction argument. Assume that there exists a subsequence $\lambda' \to 0$ such that

$$\lim_{\lambda' \to 0} |v_{\lambda'}|_{L^\infty} = \infty, \qquad (74)$$

and we shall look for a contradiction. Put $w_\lambda(x) = \frac{v_\lambda(x)}{|v_\lambda|_{L^\infty}}$. By (71), remark that w_λ satisfies

$$\lambda w_\lambda + F(x, \nabla w_\lambda, \nabla^2 w_\lambda) - \int_{\mathbf{R^N}} [w_\lambda(x+z) - w_\lambda(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla w_\lambda(x), z\rangle] c(z)dz - \frac{f(x) - \lambda u_\lambda(x_0)}{|v_\lambda|_{L^\infty}} = 0 \qquad x \in \mathbf{T^N}, \qquad (75)$$

and that

$$|w_\lambda|_{L^\infty} = 1, \quad w_\lambda(x_0) = 0 \qquad \forall \lambda \in (0,1). \qquad (76)$$

From the comparison result for (69), there exists a constant $C > 0$ such that $|\lambda u_\lambda|_{L^\infty} < C$ ($\forall \lambda \in (0,1)$), and thus from (74) and (76) there exists a constant $M > 0$ such that

$$|\lambda' w_{\lambda'} - \frac{f(x) - \lambda' u_{\lambda'}(x_0)}{|v_{\lambda'}|_{L^\infty}}| \leq M, \quad |w_{\lambda'}| \leq M.$$

Therefore, by applying Thorems 3.1 and 3.2 to (75) for $g = -(\lambda' w_{\lambda'} - \frac{f(x)-\lambda' u_{\lambda'}(x_0)}{|v_{\lambda'}|_{L^\infty}})$ and $\theta_0 = 0$, we know that there exist $\theta \in (0,1)$ and a constant $C_\theta > 0$ such that

$$|w_{\lambda'}(x) - w_{\lambda'}(y)| \le C_\theta |x-y|^\theta \quad \forall x, y \in \mathbf{T^N}.$$

So, by the Ascoli-Alzera theorem, there exists an Hölder continuous function w such that

$$\lim_{\lambda' \to 0} w_{\lambda'}(x) = w(x) \quad \text{uniformly in} \quad \mathbf{T^N},$$

and from (76)

$$|w|_{L^\infty} = 1, \quad w(x_0) = 0.$$

Moreover, by putting $\lambda = \lambda'$ in (75), and by passing $\lambda' \to 0$, since the limit procedure of viscosity solutions, introduced by Barles and Perthame [11] (see also [5] and [16]), is valid for the present nonlocal case, we see that w is a viscosity solution of

$$F(x, \nabla w, \nabla^2 w) - \int_{\mathbf{R^N}} [w(x+z) - w(x) - \mathbf{1}_{|z|\le 1} \langle \nabla w(x), z \rangle] c(z) dz = 0 \qquad \forall x \in \mathbf{T^N}. \tag{77}$$

However, since F satisfies (65), by the strong maximum principle (Theorem 5.1), and by taking account that w is periodic in $\mathbf{T^N}$, we see that w is almost everywhere constant in $\mathbf{T^N}$. This contradicts to the fact that w is an Hölder continuous function such that $|w|_{L^\infty} = 1$ and $w(x_0) = 0$. Therefore, the assumption (74) is false, and we have proved (73).

(Step 2.) From Step 1, we see that v_λ ($\lambda \in (0,1)$) satisfies (73) and

$$F(x, \nabla v_\lambda, \nabla^2 v_\lambda) - \int_{\mathbf{R^N}} [v_\lambda(x+z) - v_\lambda(x) - \mathbf{1}_{|z|\le 1} \langle \nabla v_\lambda(x), z \rangle] c(z) dz - (f - \lambda u_\lambda(x_0) - \lambda v_\lambda) = 0 \quad \text{in} \quad \mathbf{T^N}, \tag{78}$$

and there exists $M > 0$ such that

$$|f(x) - \lambda u_\lambda(x_0) - \lambda v_\lambda(x)| < M \quad \forall x \in \mathbf{T^N}, \quad \forall \lambda \in (0,1).$$

We apply again the result in Theorems 3.1 and 3.2 to (78), and see that there exist $\theta \in (0,1)$ and a constant $C_\theta > 0$ such that

$$|v_\lambda(x) - v_\lambda(y)| \le C_\theta |x-y|^\theta \quad \forall x, y \in \mathbf{T^N}, \quad \forall \lambda \in (0,1).$$

So, we can take a subsequence $\lambda' \to 0$ of $\lambda \to 0$ such that

$$\lim_{\lambda'\to 0} v_{\lambda'}(x) = \lim_{\lambda'\to 0}(u_{\lambda'}(x) - u_{\lambda'}(x_0)) = \exists u(x) \quad \text{uniformly in} \quad \mathbf{T^N},$$

$$\lim_{\lambda'\to 0} \lambda' u_{\lambda'}(x) = \lim_{\lambda'\to 0} \lambda' u_{\lambda'}(x_0) = d_f \quad \text{uniformly in} \quad \mathbf{T^N}.$$

In the next step, we shall prove that the limit d_f is independent on the choise of the subsequence $\lambda' \to 0$.

(Step 3.) We shall prove the uniqueness of the limit number d_f obtained in Step 2. Let (d_f, u), and (d'_f, u') $(d_f \neq d'_f)$ be two pairs of the limit numbers and the limit functions. Thus,

$$d_f + F(x, \nabla u, \nabla^2 u) - \int_{\mathbf{R^N}} [u(x+z) - u(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u(x), z\rangle] c(z)dz - f(x) = 0 \quad \text{in} \quad \mathbf{T^N},$$

and

$$d'_f + F(x, \nabla u', \nabla^2 u') - \int_{\mathbf{R^N}} [u'(x+z) - u'(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u'(x), z\rangle] c(z)dz - f(x) = 0 \quad \text{in} \quad \mathbf{T^N}.$$

We may assume that $d'_f < d_f$, and by adding a constant if necessary we may also assume that $u > u'$. For any small $\nu > 0$, by choosing $\lambda > 0$ small enough we see that u and u' are respectively a viscosity subsolution and a viscosity supersolution of the following problems.

$$\lambda u + F(x, \nabla u, \nabla^2 u) - \int_{\mathbf{R^N}} [u(x+z) - u(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u(x), z\rangle] c(z)dz - f(x) \leq \nu - d_f \quad \text{in} \quad \mathbf{T^N}.$$
$$\lambda u' + F(x, \nabla u', \nabla^2 u') - \int_{\mathbf{R^N}} [u'(x+z) - u'(x)$$
$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u'(x), z\rangle] c(z)dz - f(x) \geq -\nu - d'_f \quad \text{in} \quad \mathbf{T^N}.$$

Then, from the comparison result

$$0 < \lambda(u - u')(x) \leq d'_f - d_f < 0,$$

which is a contradiction. Thus, d_f is the unique number such that (68) has a viscosity solution. We have proved the claim of (i).

(ii) We treat the case that the partial differential operator is F. (The proof for the case of H is same, and we omit it.) Let $v_\lambda = u_\lambda - u_\lambda(x_0)$, and put $|v_\lambda|_\infty = \frac{C_\lambda}{\lambda}$. We shall prove the claim in the following three steps.
(Step 1.) If for a subsequence $\lambda' \to 0$, $\lim_{\lambda' \to 0} C_{\lambda'} = 0$, then

$$|\lambda' u_{\lambda'}(x) - \lambda' u_{\lambda'}(x_0)|_\infty = \lambda' |v_{\lambda'}|_\infty = C_{\lambda'} \to 0,$$

which implies the existence of a constant

$$d_f = \lim_{\lambda' \to 0} \lambda' u_{\lambda'}(x) = \lim_{\lambda' \to 0} \lambda' u_{\lambda'}(x_0) \qquad \text{uniformly in} \quad \mathbf{T^N}.$$

(Step 2.) Now, assume that for any subsequence $\lambda' \to 0$, $C_{\lambda'}$ does not converge to zero. That is, there exists a number $C_0 > 0$ such that $\liminf_{\lambda \to 0} C_\lambda \geq C_0 > 0$. From the comparison result for (69), $|v_\lambda|_\infty \leq \frac{2M}{\lambda}$, and thus $0 < C_\lambda \leq 2M$ $(\forall \lambda \in (0,1))$ holds. Hence, we can take a subsequence $\lambda' \to 0$ such that $\lim_{\lambda' \to 0} C_{\lambda'} = \overline{C}$ $(\lim_{\lambda' \to 0} \lambda' |u_{\lambda'} - u_{\lambda'}(x_0)|_\infty = \overline{C})$, where $C_0 \leq \overline{C} \leq 2M$. For simplicity, we shall use λ in place of λ'. Then, for $w_\lambda = \frac{v_\lambda}{|v_\lambda|_\infty}$ we have

$$\lambda w_\lambda + F(x, \nabla w_\lambda, \nabla^2 w_\lambda) - \int_{\mathbf{R^N}} [w_\lambda(x+z) - w_\lambda(x)$$

$$-\langle \nabla w_\lambda(x), z \rangle] c(z) dz - \frac{\lambda}{C_\lambda}(f(x) - \lambda u_\lambda(x_0)) = 0 \quad \text{in} \quad \mathbf{T^N}, \qquad (79)$$

where $|w_\lambda|_\infty = 1$, $w_\lambda(x_0) = 0$, $\lim_{\lambda \to 0} C_\lambda = \overline{C}$. Here, we claim that for a constant $\theta \in (0, \theta_0)$, there exists $C_\theta > 0$ independent on $\lambda \in (0,1)$ such that

$$|w_\lambda(x) - w_\lambda(y)| \leq C_\theta |x-y|^\theta, \qquad (80)$$

which can be proved by the similar contradiction argument used in the proof of Theorem 4.2, and which we omit here.

From (80), there exists a subsequence $\lambda' \to 0$ such that $\lim_{\lambda' \to 0} w_{\lambda'}(x) = \exists w(x)$, where the limit w is also Hölder continuous, $|w|_\infty = 1$, $w(x_0) = 0$ and is the viscosity solution of

$$F(x, \nabla w, \nabla^2 w) - \int_{\mathbf{R^N}} [w(x+z) - w(x) - \langle \nabla w(x), z \rangle] c(z) = 0 \quad \text{in} \quad \mathbf{T^N}.$$

However, the strong maximum principle (Theorem 5.1) asserts that w is almost everywhere constant, which is a contradiction. Therefore, $\liminf_{\lambda'\to 0} C_{\lambda'} = \overline{C} > 0$ is false.
(Step 3) From Steps 1 and 2, we see that there exists a subsequence $\lambda' \to 0$ such that $\lim_{\lambda'\to 0} \lambda' u_{\lambda'}(x) = d_f$ uniformly in $\mathbf{T^N}$. Therefore, for any $\nu > 0$ there exists $\lambda' > 0$ small enough such that

$$\begin{aligned} d_f + F(x, \nabla u_{\lambda'}, \nabla^2 u_{\lambda'}) - \int_{\mathbf{R^N}} [u_{\lambda'}(x+z) \\ -u_{\lambda'}(x) - \mathbf{1}_{|z|\leq 1}\langle \nabla u_{\lambda'}(x), z\rangle] c(z)dz - f(x) \leq \nu \quad x \in \mathbf{T^N}, \\ d_f + F(x, \nabla u_{\lambda'}, \nabla^2 u_{\lambda'}) - \int_{\mathbf{R^N}} [u_{\lambda'}(x+z) \\ -u_{\lambda'}(x) - \mathbf{1}_{|z|\leq 1}\langle \nabla u_{\lambda'}(x), z\rangle] c(z)dz - f(x) \geq -\nu \qquad x \in \mathbf{T^N}. \end{aligned} \tag{81}$$

The uniqueness of d_f can be proved in a similar way to the proof for (i).

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